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ON THE ZEROS OF PAIRS OF LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. Suppose that f is meromorphic in the plane and that F and G are given by

$$F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)}, \qquad G = f^{(k)} + \sum_{j=0}^{k-1} b_j f^{(j)},$$

with $k \geq 1$ and the a_j, b_j rational functions, such that $a_j \not\equiv b_j$ for at least one j. We classify those f for which F and G have only finitely many zeros.

1. Introduction

The study of zeros of linear differential polynomials has a long history, going back to the fundamental work of Pólya [28] on entire and meromorphic functions and their derivatives. The following theorem was proved by the first author and Hennekemper and Polloczek [5], [7] for $k \geq 3$ and by the second author [20] for $k = 2$, and confirmed a conjecture of Hayman [9], [10], [11] from 1959.

Theorem A. *Suppose that f is meromorphic in the plane and that f and $f^{(k)}$ have only finitely many zeros, for some $k \geq 2$. Then we have $f(z) = R(z)e^{P(z)}$, with R a rational function and P a polynomial. In particular, f has finite order and finitely many poles.*

Refinements of this theorem may be found in [6], [20], [21], [23], while simple examples show that no comparable result holds for $k = 1$ (see however [4]). A natural generalization of Theorem A involves replacing the k'th derivative $f^{(k)}$ by a linear differential polynomial

$$F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)}, \tag{1}$$

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with coefficients a_j which are rational functions. Thus the first author and Hellerstein proved in [6] that if f is meromorphic in the plane and

$$N(r,1/f)+N(r,1/F)=o\big(T(r,f'/f)\big),\qquad r\to\infty,$$

in which $k\ge 3$ and F is given by (1) with polynomial coefficients a_j, and in which the notation is that of [10], then f'/f has finite order. Subsequent papers [3], [29] determined all functions f meromorphic in the plane for which f and F, subject to the above assumptions, have no zeros, while the papers [20], [22] give a rather more complicated classification of all functions f meromorphic in the plane such that f and $f''+a_1f'+a_0f$ have only finitely many zeros, for any rational functions a_1, a_0. Related results appear in [14], [19], [27] and elsewhere.

With regard to these results, it seems reasonable to ask how essential the hypothesis on the zeros of f really is. Of course, it is easy to give examples of entire f for which F, as given by (1), has no zeros: just set $F=e^P$, with P a polynomial, and solve the resulting differential equation for f. However, some conclusion regarding poles might be expected, and the following theorem [24], [25], [26] summarizes some results in this direction.

Theorem B. *Suppose that f is meromorphic of finite order in the plane, and that f'' has only finitely many zeros. Then*

$$\overline{N}(r,f)=O(\log r)^3,\qquad r\to\infty.$$

If, in addition, $T(r,f)=O(r)$ or $N(r,1/f')=o(r^{1/2})$ as $r\to\infty$, then f has only finitely many poles.

On the other hand, examples of meromorphic f having infinite order, such that f' and f'' have no zeros, while f has an arbitrary set of poles, were given in [24], and we show in the next section how to construct examples of functions f and linear differential polynomials F in f, such that F and F' have no zeros, while f has an arbitrary set of poles. Thus the zeros of a single linear differential polynomial in f do not suffice to determine f.

In the present paper, we consider two linear differential polynomials

$$F=L_k(f)=f^{(k)}+\sum_{j=0}^{k-1}a_jf^{(j)},\qquad G=M_k(f)=f^{(k)}+\sum_{j=0}^{k-1}b_jf^{(j)},$$

in a meromorphic function f, with k a positive integer and the a_j and b_j rational functions, and with $a_j\not\equiv b_j$ for at least one j. There is a well-known reduction procedure [17], described in Lemma 1 below, to obtain linear differential operators P, Q, H with coefficients which are rational functions, such that $L_k=P(H)$ and $M_k=Q(H)$ and the common (local) solutions of the homogeneous equations

$$L_k(w)=0,\qquad M_k(w)=0 \tag{2}$$

are precisely the (local) solutions of $H(w)=0$. This allows us to concentrate on the case where the equations (2) have no non-trivial common (local) solution, that is, no common (local) solution other than the trivial solution $w\equiv 0$, for in the contrary case we may regard F and G as linear differential polynomials in $H(f)$. Our main result is then the following.

Theorem 1. *Let k be a positive integer and let $a_0,\dots,a_{k-1}$ and $b_0,\dots,b_{k-1}$ be rational functions with $a_j\not\equiv b_j$ for at least one j. Assume that the equations*

$$w^{(k)}+\sum_{j=0}^{k-1}a_jw^{(j)}=0,\qquad w^{(k)}+\sum_{j=0}^{k-1}b_jw^{(j)}=0,$$

have no non-trivial common (local) solution. Let f be meromorphic in the plane such that

$$F=f^{(k)}+\sum_{j=0}^{k-1}a_jf^{(j)},\qquad G=f^{(k)}+\sum_{j=0}^{k-1}b_jf^{(j)}$$

both have only finitely many zeros. Then f has finite order and finitely many zeros and f'/f has a representation

$$\frac{f'(z)}{f(z)}=Y(z)+\frac{P_0\big(Q(z)+\log S(z)\big)\big(Q'(z)+S'(z)/S(z)\big)}{S(z)e^{Q(z)}-1},\tag{3}$$

in which S and Y are rational functions and Q and P_0 are polynomials, and at least one of P_0 and S is constant.

In the next section we will give examples showing that (3) can indeed occur. Our approach to proving Theorem 1 exploits the fact that F and G have, with finitely many exceptions, the same poles, and proceeds via the rather surprising conclusion that f itself has finitely many zeros. This allows us to use the machinery developed in [5], [6], [7].

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2. Examples

2.1. Example. Let $a_0,\dots,a_{k-2}$ and P be polynomials, and let f be a solution of the equation

$$f^{(k-1)}+\sum_{j=0}^{k-2}a_jf^{(j)}=K=e^P.$$

Let c, d be distinct constants. Then $F=K'+cK$ and $G=K'+dK$ are both linear differential polynomials of order k in f, having finitely many zeros. However, here F and G should, according to the reduction procedure referred to in the discussion of the system (2), more correctly be regarded as linear differential polynomials in K.

2.2. Example. Setting $f(z) = \tan z$ we find that $F = f'' - 2if'$ and $G = f'' + 2if'$ are both zero-free. This example does not, however, contradict Theorem 1 since the equations $w'' - 2iw' = 0$, $w'' + 2iw' = 0$ have the non-trivial common solution $w = 1$, and F and G are more properly regarded as linear differential polynomials in f'.

2.3. Example. Let P and P_1 be polynomials, with P non-constant and P_1 not identically zero, chosen so that $P_1(P)$ is a non-positive integer at every zero of $e^P - 1$. For example, we may take $P_1(P) = P^2 4^{-1}\pi^{-2}$. Then

$$f'/f = T(e^P - 1)^{-1} = -P_1(P)P' + P_1(P)P'e^P(e^P - 1)^{-1}, \qquad T = P_1(P)P',$$

defines a meromorphic function having no zeros, and poles at all but finitely many zeros of $e^P - 1$, while the equations

$$f'(z) = 0, \qquad f'(z) + T(z)f(z) = 0$$

each have only finitely many solutions z. Further, with a and b rational functions we define L by

$$L/f = f''/f + af'/f + b = S(e^P - 1)^{-2},$$

where

$$S = be^{2P} + e^P(T' - TP' + aT - 2b) + (T^2 - T' - aT + b),$$

and L cannot vanish identically, since f has infinitely many poles. There are thus three ways to ensure that L/f has only finitely many zeros, the same then being true of L. We can either solve simultaneously both equations

$$aT - 2b = TP' - T', \qquad -aT + b = T' - T^2, \tag{4}$$

for a and b, using the fact that the determinant of the coefficients is $-T$, which is not identically zero, or we can set $b = 0$, and solve either of the equations (4) for a. To see that a non-zero rational function Y can arise in (3), we need only write $f = Ue^V g$, with U a rational function and V a polynomial, so that there are linear differential polynomials G_1, G_2 in g, with coefficients which are rational functions and with G_1/G_2 non-constant, each having finitely many zeros.

2.4. Example. Let c be a constant, let $k \geq 1$ and let $A_0, \ldots, A_k$ be polynomials with $A_k = 1$, and define the operator L by

$$L = \sum_{j=0}^{k} A_j D^j, \qquad D = d/dz.$$

Let a_n and M_n be sequences, such that each M_n is a positive integer, while the complex sequence (a_n) tends to infinity, without repetition, as $n \to \infty$. Define rational functions $R_n(z)$ by

$$R_n(z) = L\big((z - a_n)^{-M_n}\big).$$

Then R_n has a pole of order $M_n + k$ at a_n. Let g be an entire function having a simple zero at each a_n, and no other zeros. Using Mittag–Leffler interpolation, choose an entire function h such that we have, for each n,

$$c + g(z)^{-1}e^{h(z)} = R_n'(z)/R_n(z) + O(|z - a_n|^{M_n+k-1})$$

as z tends to a_n. Define H by

$$H'/H = c + g^{-1}e^h.$$

Then there are non-zero constants b_n such that we have, for each n,

$$H(z) = b_n R_n(z)\big(1 + O(|z - a_n|^{M_n+k})\big) = b_n R_n(z) + O(1)$$

as z tends to a_n. Hence there is a function h_n analytic at a_n such that $H(z) - b_n R_n(z) = h_n(z)$ on a punctured neighbourhood U_n of a_n. It follows that if w is a solution of the equation $L(w) = H(z) - b_n R_n(z)$ on a simply connected subdomain V_n of U_n then w has an analytic extension to a neighbourhood of a_n. If f_1 is a solution of the equation $L(f_1) = H$ on V_n then f_1 may be written in the form

$$f_1(z) = b_n(z - a_n)^{-M_n} + w(z) + v(z),$$

in which $L(v) = 0$ so that v is the restriction to V_n of an entire function. It follows that f_1 has an analytic extension to U_n with a pole at a_n. Therefore every local solution f of $L(f) = H$ extends to a function meromorphic in the plane and, since every zero of g is a pole of H, both $H = L(f)$ and $H' - cH$ have no zeros.

3. Preliminaries

The following lemma is well known [17, p. 126].

Lemma 1. *Let k, n be non-negative integers with $k \geq n$ and let D denote d/dz, and let linear differential operators L_1, L_2 of orders k, n be defined by*

$$L_1 = \sum_{j=0}^{k} a_j D^j, \qquad L_2 = \sum_{j=0}^{n} b_j D^j,$$

in which $a_0, \dots, a_k$, $b_0, \dots, b_n$ are rational functions with $a_k b_n \not\equiv 0$. Then there exist an integer q with $0 \leq q \leq n$ and an operator $H = \sum_{j=0}^{q} c_j D^j$, with the coefficients c_j rational functions and $c_q \not\equiv 0$, and linear differential operators Q_1, Q_2, P_1, P_2 with rational functions as coefficients, such that

$$L_1 = Q_1(H), \qquad L_2 = Q_2(H), \qquad P_1(L_1) + P_2(L_2) = H,$$

in which the parentheses denote composition. Further, if w is meromorphic on some domain U, we have $H(w) \equiv 0$ on U if and only if $L_1(w) \equiv L_2(w) \equiv 0$ on U. Moreover, the operators Q_1, Q_2 have orders $k - q$, $n - q$ respectively, while the operators P_1, P_2 both have order at most k.

Proof. This is just the Euclidean algorithm for linear differential operators but, since we need the estimate for the orders of P_1 and P_2, we present a proof. We proceed by induction on n, there being nothing at all to prove when $n=0$, as in this case H is the identity operator. Assuming the result true when one of the operators has order less than n, we apply the division algorithm [17, p. 126] for linear differential operators in order to write

$$L_1 = L(L_2) + M_1$$

with L and M_1 each a linear differential operator with rational functions as coefficients, and in which M_1 either is the zero operator or has order less than n. Plainly, the order of L is $k-n$. If M_1 is the zero operator we write $H = L_2$ and $Q_1 = L$, and P_1 is the zero operator, with P_2 and Q_2 the identity.

Now assume that M_1 is not the zero operator. The induction hypothesis gives us operators H, p_1, p_2, q_1, q_2 such that the orders of p_1 and p_2 are at most n, and such that

$$L_2 = q_2(H), \qquad M_1 = q_1(H), \qquad p_1(M_1) + p_2(L_2) = H.$$

Now we set $Q_1 = L(q_2) + q_1$, $Q_2 = q_2$ and we have

$$H = p_1(L_1) + \big(p_2 - p_1(L)\big)(L_2).$$

Thus $P_1 = p_1$ and $P_2 = p_2 - p_1(L)$ have order at most k. The remaining assertion is obvious.

The next lemma is also fairly standard.

Lemma 2. *There exists a positive constant c with the following properties. Suppose that f is transcendental and meromorphic in the plane, and that r is large and $N > 1$. Then we have*

$$\big|\log|f(z)|\big| \le cN^2 T(r,f)$$

for all z with $\frac{1}{2}r \le |z| \le re^{-2/N}$ and lying outside a union of discs having sum of radii at most $4erN^{-2}$.

Proof. We denote by d positive constants not depending on f, r, N. Let $r_j = re^{-j/N}$, $j = 1, 2$. Then provided r is large enough we have

$$n(r_1, f) + n(r_1, 1/f) \le \big(\log(r/r_1)\big)^{-1}\big(2T(r,f) + \log|1/f(0)|\big) \le dNT(r,f), \tag{5}$$

with minor modifications if $f(0) = 0, \infty$. Let the zeros and poles of f in $\frac{1}{4}r \le |z| \le r_1$ be $a_1, \dots, a_n$ and $b_1, \dots, b_m$, respectively, repeated according to multiplicity, and write

$$f(z) = g(z)F(z)G(z)^{-1}, \qquad F(z) = \prod_{j=1}^{m}(1 - z/b_j), \qquad G(z) = \prod_{k=1}^{n}(1 - z/a_k),$$

so that g is analytic and non-zero in $\frac{1}{4}r \le |z| \le r_1$. For $|z| \le r$ we have, using (5),

$$\log^+ |F(z)| + \log^+ |G(z)| \le d(m+n) \le dNT(r,f). \tag{6}$$

We also have [13, p. 366]

$$\log\left|\prod_{k=1}^{n}(z-a_k)\right| \ge n\log(rN^{-2})$$

outside a union E_1 of discs having sum of radii at most $2erN^{-2}$, so that for z satisfying $\frac{1}{2}r \le |z| \le r_1$ but lying outside E_1 we have

$$\begin{aligned} \log|G(z)| &\ge n\log(rN^{-2}) - \sum_{k=1}^{n}\log|a_k| \\ &\ge n\log(rN^{-2}) - n\log r \ge -dN^2T(r,f), \end{aligned} \tag{7}$$

using (5). Using the fact that $F(0) = G(0) = 1$, we clearly have

$$T(r,g) \le dNT(r,f), \tag{8}$$

by (6). Finally, a standard application of the Poisson–Jensen formula to $g(\zeta)$ in $|\zeta| \le r_1$ gives, for $\frac{1}{2}r \le |z| \le r_2$,

$$\begin{aligned} \big|\log|g(z)|\big| &\le \frac{r_1+r_2}{r_1-r_2}\big(m(r_1,g)+m(r_1,1/g)\big) + dn(r_1,g) + dn(r_1,1/g) + O(\log r) \\ &\le dN^2T(r,f), \end{aligned}$$

using (5) and (8) and, combining the last estimate with (6) and (7), the result follows.

We use the following notation in the next lemma and henceforth. If g is meromorphic in $0 \le r_1 \le |z| < \infty$ then by a result of Valiron [30, p. 15] we may write $g(z) = z^N h(z) g_1(z)$, in which g_1 is meromorphic in the plane, N is an integer, and h is analytic in $|z| \ge r_1$ with $h(\infty) = 1$. With $n(r,g)$ the number of poles of g in $r_1 \le |z| \le r$, the Nevanlinna characteristic is defined for $r \ge r_1$ by [2, p. 89]

$$\begin{aligned} &T(r,g) = m(r,g) + N(r,g) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |g(re^{i\theta})|\,d\theta + N(r,g), \\ &N(r,g) = \int_R^r n(t,g)\,\frac{dt}{t} \end{aligned} \tag{9}$$

and we have $T(r,g) = T(r,1/g) + O(\log r)$. Further, $S(r,g)$ will denote any quantity such that

$$S(r,g) = O\big(\log^+ T(r,g) + \log r\big) \qquad \text{(n.e.)}, \tag{10}$$

in which (n.e.) ('nearly everywhere') means as r tends to infinity outside a set of finite measure. In particular, $m(r,g'/g) = S(r,g)$.

We denote sectorial regions using

$$S^*(r,\alpha,\beta) = \{z : |z| > r,\ \alpha < \arg z < \beta\},$$

this a region on the Riemann surface of $\log z$ if $\beta - \alpha > 2\pi$.

Lemma 3. *Suppose that $f_1,\dots,f_N$ are functions analytic in the region $S = S^*(r_0,-\pi,\pi)$ and each admitting unrestricted analytic continuation in $|z| > r_0$, the continuations satisfying*

$$\log^+\log^+|f_j(z)| = O(\log|z|) \tag{11}$$

on $S^*(r_0,-2\pi,2\pi)$. Suppose that g is meromorphic in $|z| > r_0$. Suppose further that for some positive integer Q, each of the functions $g_1,\dots,g_k$ on S is a polynomial in the $f_j^{(m)}$, $g^{(m)}$, $1 \le j \le k$, $0 \le m \le Q$. Suppose finally that $g_1,\dots,g_k$ are linearly independent solutions in S of an equation

$$w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0, \tag{12}$$

in which the A_j are meromorphic in $|z| > r_0$. Then we have, for $j = 0,\dots,k-1$,

$$m(r,A_j) = S(r,g). \tag{13}$$

Proof. In this proof we use characteristic functions defined as in (9). When the f_j are meromorphic in $|z| > r_0$, the estimate (13) is well known [6]. We first note that each g_μ, being a polynomial in the $f_j^{(m)}$ and $g^{(m)}$, may be written as a quotient of function elements admitting unrestricted analytic continuation in $|z| > r_0$, and that continuing g_j in this way around any curve homotopic to zero in $|z| > r_0$ leads back to g_j. Since each g_j solves (12), there exist constants $c_{j,m}$ such that continuing g_j once counter-clockwise around any circle $|z| = r > r_0$ leads to

$$\sum_{m=1}^{k} c_{j,m} g_m.$$

The usual eigenvalue argument [17, p. 358] then gives a solution g^* of (12), which we can assume without loss of generality is g_1, such that under this continuation g_1 leads to a constant multiple of itself. Thus, for some constant c, the function

$$h_1 = z^c g_1$$

is meromorphic in $|z| > r_0$. We then have, for $j = 1, \dots, k$,

$$\begin{aligned} m(r, g_1^{(j)}/g_1) &\leq \sum_{m=1}^{j} m(r, h_1^{(m)}/h_1) + O(\log r) \\ &\leq O\big(\log^+ T(r, h_1) + \log r\big) \qquad \text{(n.e.)}. \end{aligned} \tag{14}$$

Further, (11) gives, for $1 \leq j \leq k$,

$$\log^+ \log^+ |f_j^{(m)}(z)| = O(\log |z|)$$

on $S^*(2r_0, -\pi, \pi)$ and recalling the representation of the g_μ as polynomials in the $f_j^{(m)}$ and $g^{(m)}$ we get, for some positive constant M,

$$T(r, h_1) \leq m(r, h_1) + O\big(N(r, g)\big) \leq O\big(T(r, g) + r^M\big) \qquad \text{(n.e.)}. \tag{15}$$

We now proceed by induction on k, and for $k = 1$ the result already follows, since $A_0 = -g_1'/g_1$. Assuming now that the result is true for $k - 1$, we apply the familiar reduction of order method [17] to write $v_j = g_j/g_1$ for $j = 2, \dots, k$, so that each v_j solves an equation

$$v^{(k)} + \sum_{m=1}^{k-1} B_m v^{(m)} = 0, \tag{16}$$

and the B_m are meromorphic in $|z| > r_0$ and can be calculated from the coefficients $A_1, \dots, A_{k-1}$ and the $g_1^{(j)}/g_1$ as follows. We have, with $A_k = 1$,

$$B_m = A_m + \sum_{j=m+1}^{k} A_j P_{j,m}(g_1'/g_1), \qquad m = 1, \dots, k-1. \tag{17}$$

Here each $P_{j,m}(g_1'/g_1)$ is a differential polynomial in g_1'/g_1 with constant coefficients. We regard (16) as an equation of order $k-1$ in the $w_j = v_j'$ and we then write $y_j = w_j g_1^2 = g_j' g_1 - g_j g_1'$. The y_j solve

$$y^{(k-1)} + \sum_{m=0}^{k-2} C_m y^{(m)} = 0,$$

with coefficients C_m meromorphic in $|z| > r_0$, and the y_j are themselves polynomials in the $f_j^{(m)}$ and $g^{(m)}$. Further, with $B_k = 1$ we have

$$C_m = B_{m+1} + \sum_{j=m+2}^{k} B_j Q_{j,m}(g_1'/g_1), \qquad m = 0, \dots, k-2, \tag{18}$$

in which each $Q_{j,m}(g_1'/g_1)$ is a differential polynomial in g_1'/g_1, with constant coefficients.

From the induction hypothesis we deduce that $m(r, C_m) = S(r, g)$ for each m, so that the same is true of the B_m, using (14), (15) and (18). We now have $m(r, A_m) = S(r, g)$ for $m = 1, \dots, k-1$, using (17), and (12) and (14) and (15) give $m(r, A_0) = S(r, g)$. The induction is complete and the lemma is proved.

Lemma 4. *Let $k \geq 1$ be an integer and let $f_1, \dots, f_k$, G, H and $a_0, \dots, a_{k-1}$, $A_0, \dots, A_{k-1}$ all be meromorphic in a domain U. Suppose that $f_1, \dots, f_k$ are linearly independent solutions in U of*

$$L_k(w) = w^{(k)} + \sum_{j=0}^{k-1} a_j w^{(j)} = 0.$$

Suppose further that the functions $f_1 H + f_1' G, \dots, f_k H + f_k' G$ are linearly independent solutions in U of

$$M_k(w) = w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0.$$

Then we have, in U, setting $A_k = 1$ and $A_{-1} = a_{-1} = 0$,

$$\begin{aligned} kH' + (A_{k-1} - a_{k-1})H = &-\left(\tfrac{1}{2}k(k-1)G'' + (k-1)A_{k-1}G' + A_{k-2}G\right) \\ &+ a_{k-1}(A_{k-1}G + kG') + G(a_{k-1}' + a_{k-2} - a_{k-1}^2). \end{aligned} \tag{19}$$

Proof. When $a_{k-1} = 0$ this is a special case of Lemma 6 of [6]. Since $M_k(f_j H + f_j' G) = 0$ we have

$$M_k(f_j H) = -M_k(f_j' G). \tag{20}$$

For integers n and m, we use the notation

$${}^nC_m = \frac{n!}{m!(n-m)!}$$

when $0 \leq m \leq n$, and

$${}^nC_m = 0$$

otherwise. We also write, for $0 \le \mu \le k$,

$$M_{k,\mu}(w) = \sum_{m=\mu}^{k} ({}^mC_\mu) A_m w^{(m-\mu)}, \qquad M_{k,-1}(w) = 0.$$

Thus, for $j = 1, \dots, k$,

$$\begin{aligned} M_k(f_j H) &= \sum_{m=0}^{k} A_m \sum_{\mu=0}^{k} ({}^mC_\mu) f_j^{(\mu)} H^{(m-\mu)} = \sum_{\mu=0}^{k} f_j^{(\mu)} M_{k,\mu}(H) \\ &= \sum_{\mu=0}^{k-1} f_j^{(\mu)} \big(M_{k,\mu}(H) - a_\mu H \big). \end{aligned} \tag{21}$$

We also have

$$\begin{aligned} M_k(f_j' G) &= \sum_{\mu=0}^{k} f_j^{(\mu+1)} M_{k,\mu}(G) \\ &= \sum_{\mu=0}^{k-2} f_j^{(\mu+1)} M_{k,\mu}(G) + f_j^{(k)} M_{k,k-1}(G) + f_j^{(k+1)} G \\ &= \sum_{\mu=0}^{k-1} f_j^{(\mu)} \big(M_{k,\mu-1}(G) - a_\mu M_{k,k-1}(G) + (a_\mu a_{k-1} - a_\mu' - a_{\mu-1}) G \big). \end{aligned} \tag{22}$$

Since the Wronskian determinant of the f_j is not identically zero, the coefficient of $f_j^{(\mu)}$ on the right-hand-side of (21) and that on the right-hand-side of (22) must have sum 0, by (20). Now $\mu = k-1$ gives (19).

The following lemma is from [22].

Lemma A. *Let c, M, N be positive constants and let $Q(z)$ be analytic and satisfy $|Q(z)| \le M + |z|^M$ in a half-plane $\mathrm{Re}(z) \ge c$. Suppose that $Q(n)$ is an integer for all integers $n \ge N$. Then Q is a polynomial.*

Lemma 5. *Suppose that R, S are rational functions, with R not identically zero, that P, P_1 are polynomials, with P_1 non-constant. Suppose that we have $P_1\big(P(z) + \log R(z)\big) \equiv S(z)$ in some domain U. Then R is constant.*

Proof. By the hypotheses there is an equation

$$\sum_{j=0}^{q} a_j(z) w^j = 0, \tag{23}$$

with polynomial coefficients a_j, not all 0, having a local solution $w = \log R(z)$. The analytic continuations of $\log R(z)$ all satisfy the same equation. But $\log R(z)$ adds an integer multiple of $2\pi i$ as we continue once around a zero or pole of R and, since the solution of (23) has at most q branches, we conclude that R has no zeros or poles and is constant.

4. An estimate for logarithmic derivatives

Lemma 6. *Suppose that $k \geq 1$ and that f is meromorphic in the plane and that*

$$F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)}, \qquad G = f^{(k)} + \sum_{j=0}^{k-1} b_j f^{(j)}, \tag{24}$$

with the a_j and b_j rational functions. Then either F/G is constant or

$$m(r, f'/f) \leq \overline{N}(r, 1/F) + \overline{N}(r, 1/G) + S(r, f'/f). \tag{25}$$

Proof. Let $f_1, \ldots, f_k$ be linearly independent solutions of the equation

$$w^{(k)} + \sum_{j=0}^{k-1} a_j w^{(j)} = 0 \tag{26}$$

in the domain $S = S^*(r_0, -\pi, \pi)$. Then the f_j all admit unrestricted analytic continuation in $|z| > r_0$, provided r_0 is large enough. Let

$$W = W(f_1, \ldots, f_k), \qquad W'/W = -a_{k-1} \tag{27}$$

in S. Then we have, in S,

$$W(f_1, \ldots, f_k, f) = WF$$

and so

$$W\big((f_1/f)', \ldots, (f_k/f)'\big) = (-1)^k W F f^{-k-1}$$

and

$$W(w_1, \ldots w_k) = WF/f, \qquad w_j = -f_j' + f_j f'/f.$$

Thus the w_j are linearly independent solutions in S of an equation

$$w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0, \qquad A_{k-1} = -W'/W + f'/f - F'/F. \tag{28}$$

We assert that the A_j are meromorphic in $|z| > r_0$, establishing this in the standard way by noting that if A_j^* and w_m^* are respectively the function elements obtained by analytically continuing A_j and w_m once counter-clockwise around $|z| = r > r_0$, then w_m^* is a linear combination of the w_j in S, and $w_1^*, \ldots, w_k^*$ are linearly independent by the law of permanence of functional relations. Since we have

$$\sum_{j=0}^{k-1} (A_j^* - A_j) w_m^{(j)} = 0$$

in S for $1 \le m \le k$, we deduce that $A_j^* = A_j$ on S. Our assertion established, we have, by Lemma 3,

$$m(r, A_j) = S(r, f'/f), \qquad j = 0, \dots, k-1. \tag{29}$$

We may apply Lemma 4, with $H = f'/f$ and $G = -1$, to obtain, using (27), (28) and (29),

$$k(f'/f)' + (A_{k-1} - a_{k-1})f'/f = k(f'/f)' + (f'/f - F'/F)f'/f = C,$$
$$m(r, C) = S(r, f'/f).$$

The same argument with the a_j replaced by b_j gives

$$k(f'/f)' + (B_{k-1} - b_{k-1})f'/f = k(f'/f)' + (f'/f - G'/G)f'/f = D,$$
$$m(r, B_{k-1}) + m(r, D) = S(r, f'/f).$$

We therefore have

$$A^* f'/f = E^*,$$
$$A^* = (A_{k-1} - a_{k-1}) - (B_{k-1} - b_{k-1}) = G'/G - F'/F,$$
$$m(r, A^*) + m(r, E^*) = S(r, f'/f),$$

and either F/G is constant or $A^* \not\equiv 0$, in which case we obtain (25), on writing $f'/f = E^*/A^*$ and

$$m(r, f'/f) \le m(r, E^*) + m(r, A^*) + N(r, A^*) + O(1).$$

5. Estimates for counting functions

We use the following notation throughout this section. Let k be a positive integer and let f be a meromorphic function in the plane. Let F and G be given by (24), with the a_j, b_j rational functions and $a_j - b_j \not\equiv 0$ for at least one j. Assume that neither F nor G vanishes identically. Define V, E by

$$F = VG, \qquad E = G - F = (1 - V)G = \sum_{j=0}^{k-1} (b_j - a_j) f^{(j)}. \tag{30}$$

We begin with some basic estimates. Dividing the first relation of (30) through by f and writing each $f^{(j)}/f$ as a differential polynomial in f'/f, we see at once that

$$T(r, F/f) + T(r, G/f) \le O\big(T(r, f'/f)\big) + S(r, f'/f). \tag{31}$$

Since all but finitely many zeros and poles of V arise from zeros of F and G, we also have

$$\begin{aligned} m(r,V'/V) &= S(r,f'/f), \\ T(r,V'/V) &\le \overline{N}(r,1/F)+\overline{N}(r,1/G)+S(r,f'/f). \end{aligned} \tag{32}$$

We now estimate $m(r,F'/F)$ and $m(r,G'/G)$. From (31) and the relations

$$F'/F=(F/f)'/(F/f)+f'/f, \qquad V'/V=F'/F-G'/G,$$

we see that any term which is $S(r,F'/F)$, or $S(r,G'/G)$ or $S(r,V'/V)$, is an $S(r,f'/f)$, while if V is non-constant then, using Lemma 6,

$$m(r,F'/F)\le m(r,f'/f)+S(r,f'/f)\le \overline{N}(r,1/F)+\overline{N}(r,1/G)+S(r,f'/f). \tag{33}$$

The same estimate plainly holds with F'/F replaced by G'/G.

Lemma 7. *Let V be as in* (30). *Then there exists an integer q with $1\le q\le k$ such that G satisfies*

$$0=(1-V)G^{(q)}+\sum_{j=0}^{q-1}T_j(V)G^{(j)}, \tag{34}$$

in which each $T_j(V)$ has a representation

$$T_j(V)=\alpha_j+\sum_{m=0}^{q}\beta_{m,j}V^{(m)},$$

with the coefficients α_j and $\beta_{m,j}$ rational functions.

Proof. We begin by recalling (30). Thus, for $s=0,\dots,k$, we have

$$E^{(s)}=(1-V)G^{(s)}-\sum_{j=1}^{s}\frac{s!}{j!(s-j)!}V^{(j)}G^{(s-j)}. \tag{35}$$

Using the division algorithm for linear differential operators, each $E^{(s)}$ may be written in the form

$$E^{(s)}=\sum_{m=0}^{s-1}c_{m,s}G^{(m)}+\sum_{m=0}^{k-1}d_{m,s}f^{(m)}, \qquad s=0,\dots,k,$$

the first sum not appearing when $s = 0$, and with the $c_{m,s}$ and $d_{m,s}$ rational functions. Now the k by $k+1$ matrix $(d_{m,s})$ has rank at most k and so by elementary linear algebra its columns are linearly dependent over the field of rational functions. Hence there are rational functions δ_s, not all identically zero, such that

$$\sum_{s=0}^{k} \delta_s d_{m,s} \equiv 0$$

for $0 \le m \le k-1$. Thus, if q is the largest s such that $\delta_s \not\equiv 0$, then $q > 0$ because, by hypothesis, at least one $d_{m,0}$ is non-zero, and we have

$$\sum_{s=0}^{q} \delta_s E^{(s)} = \sum_{t=0}^{q-1} d_t G^{(t)},$$

with coefficients d_t which are rational functions. Replacing the $E^{(s)}$ using (35), we have an equation as asserted.

We now make some estimates for the number of zeros and poles of f, under certain assumptions on the coefficients.

Lemma 8. *If* $a_{k-1} \equiv b_{k-1}$ *then*

$$\overline{N}(r,f) \le \overline{N}(r,1/F) + \overline{N}(r,1/G) + S(r,f'/f).$$

To prove Lemma 8 we write, using (30),

$$(1-V)f^{(k)} + \sum_{j=0}^{k-1}(a_j - Vb_j)f^{(j)} = 0. \tag{36}$$

Suppose that z is large and that $f(z) = \infty$. Then $V(z) = 1$ and, by the hypothesis that $a_{k-1} \equiv b_{k-1}$, we have $a_{k-1}(z) - V(z)b_{k-1}(z) = 0$. Dividing (36) through by $f^{(k-1)}$, we see that $(1-V)f^{(k)}/f^{(k-1)}$ vanishes at z. Thus $V'(z) = V(z) - 1 = 0$ and the result follows from (32).

Lemma 9. *Suppose that the equations*

$$L_1(w) = w^{(k)} + \sum_{j=0}^{k-1} a_j w^{(j)} = 0, \qquad L_2(w) = w^{(k)} + \sum_{j=0}^{k-1} b_j w^{(j)} = 0$$

have no non-trivial common (local) solution, and that V *is non-constant. Then we have*

$$N(r,1/f) \le 2N(r,1/F) + 4k\overline{N}(r,1/F) + 5k\overline{N}(r,1/G) + S(r,f'/f) \qquad (n.e.). \tag{37}$$

Proof. Since the equations $L_1(w) = 0$, $L_2(w) = 0$ have no non-trivial common (local) solution, it follows using Lemma 1 that there exist a rational function b, not identically zero, and linear differential operators L, M with coefficients which are rational functions and with order at most k, such that we have $b = L(L_1) + M(L_2)$ and hence

$$bf = L(F) + M(G),$$

and using (30) we write the last relation in the form

$$bf/F = L(F)/F + M(G)/F = L(F)/F + M(G)/GV.$$

But $L(F)/F$ may be written as a polynomial of degree at most k in F'/F and its derivatives, with coefficients which are rational functions. Using (33) applied to F and G and the remark preceding (33), this gives

$$m(r, f/F) \le m(r, 1/V) + 2k\big(\overline{N}(r, 1/F) + \overline{N}(r, 1/G)\big) + S(r, f'/f). \tag{38}$$

We now write the equation (34) in the form

$$\begin{aligned} 1/V &= V_1/V_2, \\ V_1 &= \left(G^{(q)}/G + \sum_{j=0}^{q-1} M_j G^{(j)}/G\right), \\ V_2 &= \left(G^{(q)}/G + \sum_{j=0}^{q-1} N_j G^{(j)}/G\right). \end{aligned}$$

Here each M_j is a differential polynomial in V'/V, with coefficients which are rational functions, and each N_j is a rational function. We now have, by (32) and (33), applied to G,

$$m(r, V_j) \le q\big(\overline{N}(r, 1/F) + \overline{N}(r, 1/G)\big) + S(r, f'/f)$$

for $j = 1, 2$, as well as

$$N(r, V_2) \le q\big(\overline{N}(r, f) + \overline{N}(r, 1/G)\big) + S(r, f'/f).$$

Combining these estimates with (38), we have

$$m(r, f/F) \le q\overline{N}(r, f) + (2k+2q)\overline{N}(r, 1/F) + (2k+3q)\overline{N}(r, 1/G) + S(r, f'/f). \tag{39}$$

Since $1/f = (F/f)(1/F)$ and since each pole z of f with z large is a pole of F/f of order k, we can now write

$$\begin{aligned} N(r, 1/f) + k\overline{N}(r, f) &\le N(r, F/f) + N(r, 1/F) + O(\log r) \\ &\le T(r, f/F) + N(r, 1/F) + O(\log r) \\ &\le m(r, f/F) + 2N(r, 1/F) + O(\log r) \end{aligned}$$

to obtain (37), using (39) and the fact that $q \le k$.

Lemma 10. *With the hypotheses of Lemma* 9, *suppose that*

$$N(r,1/F)+N(r,1/G)=S(r,f'/f).$$

Then

$$N(r,1/f)=S(r,f'/f),$$

and

$$T(r,f'/f)\le \overline{N}(r,f)+S(r,f'/f)\le T(r,V)+S(r,f'/f).$$

The proof is obvious, using Lemmas 6 and 9 and recalling that $V=1$ at all but finitely many poles of f.

6. A growth lemma

Lemma 11. *Suppose that f is meromorphic in the plane and that F and G are given by* (24), *with $k\ge 1$ and with the a_j and b_j rational functions, such that $a_j\not\equiv b_j$ for at least one j. Suppose that F and G have only finitely many zeros. Then f has finite order.*

We remark that when $k\ge 3$ and at least one of F and G has polynomial coefficients, it already follows from the hypotheses and Theorem 2 of [6] that f'/f has finite order.

Proof of Lemma 11. Suppose that k and the functions $f(z)$, $F(z)$, $G(z)$ are as in the hypotheses, and suppose that f is meromorphic of infinite order in the plane. Then F/G is transcendental, because otherwise f would be a solution of a homogeneous linear differential equation with rational functions as coefficients, and f would have finite order. With the notation of Lemma 1, and with $a_k=b_k=1$, we may write $F=L_1(f)$, $G=L_2(f)$. Let H be the operator of Lemma 1.

Suppose first that H has positive order. In this case we set $g_1=H(f)=\sum_{j=0}^{q}c_jf^{(j)}$, in which $0\le q\le k-1$, and the c_j are rational functions, with $c_q\not\equiv 0$. By Lemma 1 there are differential operators Q_j and P_j, with coefficients which are rational functions, such that

$$g_1=P_1(F)+P_2(G),\qquad F=Q_1(g_1),\qquad G=Q_2(g_1),$$

and, dividing through by the leading coefficients, there are linear differential operators Q_1^*, Q_2^*, each of form

$$Q_j^*=D^{k-q}+\sum_{m=0}^{k-q-1}a_{j,m}D^m$$

and having coefficients which are rational functions, such that $Q_1^*(g_1)$ and $Q_2^*(g_1)$ both have finitely many zeros.

Lemma 12. *If g_1 has finite order then so has f.*

Proof. Assume that g_1 has finite order. Then since all but finitely many poles of f are poles of g_1/f of multiplicity q, it follows that $N(r,f)$ has finite order and we can write $f = f_1/f_2 = f_1 u_2$, with the f_j entire, f_2 of finite order. For $|z|$ outside a set F_0 of finite measure, standard estimates from [10, p. 22] or [8] give $f_2^{(m)}(z)/f_2(z) = O(|z|^{d_1})$, for $1 \le m \le q$, in which d_1 is a positive constant. Substituting $f = f_1 u_1$ into the equation $H(f) = g_1$ and dividing through by $f_1 u_1$, we obtain an equation

$$f_1^{(q)}/f_1 + \sum_{j=0}^{q-1} A_j f_1^{(j)}/f_1 = g_1/f = g_1 f_2/f_1,$$

in which the coefficients A_j satisfy $A_j(z) = O(|z|^{d_2})$, for $0 \le j \le q-1$, and for $|z|$ outside F_0, where d_2 is a positive constant. A standard application of the Wiman–Valiron theory [12, Theorem 12] (see also [30]) now shows that f_1 has finite order and so has f. This proves Lemma 12.

Returning to the proof of Lemma 11, we may assume henceforth that H has order 0, that is, that the equations

$$L_1(w) = 0, \qquad L_2(w) = 0$$

have no non-trivial common (local) solution. Then by Lemma 10 we have

$$T(r, f'/f) \le cT(r, F/G) \qquad \text{(n.e.)}, \tag{40}$$

using c throughout this proof to denote a positive constant, not necessarily the same at each occurrence.

Since F and G are given by (24), we may write

$$\begin{aligned} &F(z) = R(z)e^{P(z)}G(z), \\ &E(z) = G(z) - F(z) = \big(1 - R(z)e^{P(z)}\big)G(z) = \sum_{j=0}^{k-1} B_j(z) f^{(j)}(z), \end{aligned} \tag{41}$$

with P entire, and with R and the B_j rational functions. If $|z|$ is large and f has a pole of multiplicity n at z then, dividing the equation $F = Re^P G$ through by $f^{(k-1)}$, we obtain

$$R(z)e^{P(z)} = 1, \qquad (n+k-1)\big(R'(z)/R(z) + P'(z)\big) = b_{k-1}(z) - a_{k-1}(z) \tag{42}$$

and so

$$\log^+ |P'(z)| = O(\log^+ |z|). \tag{43}$$

Also, if $R(z)e^{P(z)} = 1$ and $|z|$ is large, then either f has a pole at z, or $E(z) = 0$.

Suppose that P is a polynomial. Since $R(z)e^{P(z)} = 1$ at all but finitely many poles of f we have $\overline{N}(r, f) \le T(r, Re^P) + O(\log r)$. But by (42) the multiplicity n of a pole of f at z is bounded by a power of $|z|$. Therefore

$$\log^+ N(r, f) = O(\log r)$$

and we have $f = f_1/f_2$ in which the f_j are entire and f_2 is not identically zero but has finite order. There then exists a subset E^* of $(1, \infty)$ of infinite logarithmic measure such that for $|z| = r$ in E^* we have

$$|R(z)e^{P(z)} - 1| \ge r^{-c}, \qquad |f_2^{(j)}(z)/f_2(z)| \le r^c, \qquad 1 \le j \le k,$$

the easiest way to establish this being to write

$$1/(Re^P - 1) = R^{-1}e^{-P}(1 - R^{-1}e^{-P})^{-1}$$

and then use standard estimates [10, p. 22] for the logarithmic derivative of the function $1 - R^{-1}e^{-P}$. As in the proof of Lemma 12 a standard application of the Wiman–Valiron theory [12, Theorem 12] to the relation $G = (1 - Re^P)^{-1}E$ shows that f_1 has finite order and so does f. Therefore we may assume for the rest of the proof that P is transcendental.

Take a large positive r_0, normal for P with respect to the Wiman–Valiron theory [12], [30], and such that, using (40) and (41),

$$T(r_0, f^{(j)}/f) < cT(r_0, F/G) = cT(r_0, Re^P) \tag{44}$$

for $j = 1, \dots, k$. For non-zero complex v, and positive K, we define the logarithmic rectangle

$$D(v, K) = \{u = ve^\tau : |\operatorname{Re}(\tau)| \le KN^{-2/3},\ |\operatorname{Im}(\tau)| \le KN^{-2/3}\},$$

in which $N = \nu(r_0, P)$ is the central index of P, and is large if r_0 is large.

By Lemma 2, (41) and (44) we have, for $j = 1, \dots, k$,

$$\log|f^{(j)}(z)/f(z)| \le cN^2T(r_0, Re^P), \quad \log|E(z)/f(z)| \ge -cN^2T(r_0, Re^P), \tag{45}$$

for all z with $\frac{1}{2}r_0 \le |z| \le r_0e^{-2/N}$ and lying outside a union D_0 of open discs having sum of radii at most cr_0N^{-2}, so that there is a subset D_1 of $[0, 2\pi]$, having measure at most cN^{-2}, such that some determination of $\arg\zeta$ is in D_1 for every ζ in D_0.

Choose z_0 with $|z_0| = r_0$ and $|P(z_0)| = M(r_0, P)$. On $D(z_0, 128)$ we have [12, Theorem 12]

$$P(z) + \log R(z) = P(z)\big(1 + o(1)\big) = P(z_0)(z/z_0)^N\big(1 + o(1)\big) = \alpha\zeta^N, \tag{46}$$

(47) $\alpha = P(z_0)z_0^{-N}, \qquad \zeta = z\big(1 + o(1/N)\big), \qquad P'(z)/P(z) = \big(1 + o(1)\big)N/z.$

In particular, $P'(z)$ is large on $D(z_0, 128)$ so that, using (43), there are no poles of f in $D(z_0, 128)$, and by (41) every zero of $R(z)e^{P(z)} - 1$ in $D(z_0, 128)$ is simple and is a zero of E.

On $D(z_0, 128)$ we write

$$z = z_0 e^{\tau}, \qquad \zeta = z_0 e^{\sigma}, \qquad \sigma = \tau + o(1/N),$$

so that

$$\frac{d\sigma}{d\tau} = 1 + o(N^{-1/3})$$

and, by convexity, σ is a univalent function of τ for $|\operatorname{Re}(\tau)| \le 64N^{-2/3}$, $|\operatorname{Im}(\tau)| \le 64N^{-2/3}$. Further,

$$\frac{d\zeta}{dz} = \frac{\zeta}{z}\frac{d\sigma}{d\tau} = 1 + o(N^{-1/3}) \tag{48}$$

on $D(z_0, 64)$. In addition, the image of $D(z_0, 64)$ under $\zeta = \zeta(z)$ contains $D(z_0, 32)$, and $\alpha\zeta^N$ is large for ζ in $D(z_0, 32)$.

If c_0 is a positive constant there exists a positive constant c_1 such that on each circle $|w| = (2n + 1)\pi$, with n a positive integer, and on the ray $\arg w = 0$, $|w| \ge c_0$, we have $|e^w - 1| \ge c_1$. We choose σ_0 such that

$$\sigma_0 \in [-16N^{-1}, -8N^{-1}], \qquad |\alpha z_0^N e^{N\sigma_0}| = (2n + 1)\pi$$

for some integer n, and we choose m_1, n_1, m_2, n_2 such that

$$m_1, n_1, m_2, n_2 \in [4N^{-2/3}, 8N^{-2/3}]$$

and

$$\arg(\alpha\zeta^N) = 0 \qquad \text{for } \zeta = z_0 e^{\sigma},\ \operatorname{Im}(\sigma) \in \{-n_1, n_2\},$$

and such that $|(\alpha/\pi)z_0^N e^{N\sigma}|$ is an odd integer for $\operatorname{Re}(\sigma)$ in $\{\sigma_0 - m_1, \sigma_0 + m_2\}$. Thus on the boundary of the logarithmic rectangle

$$B = \{\zeta = z_0 e^{\sigma} : \sigma_0 - m_1 \le \operatorname{Re}(\sigma) \le \sigma_0 + m_2,\ -n_1 \le \operatorname{Im}(\sigma) \le n_2\},$$

and on the arc L_0 given by

$$\zeta = z_0 e^{\sigma_0 + i\lambda}, \qquad -n_1 \le \lambda \le n_2, \tag{49}$$

we have

$$|e^{\alpha\zeta^N} - 1| \ge c_1 > 0. \tag{50}$$

Now L_0 lies in $|\zeta| \le r_0 e^{-8/N}$ and so using (47) the image $z(L_0)$ of L_0 under the mapping $z = z(\zeta)$ lies in

$$|z| \le r_0 e^{-8/N}\big(1 + o(1)/N\big) \le r_0 e^{-2/N}.$$

Further, if L_1 is the sub-arc of L_0 given by $-1/N \le \lambda \le 1/N$ in (49), then the variation of $\arg z$ on $z(L_1)$ is, by (47), at least $\big(c - o(1)\big)/N$, and using the remark following (45) we may therefore choose ζ_1 lying on L_1, such that the inequalities of (45) all hold at $z_1 = z(\zeta_1)$. Note that $D(\zeta_1, 2)$ is contained in B, provided r_0 is large enough, while B in turn lies in $D(z_0, 16)$, and $z(B)$ lies in $D(z_0, 32)$.

Lemma 13. *The number of zeros of* $e^{\alpha\zeta^N}-1$ *in* $D(\zeta_1,1)$ *is at least* $ce^{N^{1/3}}M(r_0,P)$.

Proof. We have $|\zeta_1|=r_0e^{\sigma_0}$ and σ_0 is in $[-16/N,-8/N]$, so that $|\alpha\zeta_1^N|=|P(z_0)|e^{\gamma}$, for some γ in $[-16,-8]$, and the image of $D(\zeta_1,1)$ under $w=\alpha\zeta^N$ covers the annulus

$$|P(z_0)|e^{-N^{1/3}+\gamma}\le|w|\le|P(z_0)|e^{N^{1/3}+\gamma},$$

so that the number of zeros of $e^{\alpha\zeta^N}-1$ in $D(\zeta_1,1)$ is at least $ce^{N^{1/3}}|P(z_0)|=ce^{N^{1/3}}M(r_0,P)$. This proves Lemma 13.

We may now complete the proof of Lemma 11. Let $g(z)=f(z)/f(z_1)$, and let C be the union of L_0 and the boundary of B. Using (46) and (50) and the relation $G=(1-Re^P)^{-1}E$, we have, on $z(C)$,

$$g^{(k)}(z)=\sum_{j=0}^{k-1}s_j(z)g^{(j)}(z),\qquad s_j(z)=O(|z|^c). \tag{51}$$

Since $|dz|\le 2|d\zeta|$, by (48), the arc length of $z(C)$ is $o(r_0)$. We write the equation (51) in vector form as

$$I'(z)=A(z)I(z),\qquad I(z)=\big(g^{(k-1)}(z),\dots,g(z)\big)^T,$$

in which the k by k matrix A has entries which are $O(r_0)^c$ on $z(C)$. Writing

$$I(z)=I(z_1)+\int_{z_1}^{z}A(u)I(u)\,du,\qquad S(z)=\max\{|g^{(j)}(z)|:j=0,\dots,k-1\},$$

we have

$$S(z)\le V(z)=S(z_1)+\int_{z_1}^{z}r_0^cS(u)\,|du|,$$

and the standard Gronwall method [1, p. 35] (see also [15], [16], [20]) gives, with t denoting arc length on $z(C)$,

$$\frac{d}{dt}\big(V\big(z(t)\big)\big)\le r_0^cS\big(z(t)\big)\le r_0^cV\big(z(t)\big),$$

and so

$$S\big(z(t)\big)\le V\big(z(t)\big)\le V(z_1)\exp(r_0^ct)\le S(z_1)\exp(r_0^c).$$

We thus have, for $j=0,\dots,k-1$,

$$|g^{(j)}(z)|\le S(z)\le S(z_1)\exp(r_0^c)\le\exp\big(N^cT(r_0,Re^P)+r_0^c\big),$$

using (45), and so

$$|E(z)/f(z_1)| \le \exp\bigl(N^c T(r_0, Re^P) + r_0^c\bigr)$$

on $z(C)$. Hence the function H_1 defined by $H_1(\zeta) = E(z)/f(z_1)$ satisfies

$$\log |H_1(\zeta)| \le N^c T(r_0, Re^P)$$

for all ζ on C, and so for all ζ in B, and hence for all ζ in $D(\zeta_1, 2)$, by the maximum principle. But we also have, by (45),

$$\log |H_1(\zeta_1)| = \log |E(z_1)/f(z_1)| \ge -N^c T(r_0, Re^P).$$

Mapping $D(\zeta_1, 2)$ to the unit disc, using $w = \phi(\zeta)$, with ζ_1 mapped to 0, and writing $J(w) = H_1(\zeta)$, we have, for $0 < r < 1$,

$$T(r, 1/J) \le T(r, J) + \log |1/J(0)| \le \log M(r, J) + \log |1/H_1(\zeta_1)| \le N^c T(r_0, Re^P).$$

Thus the number of zeros of $H_1(\zeta)$ in $D(\zeta_1, 1)$ is at most $N^c T(r_0, Re^P)$. Hence, using (41) and (46) and the remark following (47), the number of zeros of $e^{\alpha \zeta^N} - 1$ in $D(\zeta_1, 1)$ is at most

$$N^c T(r_0, Re^P) \le N^c \log M(r_0, Re^P) \le N^c \bigl(M(r_0, P) + O(\log r)\bigr) \le N^c M(r_0, P).$$

This contradicts Lemma 13 and Lemma 11 is proved.

7. Proof of Theorem 1

Suppose that f and F and G are as in the hypotheses. Then we know by Lemma 11 that f has finite order. If F/G is constant then f has finitely many poles and since F has finitely many zeros we have $F = R_1 e^V$ with R_1 a rational function and V a polynomial. Since G is a constant multiple of F and since Lemma 1 gives $f = V_1(F) + V_2(G)$, in which V_1 and V_2 are linear differential operators, the coefficients of which are rational functions, we deduce that $f = R_2 e^V$ with R_2 a rational function, and f'/f is a rational function.

Assume henceforth that F/G is non-constant. It follows from Lemma 10 and (10) that f has only finitely many zeros. If f has only finitely many poles then again f'/f is a rational function. We assume henceforth that f has infinitely many poles.

We have (41), with R a rational function and P a non-constant polynomial. Since

$$m(r, f'/f) + N(r, 1/f) = O(\log r),$$

the order ρ of $T(r, f'/f)$ is the same as that of $\overline{N}(r, f)$. Writing F/f and G/f as differential polynomials in f'/f with coefficients which are rational functions, it is now clear that

$$\deg(P) \le \rho = \limsup_{r\to\infty} \frac{\log \overline{N}(r, f)}{\log r}. \tag{52}$$

Let r_0 be large and positive. We define, in the domain $U = S^*(r_0, -\pi, \pi)$, linearly independent solutions $f_1, \ldots, f_k$ of the equation (26), and the Wronskian $W = W(f_1, \ldots, f_k)$ satisfies (27) in U. We further define g, h in U by

$$g^{-k} = F/f, \qquad h = (-f'/f)g. \tag{53}$$

Then g and h are analytic in U and g, h, W and the f_j all admit unrestricted analytic continuation in $|z| > r_0$, the continuations of these functions H_m all satisfying

$$\log^+ \log^+ |H_m(z)| = O(\log |z|) \tag{54}$$

on $S^*(r_0, -2\pi, 2\pi)$.

We have

$$W(f_1, \ldots, f_k, f) = WF = Wfg^{-k}$$

and hence

$$W\big((f_1/f)', \ldots, (f_k/f)'\big) = (-1)^k W f^{-k} g^{-k}$$

and

$$W(f_1 h + f_1' g, \ldots, f_k h + f_k' g) = (-1)^k W \tag{55}$$

in U. Thus the functions $f_j h + f_j' g$, for $j = 1, \ldots, k$, are linearly independent solutions in U of an equation

$$w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0, \qquad A_{k-1} = -W'/W. \tag{56}$$

We assert that the A_j are rational functions. First, if E_1 is the set of all singular points of the equation (26) as well as of all zeros of f and F then E_1 is finite and the f_j and g and h all admit unrestricted analytic continuation in the complement Ω of E_1 in the plane. Further, since g^k and h^k are meromorphic, and since the f_j form a fundamental solution set of (26), analytic continuation of any of the functions $f_1 h + f_1' g, \ldots, f_k h + f_k' g$ once around any point of E_1 leads back to a linear combination of the same functions. By (55) and the standard representation for the A_j as quotients of determinants, we deduce that the A_j are analytic in Ω. By (54) and Lemma 3, they satisfy

$$m(r, A_j) = O(\log r), \qquad r \to \infty.$$

Thus the A_j each have at most a pole at infinity, and a similar analysis in a punctured neighbourhood of each point of E_1 shows that the A_j are rational functions.

We denote henceforth by d_j rational functions. Since each $f_j h + f_j' g$ satisfies (56) we obtain, using (26), (27), (56) again and Lemma 4,

$$h' = -\tfrac{1}{2}(k-1)g'' + d_1 g' + d_2 g. \tag{57}$$

However, we may define Y and g_1, h_1 on U by

$$Y^k = Re^P, \qquad g_1 = Yg, \qquad h_1 = Yh, \tag{58}$$

and using (41) and (53) we have

$$G = Y^{-k} F = g_1^{-k} f.$$

The same method as above gives us an equation

$$h_1' = -\tfrac{1}{2}(k-1)g_1'' + d_3 g_1' + d_4 g_1$$

in U, which leads at once to

$$h' + (Y'/Y)h = -\tfrac{1}{2}(k-1)g'' + d_5 g' + d_6 g, \tag{59}$$

using (58). Thus (57) and (59) give

$$h = d_7 g' + d_8 g, \qquad -f'/f = d_7 g'/g + d_8. \tag{60}$$

The equations (60) continue to hold under analytic continuation of g and h. Further, $d_7(z)$ is a positive integer at a pole z of f with $|z|$ large. Hence $d_7 \not\equiv -\frac{1}{2}(k-1)$. Therefore (57) and (60) together give

$$g'' + D_1 g' + D_0 g = 0,$$

with coefficients D_j which are rational functions.

Writing

$$g = uv, \qquad 2v'/v = -D_1, \tag{61}$$

the function u admits unrestricted analytic continuation in $|z| > r_0$ and solves an equation

$$u''(z) + a(z)u(z) = 0, \tag{62}$$

in which a is a rational function. We assume that either $a(z) \equiv 0$ or

$$a(z) = \alpha_m z^m \big(1 + o(1)\big), \qquad z \to \infty, \tag{63}$$

in which m is an integer and $\alpha_m \neq 0$. If $a(z) \equiv 0$ or $m \leq -2$ we can take any sectorial region U_1 given by $|z| > r_1$, $|\arg z - \theta_1| \leq \frac{1}{2}\pi$. We can estimate the number $n(r, U_1, 1/u)$ of zeros of u, and hence zeros of g, in the set $\{z \in U_1 : |z| \leq r\}$ as follows. Under the assumption $m \leq -2$ the equation (62) has a regular singular point at infinity [17], and there exist a constant d and a solution u_1 of (62), such that in the sectorial region U_1 we have

$$u_1(z) = z^d \phi(z) = z^d \big(1 + o(1)\big),$$

in which $\phi(z)$ is analytic in $|z| > r_0$ with $\phi(\infty) = 1$. A second solution of (62) may be obtained by writing

$$(u_2/u_1)' = u_1^{-2},$$

so that, subtracting a constant if necessary,

$$u_2(z)/u_1(z) = \big(1 + o(1)\big)(1 - 2d)^{-1} z^{1-2d}$$

in U_1, provided $d \neq \frac{1}{2}$, while if $d = \frac{1}{2}$ we get

$$u_2(z)/u_1(z) = \big(1 + o(1)\big) \log z.$$

Writing u as a linear combination of u_1 and u_2 in U_1 we deduce that

$$n(r, U_1, 1/u) = O(\log r), \qquad r \to \infty,$$

which contradicts (52). We may assume henceforth that $a(z) \not\equiv 0$ and $m \geq -1$ in (63).

Now asymptotic representations for the solutions of (62) are obtained by the method of Hille [15], [16], as follows. The critical rays for (62) are those rays $\arg z = \theta_0$ for which

$$\arg \alpha_m + (m + 2)\theta = 0 \qquad \text{mod } 2\pi.$$

If $\arg z = \theta_0$ is a critical ray and ε is a positive constant then in the sectorial region

$$S_0 = S^*\big(r_0, \theta_0 + \varepsilon - 2\pi/(m + 2), \theta_0 - \varepsilon + 2\pi/(m + 2)\big)$$

we write $z^* = 2r_0 e^{i\theta_0}$ and

$$(64) \qquad Z = \int_{z^*}^{z} a(t)^{1/2}\,dt = 2\alpha_m^{1/2}(m+2)^{-1} z^{(m+2)/2}\big(1+o(1)\big), \qquad z \to \infty,$$

and we have *principal* solutions u_1, u_2 of (62) satisfying

$$(65) \qquad u_j(z) = a(z)^{-1/4}\exp\big(iZ(-1)^j + o(1)\big)$$

in S_0. In one of the sectorial regions

$$S_1 = S^*\big(r_0, \theta_0+\varepsilon, \theta_0-\varepsilon+2\pi/(m+2)\big), \qquad S_2 = S^*\big(r_0, \theta_0+\varepsilon-2\pi/(m+2), \theta_0-\varepsilon\big),$$

we have $u_1(z)/u_2(z) \to 0$ as $|z| \to \infty$, and we refer to u_2 as *dominant* and u_1 as *sub-dominant* in that sectorial region, while in the other we have $u_2(z)/u_1(z) \to 0$ and u_1 is dominant. If u^* is any solution of (62), then u^* has at most finitely many zeros in $S_1 \cup S_2$. Both principal solutions u_1, u_2 admit unrestricted analytic continuation in $|z| > r_0$, although not generally without zeros.

It follows from these asymptotics that we have

$$n(r, U_1, 1/u) = O(r^{(m+2)/2}), \qquad r \to \infty,$$

for any sectorial region U_1 as above. Hence the degree n of P satisfies, by (52),

$$(66) \qquad n \le \tfrac{1}{2}(m+2).$$

We take a critical ray $\arg z = \theta_0$ of (62) such that f has infinitely many poles in $|z| > r_0$, $|\arg z - \theta_0| \le \pi/(m+2)$, and we write

$$u = C_1 u_1 - C_2 u_2$$

there, with C_1, C_2 constants, both necessarily non-zero. The function $\zeta = \pm(1/2\pi i)\log(C_2u_2/C_1u_1)$ maps the sectorial region S_0 conformally onto a region containing a half-plane $\mathrm{Re}(\zeta) \ge c$. At each point in S_0 where ζ is an integer, we have $u = 0$ and hence $f = \infty$ and hence $Re^P = 1$, so that $(P + \log R)/2\pi i$ is an integer. Writing $P + \log R$ as a function of ζ and applying Lemma A, we obtain a polynomial P_1 such that we have

$$P + \log R = P_1(\zeta).$$

But (66) and the asymptotics (64), (65) for u_1, u_2 and ζ force P_1 to be linear. Consequently there exist constants c, c^* such that $C_2u_2/C_1u_1 = c^*(Re^P)^c$. Hence $u_2'/u_2 - u_1'/u_1$ is a rational function, and so are u_2u_1 and u_1'/u_1 and u_2'/u_2. So using (61) there exist rational functions T_j such that we have

$$g'/g = T_1 + u'/u = T_2 + T_3(u_2/u_1)'(1 - C_2u_2/C_1u_1)^{-1} = T_4 + T_5\big(c^*(Re^P)^c - 1\big)^{-1},$$

and, using the second equation of (60),

$$f'/f = T_6 + T_7\big(c^*(Re^P)^c - 1\big)^{-1}.$$

By analytic continuation, R^c must be a rational function, and we can write

$$f'/f = T_6 + T_7(Se^Q - 1)^{-1},$$

with S a rational function and Q a non-constant polynomial. Examining the residue of f'/f at a zero of $Se^Q - 1$, a further application of Lemma A shows that T_7 has a representation

$$T_7 = P_2(Q + \log S)(Q' + S'/S),$$

with P_2 a polynomial, and by Lemma 5 either S or P_2 is constant.

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