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THE LIMIT OF MAPPINGS WITH FINITE DISTORTION

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Abstract. We show here that the limit mapping f of a weakly convergent sequence of mappings f_ν with finite distortion also has finite distortion and give several dimension free estimates for the dilatation of f. Our arguments are based on the weak continuity of the Jacobian determinants and the concept of polyconvexity.

1. Introduction

Let $f\colon \Omega \to \mathbf{R}^n$ be a mapping in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ where Ω is a domain in $\mathbf{R}^n$. Then the differential matrix $Df(x) \in \mathbf{R}^{n\times n}$ and its determinant $J(x,f) = \det Df(x)$ are well defined at almost every point $x \in \Omega$. Here $\mathbf{R}^{n\times n}$ denotes the space of all $n\times n$-matrices, where $n > 1$, equipped with the operator norm

$$|A| = \max\{|A\xi| : \xi \in \mathrm{S}^{n-1}\}.$$

We assume most of the time that $J(x,f) \geq 0$ a.e. and refer to such mappings f as orientation preserving. We let $\mathbf{R}^{n\times n}_+$ denote the set of matrices with positive determinant and write $\mathbf{R}^{n\times n}_+ \cup \{0\}$ when the zero matrix is included.

Definition 1.1. A mapping $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ is said to be of finite distortion if

$$Df(x) \in \mathbf{R}^{n\times n}_+ \cup \{0\}$$

for almost every $x \in \Omega$.

In what follows it is vital that the Sobolev exponent is at least the dimension of Ω so that we can integrate the Jacobian. In this case the mappings of finite distortion are actually continuous [18].

Definition 1.1 asserts that

$$|Df(x)|^n \leq K_O(x) J(x,f) \tag{1.2}$$

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where $1 \le K_O(x) < \infty$ a.e. The smallest such function defined by

$$K_O(x,f) = \frac{|Df(x)|^n}{J(x,f)} \tag{1.3}$$

if $J(x,f) \neq 0$ and 1 otherwise is called the *outer dilatation* function of f.

We shall establish the following limit theorem.

Theorem 1.4. *Suppose that $f_\nu\colon \Omega \to \mathbf{R}^n$ is a sequence of mappings of finite distortion which converges weakly in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ to f and suppose that*

$$K_O(x, f_\nu) \le M(x) < \infty \qquad \textit{for } \nu = 1, 2, \ldots \tag{1.5}$$

a.e. in Ω. Then f has finite distortion and

$$K_O(x,f) \le M(x) \tag{1.6}$$

a.e. in Ω.

Theorem 1.4 is a refinement of Reshetnyak's theorem [15] concerning mappings f_ν of bounded distortion, that is mappings which satisfy (1.5) with $M(x) \le K$ where K is a constant. In this case, weak convergence in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ implies uniform convergence on compact sets and hence, by Reshetnyak's theorem, that the limit mapping f satisfies $K_O(x,f) \le K$ instead of the pointwise bound given in (1.6).

Remark 1.7. The hypotheses of Theorem 1.4 imply a stronger conclusion than (1.6), namely the existence of a subsequence $\{f_{\nu_k}\}$ such that

$$K_O(x,f) \le \mathrm{b}{*}\lim_{k\to\infty} K_O(x, f_{\nu_k}) \tag{1.8}$$

in Ω.

The limit in (1.8) is to be understood in the sense of *biting convergence* defined in Section 2; see [1], [3] and [6]. The basic ingredient of our proof is the higher integrability of nonnegative Jacobians. For a discussion of this property for mappings with bounded distortion see [5], [7], [8], [11] and [16].

The outer dilatation function $K_O(x,f)$ has a simple geometric interpretation. If $f\colon \Omega \to \mathbf{R}^n$ has a differential $Df(x) \neq 0$, then $Df(x)$ maps the unit sphere onto an ellipsoid E and

$$K_O(x,f) = \frac{\mathrm{vol}(B_O)}{\mathrm{vol}(E)}, \tag{1.9}$$

where B_O is the smallest ball circumscribed about E. In the same way, we may define the *inner dilatation* of f at x by

$$K_I(x,f) = \frac{\mathrm{vol}(E)}{\mathrm{vol}(B_I)}, \tag{1.10}$$

where B_I is the largest ball inscribed in E. We set $K_I(x,f) = 1$ at degenerate points where $Df(x) = 0$ and we call

$$K(x,f) = \max\{K_O(x,f), K_I(x,f)\} \tag{1.11}$$

the *maximal dilatation*,

$$K_M(x,f) = \tfrac{1}{2}\big(K_O(x,f) + K_I(x,f)\big) \tag{1.12}$$

the *mean dilatation* and

$$H(x,f) = \big(K_O(x,f)\,K_I(x,f)\big)^{1/n} \tag{1.13}$$

the *linear dilatation* for f at x. The linear dilatation has the following dimension free representation

$$H(x,f) = \frac{\max\{|Df(x)\xi| : \xi \in \mathrm{S}^{n-1}\}}{\min\{|Df(x)\xi| : \xi \in \mathrm{S}^{n-1}\}} \tag{1.14}$$

at points where $Df(x) \neq 0$.

All of these dilatation functions coincide when $n = 2$; this is not the case when $n > 2$. However, the functions K_I, K_M and K have the same lower semicontinuity property as K_O when $n > 2$.

Theorem 1.15. *Theorem* 1.4 *and Remark* 1.7 *remain valid with* $K_I(x,f)$, $K_M(x,f)$ *and* $K(x,f)$ *in place of* $K_O(x,f)$.

This is not true of the geometrically appealing linear dilatation $H(x,f)$ when $n > 2$. Indeed a striking example in [10] exhibits for each $K > 1$ a sequence of mappings $f_\nu \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^n)$ such that

1. $H(x,f_\nu) \equiv K$,
2. f_ν converges uniformly to a linear map $f\colon \mathbf{R}^n \to \mathbf{R}^n$,
3. $H(x,f) \equiv K' > K$ where K' is a constant.

In light of this anomaly it is desirable to see what one can say about the linear dilatation of the limit f of a sequence of mappings. The following analogue of Theorems 1.4 and 1.15 answers a question raised at the Saariselkä Conference in June 1997. See also Section 14 in [17].

Theorem 1.16. *Suppose that* $f_\nu\colon \Omega \to \mathbf{R}^n$ *is a sequence of mappings of finite distortion which converges weakly in* $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ *to* f *and suppose that*

$$H(x,f_\nu) \le M(x) < \infty \qquad \text{for } \nu = 1,2,\dots \tag{1.17}$$

a.e. in Ω. *Then* f *has finite distortion and*

$$H(x,f) \le \tfrac{1}{2}\big(M(x) + M(x)^{n-1}\big)^{2/n} \le M(x)^{2-(2/n)} \tag{1.18}$$

a.e. in Ω.

2. Biting convergence

We shall make use of some ideas of Brooks and Chacon [6], in particular, the notion of *biting convergence* or weak convergence in measure.

Suppose that h and h_ν, $\nu = 1, 2, \ldots$, are Lebesgue measurable functions on $E \subset \mathrm{R}^n$ with values in a finite dimensional normed space $(V, \|\cdot\|)$. In our applications we shall assume that $V = \mathbf{R}$ or $V = \mathbf{R}^{n\times n}$. We say that h_ν converges to h in the *biting* sense in E if there exist an increasing sequence of measurable subsets E_k of E with

$$\bigcup_k E_k = E$$

such that for each k, the functions h and h_ν are in $L^1(E_k, V)$ for all ν and

$$\lim_{\nu\to\infty} \int_{E_k} \phi\, h_\nu\, dx = \int_{E_k} \phi\, h\, dx \tag{2.1}$$

whenever $\phi \in L^\infty(E_k)$. In other words, the sequence h_ν converges weakly to h outside arbitrarily small *bites* from E, that is outside $E \setminus E_k$ for $k = 1, 2, \ldots$. We shall call h the *biting* limit of the sequence h_ν and write

$$h_\nu \xrightarrow{\mathrm{b}} h \qquad \text{or} \qquad h = \mathrm{b}{*}\lim_{\nu\to\infty} h_\nu. \tag{2.2}$$

It is immaterial which increasing sequence of subsets E_k of E we choose to define h as long as the weak limits on these sets exist; different bites yield the same limit. We leave it to the reader to verify the following two simple properties of biting convergence.

Lemma 2.3. *If $h_\nu \xrightarrow{\mathrm{b}} h$ in E and if λ is finite and measurable in E, then $\lambda\, h_\nu \xrightarrow{\mathrm{b}} \lambda\, h$ in E.*

Lemma 2.4. *If h_ν are measurable functions in E, $\nu = 1, 2, \ldots$, and if*

$$\sup_\nu \|h_\nu(x)\| < \infty$$

a.e. in E, then h_ν contains a subsequence which converges in the biting sense in E.

We shall require the following lemma.

Lemma 2.5. *If A_ν converges weakly to A in $L^p_{\mathrm{loc}}(E, \mathbf{R}^{n\times n})$ where $1 \le p < \infty$, then there is a subsequence $\{\nu_k\}$ such that*

$$|A|^s \le \mathrm{b}{*}\lim_{k\to\infty} |A_{\nu_k}|^s \tag{2.6}$$

for $1 \le s \le p$.

Proof. Choose measurable unit vectors $\xi = \xi(x)$ and $\zeta = \zeta(x)$ such that

$$|A(x)| = \langle A(x)\xi(x), \zeta(x)\rangle \qquad \text{whence} \qquad |A_\nu(x)| \ge \langle A_\nu(x)\xi(x), \zeta(x)\rangle$$

in E. Since t^s is convex in $0 < t < \infty$,

$$|A_\nu|^s - |A|^s \ge s|A|^{s-1}(|A_\nu| - |A|) \ge s|A|^{s-1}(\langle A_\nu\xi, \zeta\rangle - \langle A\xi, \zeta\rangle) \tag{2.7}$$

in E, the right hand side of (2.7) converges to 0 in the biting sense as $\nu \to \infty$ by Lemma 2.3 and we obtain (2.6) for some subsequence $\{\nu_k\}$ by Lemma 2.4. □

3. Weak continuity of minors

Suppose that $\{f_\nu\}$ is a sequence of orientation preserving mappings

$$f_\nu = (f_\nu^1, \dots, f_\nu^n)\colon \Omega \to \mathbf{R}^n, \qquad \nu = 1, 2, \dots, \tag{3.1}$$

which converge weakly in $W^{1,n}_{\text{loc}}(\Omega, \mathbf{R}^n)$ to a mapping $f = (f^1, \dots, f^n)$. This simply means that for each $i, j = 1, 2, \dots, n$ we have

$$\lim_{\nu\to\infty} \int_\Omega \phi \, \frac{\partial f_\nu^i}{\partial x_j} \, dx = \int_\Omega \phi \, \frac{\partial f^i}{\partial x_j} \, dx \tag{3.2}$$

for each ϕ in $L_0^{n/(n-1)}(\Omega)$, the space of test functions in $L^{n/(n-1)}$ with compact support in Ω.

A similar conclusion can be drawn for arbitrary minors of the differential matrix Df. Given l-tuples $1 \le i_1 < \cdots < i_l \le n$ and $1 \le j_1 < \cdots < j_l \le n$ we let

$$\frac{\partial(f^{i_1}, \dots, f^{i_l})}{\partial(x_{j_1}, \dots, x_{j_l})}$$

denote the corresponding $l \times l$ minor of Df. We then have the following counterpart for (3.2).

Lemma 3.3 (Weak continuity). *The above hypotheses imply that*

$$\lim_{\nu\to\infty} \int_\Omega \phi \, \frac{\partial(f_\nu^{i_1}, \dots, f_\nu^{i_l})}{\partial(x_{j_1}, \dots, x_{j_l})} \, dx = \int_\Omega \phi \, \frac{\partial(f^{i_1}, \dots, f^{i_l})}{\partial(x_{j_1}, \dots, x_{j_l})} \, dx \tag{3.4}$$

for each ϕ in $L_0^{n/(n-l)}(\Omega)$ and corresponding $l \times l$ minors of Df_ν and Df, $l = 1, 2, \dots, n$.

Proof. The convergence of the minors in the case where ϕ is in $C_0^\infty(\Omega)$ follows from integration by parts and the compactness of the Sobolev imbedding. The result in this case can be traced back at least as far as [2], [12] and [14]. The extension to arbitrary ϕ in $L_0^{n/(n-l)}(\Omega)$ with $1 \le l < n$ poses no problem because $C_0^\infty(\Omega)$ is dense in $L_0^{n/(n-l)}(\Omega)$. It is the case where $l = n$ and ϕ is in $L_0^\infty(\Omega)$ that requires our mappings f_ν to be orientation preserving; for this see Corollary 1.2 in [13]. □

The case where the mappings f_ν are K-quasiregular can be handled due to the higher degree of integrability of the Jacobians [5] and [8]. For yet another approach see the biting theorem for Jacobians, Corollary 2.3 in [4] and Corollary 2.2 in [19].

Corollary 3.5 (Biting convergence). *The above hypotheses imply that*

$$\text{b}*\lim_{\nu\to\infty} \frac{\partial(f_\nu^{i_1},\dots,f_\nu^{i_l})}{\partial(x_{j_1},\dots,x_{j_l})} = \frac{\partial(f^{i_1},\dots,f^{i_l})}{\partial(x_{j_1},\dots,x_{j_l})} \tag{3.6}$$

for corresponding $l\times l$ minors of Df_ν and Df, $l=1,2,\dots,n$.

4. Dilatation functions

We introduce as in Section 1 the following quantities for a matrix A in $\mathbf{R}_+^{n\times n}$:

$$\begin{array}{ll} \textit{Outer dilatation} & K_O(A) = |A|^n/\det(A), \\ \textit{Inner dilatation} & K_I(A) = K_O(A^{-1}), \\ \textit{Mean dilatation} & K_M(A) = \tfrac12\big(K_O(A)+K_I(A)\big), \\ \textit{Maximal dilatation} & K(A) = \max\{K_O(A),K_I(A)\}, \\ \textit{Linear dilatation} & H(A) = \big(K_O(A)\,K_I(A)\big)^{1/n}. \end{array} \tag{4.1}$$

By Cramer's rule, we can express $K_I(A)$ in terms of the minors of order $(n-1)$ and the determinant of A as follows:

$$K_I(A) = \frac{|A^{\#}|^n}{\det(A)^{n-1}}. \tag{4.2}$$

Here $A^{\#}$ is the matrix in $\mathbf{R}^{n\times n}$ whose entries are co-factors of A,

$$A^{\#}_{jk} = (-1)^{j+k}\det(M_{jk}),$$

where M_{jk} is a submatrix of A obtained by deleting the jth row and kth column.

The above definitions and an elementary analysis of the eigenvalues of AA^T yield the following estimates

$$K_O\,K_I = H^n, \qquad K_O \le H^{n-1} \le K_I^{n-1}, \qquad K_I \le H^{n-1} \le K_O^{n-1}. \tag{4.3}$$

From this and the arithmetic-geometric mean inequality we obtain the following estimates for the linear dilatation H:

$$H^{n/2} \le \tfrac12(K_O+K_I) = K_M = \tfrac12\Big(K_O + \frac{H^n}{K_O}\Big) \le \tfrac12(H+H^{n-1}). \tag{4.4}$$

Equality holds in the first inequality only when $K_O = K_I$ and in the second inequality only when $K_O = K_I^{n-1}$ or $K_I = K_O^{n-1}$.

5. Polyconvexity

We show here that the dilatation functions K_O, K_I, K_M and K are polyconvex on the set $\mathbf{R}_+^{n\times n}$, that is, that they can be expressed as convex functions of minors of the matrix $A \in \mathbf{R}_+^{n\times n}$.

Lemma 5.1. *The function*

$$F(x,y) = F_{p,q}(x,y) = \frac{x^p}{y^q} \tag{5.2}$$

is convex on $\mathbf{R}_+ \times \mathbf{R}_+$ *whenever* $p \geq q+1 \geq 1$.

Proof. We must show that

$$F(x,y) - F(a,b) \geq A(x-a) + B(y-b) \tag{5.3}$$

for all $x, y, a, b \in \mathbf{R}_+$ where

$$A = \frac{\partial F}{\partial x}(a,b) = \frac{pa^{p-1}}{b^q}, \qquad B = \frac{\partial F}{\partial y}(a,b) = -\frac{qa^p}{b^{q+1}}.$$

Inequality (5.3) is an immediate consequence of the arithmetic-geometric mean inequality

$$u_1^{r_1}\, u_2^{r_2}\, u_3^{r_3} \leq r_1 u_1 + r_2 u_2 + r_3 u_3 \tag{5.4}$$

which holds whenever r_j, u_j are nonnegative for $j = 1,2,3$ with $r_1 + r_2 + r_3 = 1$. See, for example, Section 2.5 in [9] In particular if we set

$$r_1 = \frac{1}{p}, \quad r_2 = \frac{q}{p}, \quad r_3 = \frac{p-q-1}{p}, \quad u_1 = \frac{x^p}{y^q}, \quad u_2 = \frac{a^p y}{b^{q+1}}, \quad u_3 = \frac{a^p}{b^q},$$

then we obtain

$$\frac{xa^{p-1}}{b^q} = u_1^{r_1}\, u_2^{r_2}\, u_3^{r_3} \leq r_1 u_1 + r_2 u_2 + r_3 u_3 = \frac{1}{p}\frac{x^p}{y^q} + \frac{q}{p}\frac{a^p y}{b^{q+1}} + \frac{p-q-1}{p}\frac{a^p}{b^q}$$

whence

$$\frac{x^p}{y^q} - \frac{a^p}{b^q} \geq \frac{pa^{p-1}}{b^q}(x-a) - \frac{qa^p}{b^{q+1}}(y-b) \tag{5.5}$$

which is (5.3). □

We see from (4.1) and (4.2) that

$$
(5.6)\qquad
\begin{aligned}
K_O(A) &= F\big(|A|, \det(A)\big) &&\text{with } p = n \text{ and } q = 1,\\
K_I(A) &= F\big(|A^{\#}|, \det(A)\big) &&\text{with } p = n \text{ and } q = n-1.
\end{aligned}
$$

We observe next that the function $F(x,y)$ is increasing in the variable x and that x is a convex function of the minors of A, $x = |A|$ or $x = |A^{\#}|$, respectively, in (5.6). This implies that both K_O and K_I are polyconvex and hence so are the mean and maximal dilatations K_M and K. On the other hand, the linear dilatation H fails to be even rank-one convex [10].

We conclude by recording from (5.5) and (5.6) what polyconvexity means for the outer and inner dilatations:

$$
(5.7)\qquad K_O(X) - K_O(A) \ge \frac{n|A|^{n-1}}{\det(A)}\big(|X| - |A|\big) - \frac{|A|^n}{\det(A)^2}\big(\det(X) - \det(A)\big)
$$

and
(5.8)

$$
K_I(X) - K_I(A) \ge \frac{n|A^{\#}|^{n-1}}{\det(A)^{n-1}}\big(|X^{\#}| - |A^{\#}|\big) - \frac{(n-1)|A^{\#}|^n}{\det(A)^n}\big(\det(X) - \det(A)\big).
$$

6. Lower semicontinuity

For simplicity of notation we will use the symbol $\mathscr{K}(f) = \mathscr{K}(x,f)$ to denote any one of the dilatations $K_O(x,f)$, $K_I(x,f)$, $K_M(x,f)$ or $K(x,f)$. Then

$$
\mathscr{K}(x,f) = \mathscr{K}\big(Df(x)\big)
$$

whenever $J(x,f) > 0$, where $\mathscr{K}(Df)$ denotes the corresponding dilatation function of matrices defined in (4.1).

Theorem 6.1. *Suppose that $f_\nu\colon \Omega \to \mathbf{R}^n$ is a sequence of mappings of finite distortion which converge weakly in $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ to f and suppose that*

$$
(6.2)\qquad \mathscr{K}(x, f_\nu) \le M(x) < \infty \qquad \textit{for } \nu = 1, 2, \dots
$$

a.e. in Ω. Then f has finite distortion and there exists a subsequence $\{f_{\nu_k}\}$ such that

$$
(6.3)\qquad \mathscr{K}(x,f) \le \mathrm{b}{*}\lim_{k\to\infty} \mathscr{K}(x, f_{\nu_k})
$$

in Ω. In particular

$$
(6.4)\qquad \mathscr{K}(x,f) \le M(x)
$$

a.e. in Ω.

Proof. We consider first the case where $\mathscr{K}(f)$ is the outer dilatation $K_O(f)$. Then (6.2) implies that

$$|Df_\nu(x)|^n \le M(x)J(x, f_\nu)$$

a.e. in Ω while

$$\text{(6.5)} \qquad \mathrm{b}{*}\lim_{\nu\to\infty} \det(Df_\nu) = \det(Df), \qquad \mathrm{b}{*}\lim_{\nu\to\infty} M\det(Df_\nu) = M\det(Df)$$

by Corollary 3.5 and Lemma 2.3. Next by Lemma 2.5 we can choose a subsequence $\{f_{\nu_k}\}$ such that

$$\text{(6.6)} \qquad |Df|^n \le \mathrm{b}{*}\lim_{k\to\infty} |Df_{\nu_k}|^n \le \mathrm{b}{*}\lim_{k\to\infty} M\det(Df_{\nu_k}) = M\det(Df)$$

and

$$\text{(6.7)} \qquad |Df(x)|^n \le M(x)J(x, f)$$

a.e. in Ω. Thus f has finite distortion and (6.4) holds a.e. in Ω.

Finally in order to establish (6.3) we apply (5.7) to the matrices $X = Df_\nu$ and $A = Df$ to obtain

$$\text{(6.8)} \qquad K_O(f_\nu) - K_O(f) \ge M_1(|Df_\nu| - |Df|) - M_2\big(\det(Df_\nu) - \det(Df)\big),$$

where

$$\text{(6.9)} \qquad M_1 = \frac{n|Df|^{n-1}}{\det(Df)}, \qquad M_2 = \frac{|Df|^n}{\det(Df)^2},$$

a.e. in the set $E \subset \Omega$ where $\det(Df) \ne 0$. Next if we restrict our attention to the set E, then by Lemma 2.5, Corollary 3.5 and Lemma 2.4 we can choose a subsequence $\{f_{\nu_k}\}$ such that

$$\text{(6.10)} \qquad |Df| \le \mathrm{b}{*}\lim_{k\to\infty} |Df_{\nu_k}|, \qquad \det(Df) = \mathrm{b}{*}\lim_{k\to\infty} \det(Df_{\nu_k})$$

and such that $K_O(f_{\nu_k})$ converges in the biting sense. Then we obtain

$$\text{(6.11)} \qquad K_O(f) \le \mathrm{b}{*}\lim_{k\to\infty} K_O(f_{\nu_k})$$

in E from (6.8), (6.9) and (6.10). By our convention

$$\text{(6.12)} \qquad K_O(f) = 1 \le \mathrm{b}{*}\lim_{k\to\infty} K_O(f_{\nu_k})$$

a.e. in $\Omega \setminus E$, completing the proof of Theorem 6.1 for the case where $\mathscr{K} = K_O$.

Suppose next that $\mathscr{K} = K_I$. Then (4.3) yields the rough estimate

$$K_O(x, f_\nu) \le K_I^{n-1}(x, f_\nu) \le M(x)^{n-1} \tag{6.13}$$

and f has finite distortion by what was proved above. Next (4.2) and (5.8) applied to the matrices $X = Df_\nu$ and $A = Df$ yield

$$K_I(f_\nu) - K_I(f) \ge M_3(|(Df_\nu)^\#| - |(Df)^\#|) - M_4\big(\det(Df_\nu) - \det(Df)\big), \tag{6.14}$$

where

$$M_3 = \frac{n|(Df)^\#|^{n-1}}{\det(Df)^{n-1}}, \qquad M_4 = \frac{(n-1)|(Df)^\#|^n}{\det(Df)^n}, \tag{6.15}$$

a.e. in E. Then $(Df_\nu)^\#$ converges weakly to $(Df)^\#$ in $L^{n/(n-1)}_{\text{loc}}(\Omega, \mathbf{R}^{n\times n})$, there is a subsequence $\{f_{\nu_k}\}$ such that the $K_I(f_{\nu_k})$ converges in the biting sense and

$$|(Df)^\#| \le \text{b}{*}\lim_{k\to\infty} |(Df_{\nu_k})^\#|, \qquad \det(Df) = \text{b}{*}\lim_{k\to\infty} \det(Df_{\nu_k}) \tag{6.16}$$

again by Lemma 2.5 and Corollary 3.5. Then

$$K_I(f) \le \text{b}{*}\lim_{k\to\infty} K_I(f_{\nu_k}) \tag{6.17}$$

in E as above while

$$K_I(f) = 1 \le \text{b}{*}\liminf_{k\to\infty} K_I(f_{\nu_k}) \tag{6.18}$$

in $\Omega \setminus E$ since f has finite distortion. This completes the proof of Theorem 6.1 for the case where $\mathscr{K} = K_I$.

If $\mathscr{K} = K_M$, then

$$\max\{K_O(x, f_\nu), K_I(x, f_\nu)\} \le 2K_M(x, f_\nu) \le 2M(x)$$

and f has finite distortion. Next there exists a subsequence $\{f_{\nu_k}\}$ such that $K_O(f_{\nu_k})$, $K_I(f_{\nu_k})$ and $K_M(f_{\nu_k})$ converge in the biting sense, and we obtain

$$\begin{aligned} 2K_M(f) &= K_O(f) + K_I(f) \\ &\le \text{b}{*}\lim_{k\to\infty} K_O(f_{\nu_k}) + \text{b}{*}\lim_{k\to\infty} K_I(f_{\nu_k}) = 2\,\text{b}{*}\lim_{k\to\infty} K_M(f_{\nu_k}) \end{aligned}$$

from (6.3) with $\mathscr{K} = K_O$ and $\mathscr{K} = K_I$.

Finally if $\mathscr{K} = K$, then

$$\max\{K_O(x,f_\nu), K_I(x,f_\nu)\} \le K(x,f_\nu) \le M(x)$$

and we can choose a subsequence $\{f_{\nu_k}\}$ such that

$$K_O(f) \le \mathrm{b}{*}\lim_{k\to\infty} K_O(f_{\nu_k}) \le \mathrm{b}{*}\lim_{k\to\infty} K(f_{\nu_k}),$$
$$K_I(f) \le \mathrm{b}{*}\lim_{k\to\infty} K_I(f_{\nu_k}) \le \mathrm{b}{*}\lim_{k\to\infty} K(f_{\nu_k}).$$

Hence

$$K(f) \le \mathrm{b}{*}\lim_{k\to\infty} K(f_{\nu_k}) \tag{6.19}$$

completing the proof of Theorem 6.1. □

7. Conclusions

Theorem 1.4, Remark 1.7 and Theorem 1.15 of Section 1 follow from Theorem 6.1. For the proof of Theorem 1.16, (1.17) and (4.4) imply that

$$K_M(x,f_\nu) \le \tfrac{1}{2}\bigl(H(x,f_\nu) + H(x,f_\nu)^{n-1}\bigr) \le \tfrac{1}{2}(M(x) + M(x)^{n-1}) \text{ for } \nu = 1,2,\dots$$

a.e. in Ω. Hence f has finite distortion and

$$H(x,f) \le K_M(x,f)^{2/n} \le \bigl(\tfrac{1}{2}\bigl(M(x) + M(x)^{n-1}\bigr)\bigr)^{2/n} \le M(x)^{2-(2/n)}$$

a.e. in Ω by (4.4) and Theorem 1.15. □

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