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ASYMPTOTIC VOLUME FORMULAE AND HYPERBOLIC BALL PACKING

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Abstract. We prove that the volume of an n-dimensional regular spherical simplex of edge length $r < \frac{1}{2}\pi$ is asymptotically $\mathrm{e}^{-x}x^{n/2}(n+1)^{1/2}/n!$, as $n \to \infty$, where $x = \sec r - 1$. The same is true for hyperbolic simplices if we set $x = 1 - \operatorname{sech} r$ and replace e^{-x} by e^{x}.

We obtain error bounds for this asymptotic, and apply it to find an upper bound for the density of packings of balls of a given radius in hyperbolic n-space, for all sufficiently large n.

1. Introduction

The volume of a polytope Δ in an n-dimensional space of constant curvature $K = \pm 1$ satisfies Schläfli's differential equation,

$$\mathrm{d}\,\mathrm{Volume} = \frac{K}{n-1}\sum(\mathrm{Vol.\,codimension\,2\,face})\,\mathrm{d}\,(\mathrm{angle}) \tag{1}$$

the sum being taken over all faces of codimension two and their corresponding dihedral angles [10, Chapter 7, 2.2].

From this, for example, we may derive the well-known area formulae for spherical and hyperbolic triangles. However, in higher dimensions, volume calculation remains a difficult problem, even with Schläfli's formula, because it requires the volumes of the codimension two faces as a function of the dihedral angles of the original polytope.

In this paper we derive an asymptotic formula for the volume, in either hyperbolic or spherical space, of a *regular* simplex, that is a simplex whose dihedral angles (or equivalently, whose edge lengths) are all equal. We let $S_n(r)$ and $H_n(r)$ denote, respectively, the volume of a regular n-simplex with edge length r in spherical space, and in hyperbolic space. We prove

Theorem 1. *The volume*

$$S_n(r) = \frac{(n+1)^{1/2}x^{n/2}\mathrm{e}^{-x}}{n!}\big(1 + s_n(r)\big), \tag{2}$$

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where $x = \sec r - 1$ *and, for each* $r_0 < \frac{1}{2}\pi$, *there is a constant* C *for which* $|s_n(r)| < C/n$, *whenever* $r \le r_0$, *and,*

$$(3) \qquad H_n(r) = \frac{(n+1)^{1/2} x^{n/2} \mathrm{e}^x}{n!}\bigl(1 + h_n(r)\bigr),$$

where $x = 1 - \operatorname{sech} r$, *and there is a constant* C *for which* $|h_n(r)| < C/n$, *for all* $r \in [0, \infty]$.

The case $r = \infty$ $(x = 1)$ of (3) corresponds to the regular ideal simplex. The asymptotic (3), in this case, is given (without proof) by Milnor [8] (see [4] for a proof).

Of course (2) and (3) are closely related, and we obtain both as a corollary of Theorem 3 below, which also provides a related asymptotic expansion.

We also apply Theorem 3 to prove a result about ball packing in hyperbolic space. The *local density* of a packing is defined to be the ratio Vol.(B)/Vol.(D), where B is a ball in the packing, and D is the set of all points closer to B than to any other ball in the packing. Böröczky [1, Theorems 1 and 4] has shown that the local density of a packing by balls of radius $\frac{1}{2}r$ is bounded above by the $d_n(r)$, defined as follows.

Let Δ be a regular simplex in hyperbolic n-space, with edge length r, and let P be the union of $n+1$ mutually tangent balls of radius $\frac{1}{2}r$ with centres at each of the vertices of Δ. We define,

$$d_n(r) = \mathrm{Vol.}(P \cap \Delta)/\mathrm{Vol.}(\Delta).$$

The limiting value $d_n(0) = \lim_{r\to 0} d_n(r)$ is obtained by doing the same construction in Euclidean space, in which case the dimensions of Δ are immaterial.

The limiting version of Böröczky's result is that the density D_n of an n-dimensional ball packing in *Euclidean* space does not exceed $d_n(0)$. This result was proved by Rogers [9]. Asymptotically better estimates have since been found (see e.g. [5]).

The function $d_n(r)$ has been proved to be strictly increasing, for $n = 2$ by Krammer (cited in [3, §37]), and for $n = 3$ by Böröczky and Florian [2]. Kellerhals [6] raises the question of whether this monotonicity also occurs in higher dimensions. For sufficiently high dimension we answer this question affirmatively.

Theorem 2. *For all sufficiently large* n, $d_n(r)$ *is a strictly increasing function of* r.

We conjecture that this result holds in all dimensions greater than one. The methods given here allow this to be tested in principle, but only through lengthy and tedious calculations. We therefore prove only the asymptotic result.

Since, for all r, $d_n(r) < 1$, monotonicity implies that, for each n, $d_n(r)$ approaches a limit $d_n(\infty)$ as $r \to \infty$. The local density of a packing by balls of any given radius is then bounded above by $d_n(\infty)$. This result has been used, when $n = 3$, to obtain improved lower bounds for the volumes of hyperbolic 3-manifolds and orbifolds. Details may be found in [7].

We use $\lfloor x \rfloor$ to denote the greatest integer not exceeding x. Given two functions $\mathrm{f}(n)$ and $\mathrm{g}(n)$, defined on the natural numbers, we use $\mathrm{f}(n) \sim \mathrm{g}(n)$ to denote asymptotic equality. For taking half integer powers of negative numbers we adopt the convention that $(-1)^{1/2} = i$, whence $(-1)^{(2n+1)/2}$ is i or $-i$, according as n is even or odd.

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2. Simplex volumes

The edge length r and the dihedral angle θ of a regular n-dimensional spherical simplex are related by the well-known formula,

$$\sec\theta = \sec r + (n-1) \tag{4}$$

The same equation, with $\operatorname{sech} r$ replacing $\sec r$, holds for the regular hyperbolic n-simplex. Thus, in the spherical case,

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\sec r}\frac{\mathrm{d}\sec r}{\mathrm{d}\sec\theta}\frac{\mathrm{d}\sec\theta}{\mathrm{d}\theta} = \frac{\sec\theta\tan\theta}{\sec r\tan r} \\ &= \frac{\big(\sec r + (n-1)\big)\sqrt{\big(\sec r + (n-1)\big)^2 - 1}}{\sec r\sqrt{\sec^2 r - 1}}. \end{aligned} \tag{5}$$

The same equation holds in the hyperbolic case, with sech r replacing sec r and a sign change on the right hand side and under the square root in the denominator.

The faces of regular simplex all have the same edge length as the original simplex. There are $\binom{n+1}{2}$ faces of codimension two, so that, for the regular spherical simplex with dihedral angle θ, Schläfli's equation may be written

$$\frac{\mathrm{d}S_n(r)}{\mathrm{d}\theta} = \frac{1}{n-1}\binom{n+1}{2}S_{n-2}(r). \tag{6}$$

Similarly we have, in the hyperbolic case,

$$\frac{\mathrm{d}H_n(r)}{\mathrm{d}\theta} = \frac{-1}{n-1}\binom{n+1}{2}H_{n-2}(r). \tag{7}$$

If $z = \sec r - 1$, then (6) gives,

$$
\begin{aligned}
\frac{\mathrm{d}S_n(r)}{\mathrm{d}z} &= \frac{\mathrm{d}S_n(r)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}z} \\
&= \frac{n(n+1)}{2(n-1)}\frac{S_{n-2}(r)}{(n+z)\big((n+z)^2-1\big)^{1/2}} \\
&= \frac{(n+1)^{1/2}}{2(n-1)^{3/2}}\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{z}{n+1}\right)^{-1/2} \\
&\quad \times \left(1+\frac{z}{n-1}\right)^{-1/2} S_{n-2}(r).
\end{aligned}
\tag{8}
$$

We obtain the corresponding equation for hyperbolic volumes by substituting $H_n(r)$ and $H_{n-2}(r)$ for $S_n(r)$ and $S_{n-2}(r)$, respectively, setting $z = \operatorname{sech} r - 1$, and multiplying the right hand side by -1. Thus if we define a family of functions $V_n(z)$ by

$$V_0(z) = 1, \qquad V_1(z) = \operatorname{ArcSec}(1+z)$$

and, for $n \geq 2$

$$V_n'(z) = \frac{(n+1)^{1/2}}{2(n-1)^{3/2}}\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{z}{n+1}\right)^{-1/2}\left(1+\frac{z}{n-1}\right)^{-1/2} V_{n-2}(z), \tag{9}$$

with $V_n(0) = 0$, then $S_n(r) = V_n(\sec r - 1)$, and $H_n(r) = (-1)^{n/2} V_n(\operatorname{sech} r - 1)$. Rather than using the functions $V_n(z)$, it will be more convenient to work with the related family $Y_n(z)$, given by

$$Y_n(z) = \frac{n!}{(\frac{1}{2}n)!(n+1)^{1/2}} V_n(z).$$

These satisfy the simpler equation

$$Y_n'(z) = \left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{z}{n+1}\right)^{-1/2}\left(1+\frac{z}{n-1}\right)^{-1/2} Y_{n-2}(z) \tag{10}$$

for $n \geq 2$ with initial condition $Y_n(0) = 0$. Together with the initial functions,

$$Y_0(z) = 1, \qquad Y_1(z) = \left(\frac{2}{\pi}\right)^{1/2} \operatorname{ArcSec}(1+z) = \frac{z^{1/2}}{(\frac{1}{2})!}\mathrm{e}^{-z}\Big(1+\frac{7}{12}z+\cdots\Big),$$

(10) can be taken as *defining* the family $Y_n(z)$. Now we have

$$S_n(r) = \frac{(\frac{1}{2}n)!(n+1)^{1/2}}{n!} Y_n(\sec r - 1) \tag{11}$$

and

$$H_n(r) = (-1)^{-n/2} \frac{(\frac{1}{2}n)!(n+1)^{1/2}}{n!} Y_n(\operatorname{sech} r - 1). \tag{12}$$

Clearly for n even, Y_n can be defined on the complex plane with the interval $(-\infty, -1)$ deleted, and is analytic in the interior of this region. For n odd, $Y_n(z)/\sqrt{z}$ has the same properties.

Theorem 3. *For each positive integer k,*

$$Y_n(z) = \frac{z^{n/2}}{(\frac{1}{2}n)!} \mathrm{e}^{-z} \left[1 + \sum_{i=1}^{k-1} r_i^{(n)} z^i + z^k \varepsilon_k^{(n)}(z) \right] \tag{13}$$

where (setting $r_0^{(n)} = 1$) we have,

$$r_i^{(n)} \sim \lambda_i n^{-\lfloor (i+1)/2 \rfloor} \qquad \text{as } n \to \infty, \tag{14}$$

the λ_i being defined recursively by $\lambda_0 = 1$, $\lambda_1 = \frac{3}{2}$, and,

$$\lambda_i = \frac{3\lambda_{i-2}}{i} \tag{15}$$

for even $i \geq 2$ and,

$$\lambda_i = \frac{[-4\lambda_{i-3} + 3\lambda_{i-2}]}{(i-1)} - 2\lambda_{i-1} \tag{16}$$

for odd $i \geq 3$. For each compact region in $\mathbf{C} - (-\infty, -1)$,

$$|\varepsilon_k^{(n)}(z)| \leq K n^{-\lfloor (k+1)/2 \rfloor}, \tag{17}$$

where K depends only on the region chosen.

Clearly Theorem 1 follows from Theorem 3 (with $k = 1$), using (11) and (12). Setting, for $n \geq 2$,

$$\begin{aligned} h_n(z) &= \left(1 + \frac{z}{n}\right)^{-1} \left(1 + \frac{z}{n+1}\right)^{-1/2} \left(1 + \frac{z}{n-1}\right)^{-1/2} \\ &= \sum_{i=0}^{\infty} t_i^{(n)} z^i = \sum_{i=0}^{k-1} t_i^{(n)} z^i + s_k^{(n)}(z), \end{aligned} \tag{18}$$

we have $t_0^{(n)} = 1$,

$$t_1^{(n)} = -\left(\frac{2}{n} + \frac{1}{n(n^2-1)}\right), \tag{19}$$

and

$$t_i^{(n)} \sim \frac{(-1)^i(i+1)}{n^i} \qquad \text{as } n \to \infty. \tag{20}$$

Also

$$\frac{s_k^{(n)}(z)}{z^k} = O(n^{-k}) \qquad (n \to \infty), \tag{21}$$

uniformly in compact subsets of $\mathbf{C} - (-\infty, -1)$.

3. Proofs of theorems

Proof of Theorem 3. Clearly we may write $Y_n(z)$ in the form (13). We must prove (14)–(17). Setting $\varrho_k^{(n)}(z) = \mathrm{e}^{-z} z^{k+n/2} \varepsilon_k^{(n)}(z)/(\frac{1}{2}n)!$, we have,

$$Y_n(z) = \frac{z^{n/2}}{(\frac{1}{2}n)!}\mathrm{e}^{-z}\left[1 + \sum_{i=1}^{k-1} r_i^{(n)} z^i\right] + \varrho_k^{(n)}(z). \tag{22}$$

When $n \geq 2$, substituting this form of $Y_n(z)$ into (10), dividing through by $\mathrm{e}^{-z} z^{(n-2)/2}/\big(\frac{1}{2}(n-2)\big)!$, and equating in turn the coefficients of z^i $(0 \leq i \leq k-1)$, and the remainder gives,

$$r_i^{(n)}\left(1 + \frac{2i}{n}\right) - \frac{2}{n} r_{i-1}^{(n)} = \sum_{j=0}^{i} t_{i-j}^{(n)} r_j^{(n-2)} \qquad (1 \leq i \leq k-1) \tag{23}$$

and,

$$\begin{aligned} &-\frac{z^{(n-2)/2}}{\big(\frac{1}{2}(n-2)\big)!}\mathrm{e}^{-z}\left[\frac{2r_{k-1}^{(n)}}{n} z^k\right] + \varrho_k^{(n)\prime}(z) \\ &\qquad = \frac{z^{(n-2)/2}}{\big(\frac{1}{2}(n-2)\big)!}\mathrm{e}^{-z}\left(\sum_{i=0}^{k-1} r_i^{(n-2)} s_{k-i}^{(n)}(z) z^i\right) + h_n(z)\varrho_k^{(n-2)}(z). \end{aligned} \tag{24}$$

whence, using (19),

$$\begin{aligned} \varrho_k^{(n)\prime}(z) = \frac{z^{(n-2)/2}}{((n-2)/2)!}\mathrm{e}^{-z}\Bigg[&\sum_{i=0}^{k-2} r_i^{(n-2)} s_{k-i}^{(n)}(z) z^i + r_{k-1}^{(n-2)} s_2^{(n)}(z) z^{k-1} \\ &+ \frac{2}{n}(r_{k-1}^{(n)} - r_{k-1}^{(n-2)}) z^k - \frac{r_{k-1}^{(n-2)} z^k}{n(n^2-1)}\Bigg] + h_n(z)\varrho_k^{(n-2)}(z). \end{aligned} \tag{25}$$

If we define,

$$R_i^{(n)} = \binom{i+\frac{1}{2}n}{i} r_i^{(n)},$$

(23) gives

$$\begin{aligned} R_i^{(n)} - R_i^{(n-2)} &= \binom{i-1+\frac{1}{2}n}{i}\left[\sum_{j=0}^{i-1} t_{i-j}^{(n)} r_j^{(n-2)} + \frac{2}{n} r_{i-1}^{(n)}\right] \\ &= \binom{i-1+\frac{1}{2}n}{i}\left[\sum_{j=0}^{i-2} t_{i-j}^{(n)} r_j^{(n-2)} - \frac{r_{i-1}^{(n-2)}}{n(n^2-1)} + \frac{2}{n}\big(r_{i-1}^{(n)} - r_{i-1}^{(n-2)}\big)\right]. \end{aligned} \tag{26}$$

Also from (23), we have

$$r_i^{(n)} - r_i^{(n-2)} = \sum_{j=0}^{i-2} t_{i-j}^{(n)} r_j^{(n-2)} - \frac{r_{i-1}^{(n-2)}}{n(n^2-1)} + \frac{2}{n}(r_{i-1}^{(n)} - r_{i-1}^{(n-2)}) - \frac{2i}{n} r_i^{(n)}. \tag{27}$$

We prove (14) by induction on i, simultaneously with,

$$r_i^{(n)} - r_i^{(n-2)} \sim -2\left\lfloor \tfrac{1}{2}(i+1)\right\rfloor \lambda_i (n^{-\lfloor (i+3)/2 \rfloor}) \qquad \text{as } n \to \infty. \tag{28}$$

Since $r_i^{(0)} = 1$, (14) and (28) hold for $i = 0$. For $i = 1$, (26) gives

$$R_1^{(n)} - R_1^{(n-2)} = \frac{n}{2}\left(\frac{-1}{n(n^2-1)}\right) = -\frac{1}{4}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

Since $R_1^{(0)} = 1$ and $R_1^{(1)} = \frac{7}{8}$, this gives, for even and odd n, respectively,

$$R_1^{(n)} = 1 - \frac{1}{4} + \frac{1}{4(n+1)} \qquad \text{and} \qquad R_1^{(n)} = \frac{7}{8} - \frac{1}{8} + \frac{1}{4(n+1)},$$

so that, for all n,

$$R_1^{(n)} = \frac{3n+4}{4(n+1)}, \qquad r_1^{(n)} = \frac{3n+4}{2(n+1)(n+2)}.$$

It is thus clear that (14) and (28) hold for $i = 1$. For the induction step we suppose that $i \geq 2$ and (14) and (28) hold for all $i < m$. We prove (14) for $i = m$. Using (20) and the induction hypothesis, the last term in the sum in the square brackets of (26) is asymptotically $3\lambda_{m-2} n^{-\lfloor (m+3)/2 \rfloor}$. If m is even then, again using the induction hypothesis, all the other terms in the square brackets diminish at least as rapidly as $n^{-(2+m/2)}$ whence,

$$R_m^{(n)} - R_m^{(n-2)} \sim \left(\frac{3\lambda_{m-2}}{2^m m!}\right) n^{m/2-1}$$

so that,

$$R_m^{(n)} \sim \left(\frac{3\lambda_{m-2}}{2^m m!}\right)\left(\frac{n^{m/2}}{2(\frac{1}{2}m)}\right)$$

and finally,

$$r_m^{(n)} \sim \left(\frac{3\lambda_{m-2}}{m}\right) n^{-m/2}$$

so that $\lambda_m = 3\lambda_{m-2}/m$. When m is odd the induction step is similar. This time the expression in the square brackets of (26) is dominated by the terms

$$t_3^{(n)} r_{m-3}^{(n-2)} + t_2^{(n)} r_{m-2}^{(n-2)} + \frac{2}{n}(r_{m-1}^{(n)} - r_{m-1}^{(n-2)}),$$

which, using (20) and the induction hypothesis, is asymptotically,

$$\big(-4\lambda_{m-3} + 3\lambda_{m-2} - 2(m-1)\lambda_{m-1}\big) n^{-(m+3)/2}$$

and the argument continues, as for the even case.

A similar argument, using (27), is used for the induction step in the proof of (28). We omit details.

Finally we prove (17). For each compact $M \subseteq \mathbf{C} - (-\infty, -1)$, we show that there are constants, K_n for which, for all $z \in M$,

$$|\varrho_k^{(n)}(z)| \le K_n \frac{e^{|z|}|z|^{k+n/2}}{n^{\lfloor (k+1)/2 \rfloor}(\frac{1}{2}n)!}, \tag{29}$$

where the sequence $\{K_n\}$ is bounded above. The bound (17) then follows from (29). For $n = 0, 1$ it is easy to see that there exist constants K_0 and K_1 respectively, for which (29) holds. For $n \ge 2$, (25), along with (14), (21) and (28), gives,

$$|\varrho_k^{(n)\prime}(z)| \le C_1 \frac{e^{|z|}|z|^{k+(n-2)/2}}{n^{\lfloor (k+3)/2 \rfloor}\big(\frac{1}{2}(n-2)\big)!} + |h_n(z)\varrho_k^{(n-2)}(z)|, \tag{30}$$

where C_1 is a constant depending only on M. Using the induction hypothesis,

$$|\varrho_k^{(n)\prime}(z)| \le \frac{e^{|z|}|z|^{k+(n-2)/2}}{n^{\lfloor (k+1)/2 \rfloor}\big(\frac{1}{2}(n-2)\big)!}\left(\frac{C_1}{n} + |h_n(z)| K_{n-2}\left(\frac{n}{n-2}\right)^{\lfloor (k+1)/2 \rfloor}\right). \tag{31}$$

Using (21), there is a constant C_2, depending only on M, for which,

$$|h_n(z)| \leq \left(1 + \frac{|z|}{(k+\frac{1}{2}n)}\right)\left(1 + \frac{C_2}{n^2}\right)$$

whence,

$$\begin{aligned} |\varrho_k^{(n)\prime}(z)| \leq {} & \frac{\mathrm{e}^{|z|}|z|^{k+(n-2)/2}[1+|z|/(k+\frac{1}{2}n)]}{n^{\lfloor (k+1)/2 \rfloor}\left(\frac{1}{2}(n-2)\right)!} \\ & \times \left(\frac{C_1}{n} + \left(1+\frac{C_2}{n^2}\right)K_{n-2}\left(\frac{n}{n-2}\right)^{\lfloor (k+1)/2 \rfloor}\right). \end{aligned} \tag{32}$$

Integrating then gives,

$$\begin{aligned} |\varrho_k^{(n)}(z)| &\leq \frac{\mathrm{e}^{|z|}|z|^{k+n/2}}{n^{\lfloor (k+1)/2 \rfloor}(\frac{1}{2}n)!}\left(\frac{n}{2k+n}\right)\left(\frac{C_1}{n} + \left(1+\frac{C_2}{n^2}\right)K_{n-2}\left(\frac{n}{n-2}\right)^{\lfloor (k+1)/2 \rfloor}\right) \\ &= K_n \frac{\mathrm{e}^{|z|}|z|^{k+n/2}}{n^{\lfloor (k+1)/2 \rfloor}(\frac{1}{2}n)!}, \end{aligned} \tag{33}$$

where

$$K_n = \left(\frac{n}{2k+n}\right)\left(\frac{C_1}{n} + \left(1+\frac{C_2}{n^2}\right)K_{n-2}\left(\frac{n}{n-2}\right)^{\lfloor (k+1)/2 \rfloor}\right).$$

Since, as $n \to \infty$, $n/(2k+n)$ and $\big(n/(n-2)\big)^{\lfloor (k+1)/2 \rfloor}$ are asymptotically $1-2k/n$ and $1+(2/n)\lfloor (k+1)/n \rfloor$, respectively, one can readily verify that the sequence $\{K_n\}$ approaches a finite limit. ◻

Proof of Theorem 2. Let Δ be a regular hyperbolic simplex Δ with edge length r and dihedral angle θ. The link of Δ is the $(n-1)$-dimensional spherical simplex obtained by intersecting it with a small sphere centred at one of its vertices, and scaling to obtain a simplex in the unit sphere. The link of Δ thus has the same dihedral angle as Δ, but a different edge length, which we denote by $\tilde{r}$.

Now let B be a hyperbolic ball of radius $\frac{1}{2}r$ centred at a vertex of Δ. In terms of volume, the intersection $B \cap \Delta$ is the same proportion of B as the link of Δ is of the unit $(n-1)$-sphere, S^{n-1}. Recalling that the volume of a hyperbolic n-ball of radius x is Volume$\,(S^{n-1}) \int_0^x \sinh^{n-1} t\,\mathrm{d}t$, the volume of $B \cap \Delta$ is given by

$$S_{n-1}(\tilde{r}) \int_0^{r/2} \sinh^{n-1} t\,\mathrm{d}t,$$

whence, adding $n+1$ disjoint regions with this volume, and dividing by the volume of Δ, we obtain

$$d_n(r) = (n+1)\frac{S_{n-1}(\tilde{r})}{H_n(r)}\int_0^{r/2}\sinh^{n-1} t\,\mathrm{d}t. \tag{34}$$

We set,

$$x = \operatorname{sech} r = \sec\tilde{r} - 1,$$

and

$$y = 1 - \operatorname{sech} r = 1 - x,$$

and also let

$$c = \cosh\left(\tfrac{1}{2}r\right) = \sqrt{\tfrac{1}{2}(x^{-1}+1)},$$
$$s = \sinh\left(\tfrac{1}{2}r\right) = \sqrt{\tfrac{1}{2}(x^{-1}-1)}.$$

The change of variable, $u = \sinh t$, and integration by parts gives

$$\begin{aligned}\int_0^{r/2}\sinh^{n-1} t\,\mathrm{d}t &= \int_0^s u^{n-1}(1+u^2)^{-1/2}\,\mathrm{d}u\\ &= \frac{1}{n}\left[\frac{s^n}{c} + \int_0^s u^{n+1}(1+u^2)^{-3/2}\,\mathrm{d}u\right]\\ &\le \frac{s^n}{nc}\left[1 + \frac{cs^2}{(n+2)}\right].\end{aligned} \tag{35}$$

Also, for $n \ge 6$,

$$\begin{aligned}\int_0^{r/2}\sinh^{n-1} t\,\mathrm{d}t &= \int_0^s u^{n-2}(1+u^{-2})^{-1/2}\,\mathrm{d}u\\ &= \frac{1}{n-1}\left[\frac{s^n}{c} - \int_0^s u^{n-4}(1+u^{-2})^{-3/2}\,\mathrm{d}u\right]\\ &= \frac{1}{n-1}\left[\frac{s^n}{c} - \frac{s^n}{(n-3)c^3} + \frac{3}{(n-3)}\int_0^s u^{n-6}(1+u^{-2})^{-5/2}\,\mathrm{d}u\right]\\ &\le \frac{s^n}{(n-1)c}\left[1 - \frac{1}{(n-3)c^2} + \frac{3}{(n-3)(n-5)c^4}\right].\end{aligned} \tag{36}$$

Now differentiating (34) with respect to θ, using (6) and (7), we obtain, when

$n \geq 3$
(37)

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\theta} d_n(r) = \frac{-(n+1)}{H_n(r)^2} \Bigg[& \frac{\mathrm{d}S_{n-1}(\tilde{r})}{\mathrm{d}\theta} \left(\int_0^{r/2} \sinh^{n-1} t \,\mathrm{d}t \right) H_n(r) \\
& + S_{n-1}(\tilde{r}) \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\int_0^{r/2} \sinh^{n-1} t \,\mathrm{d}t \right) H_n(r) \\
& - \frac{\mathrm{d}H_n(r)}{\mathrm{d}\theta} S_{n-1}(\tilde{r}) \int_0^{r/2} \sinh^{n-1} t \,\mathrm{d}t \Bigg] \\
= & \frac{(n+1) S_{n-1}(\tilde{r}) \int_0^{r/2} \sinh^{n-1} t \,\mathrm{d}t}{2H_n(r)} \\
& \times \left[-\frac{n(n-1)S_{n-3}(\tilde{r})}{(n-2)S_{n-1}(\tilde{r})} - \frac{\sinh^{n-1}(\frac{1}{2}r)}{\int_0^{r/2} \sinh^{n-1} t \,\mathrm{d}t} \frac{\mathrm{d}r}{\mathrm{d}\theta} - \frac{n(n+1)H_{n-2}(r)}{(n-1)H_n(r)} \right].
\end{aligned}
$$

We denote the expression in square brackets in (37) by X, and its three summands by X_1, X_2 and X_3 respectively. Since r is a strictly decreasing function of θ, we must prove that X is positive for all $r > 0$. Using the hyperbolic version of (11), (12) and (4) we may rewrite X as

$$
\begin{aligned}
& -2n^{1/2}(n-1)(n-2)^{1/2} \frac{Y_{n-3}(x)}{Y_{n-1}(x)} + \frac{s^{n-1}(x+n-1)\big((x+n-1)^2-1\big)^{1/2}}{x(1-x^2)^{1/2} \int_0^{r/2} \sinh^{n-1} t \,\mathrm{d}t} \\
& + 2(n+1)^{1/2} n(n-1)^{1/2} \frac{Y_{n-2}(-y)}{Y_n(-y)}.
\end{aligned} \tag{38}
$$

Showing that X is positive requires some care, as a lot of cancellation occurs in the sum. For each n, the terms X_1 and X_2 both go to infinity as $x \to 0$, as do X_2 and X_3 when $y \to 0$, but the sum is bounded in $(0,1)$. We will also show that the sum grows like n^2 as $n \to \infty$, even though the individual terms grow like n^3.

We set

$$\eta_n(z) = \big[\big(2zY_{n-2}(z)\big)/\big(nY_n(z)\big)\big] - 1.$$

By Theorem 3 (with $k = 3$), $\eta_n(t)/t$ diminishes like n^{-2}, uniformly for $t \in [-1,1]$. We have,

$$X_1 = -\frac{n^{1/2}(n-1)^2(n-2)^{1/2}}{x}\big(1+\eta_{n-1}(x)\big), \tag{39}$$

and, using (36), for $n \geq 6$,

(40)

$$X_2 \geq x^{-1}y^{-1}n^{1/2}(n-1)^2(n-2)^{1/2}\left(1+\frac{x}{n-1}\right)\left(1+\frac{x}{n}\right)^{1/2}\left(1+\frac{x}{n-2}\right)^{1/2} \times\left[1-\frac{1}{(n-3)c^2}+\frac{3}{(n-3)(n-5)c^4}\right]^{-1},$$

whence

(41)

$$\begin{aligned} X_1+X_2 &\geq \frac{1}{xy}n^{1/2}(n-1)^2(n-2)^{1/2}(1-y+3x/n+xy(1+x)^{-1}/n+\varepsilon_1 n^{-2}x) \\ &= \frac{1}{y}n^{1/2}(n-1)^2(n-2)^{1/2}(1+3/n+y(1+x)^{-1}/n+\varepsilon_1 n^{-2}) \\ &= \frac{1}{y}n^3(1+y(1+x)^{-1}/n+\varepsilon_3 n^{-2}) \end{aligned}$$

where terms of the form ε_i are bounded uniformly for $x \in [0,1]$, and for all $n \geq 6$.

Since,

$$X_3 = -\frac{1}{y}n^3(1-1/n^2)^{1/2}\big(1+\eta_n(-y)\big), \tag{42}$$

for each $x_0 < 1$, there is an n_0 for which X is positive for all $x \leq x_0$, and $n \geq n_0$.

We now consider the behaviour of X, when x is bounded below, say $x \geq x_1 > 0$. We have, using (35)

(43)

$$\begin{aligned} X_2 &= \frac{s^{n-1}(n-y)\big((n-y)^2-1\big)^{1/2}}{x(1-x^2)^{1/2}\int_0^{r/2}\sinh^{n-1}t\,\mathrm{d}t} \\ &= \frac{n(n^2-1)^{1/2}s^{n-1}}{x(1-x^2)^{1/2}\int_0^{r/2}\sinh^{n-1}t\,\mathrm{d}t}\left(1-\frac{y}{n}\right)\left(1-\frac{y}{n+1}\right)^{1/2}\left(1-\frac{y}{n-1}\right)^{1/2} \\ &\geq \frac{n^2(n^2-1)^{1/2}}{xy}\left[1+\frac{cs^2}{(n+2)}\right]^{-1}\left[1-\frac{2(2n^2-1)y}{n(n^2-1)}\right]^{1/2} \\ &= \frac{n^2(n^2-1)^{1/2}}{xy}\left[1+\frac{cy}{2(n+2)x}\right]^{-1}\left[1-\frac{2(2n^2-1)y}{n(n^2-1)}\right]^{1/2} \\ &= \frac{n^2(n^2-1)^{1/2}}{xy}\left[1-\frac{cy}{2nx}-\frac{2y}{n}+\frac{\varepsilon_3 y}{n^2}\right] \\ &= \frac{n^2(n^2-1)^{1/2}}{y}\left[1+\frac{y}{x}-\frac{cy}{2nx^2}-\frac{2y}{nx}+\frac{\varepsilon_3 y}{n^2 x}\right], \end{aligned}$$

where here, and in the sequel, terms of the form ε_i are bounded uniformly in n and $x \geq x_1$.

Hence

$$
\begin{aligned}
X_2 + X_3 &\geq \frac{1}{y} n^2 (n^2-1)^{1/2} \left[\frac{y}{x} - \frac{cy}{2nx^2} - \frac{2y}{nx} + \frac{\varepsilon_4 y}{n^2 x} \right] \\
&= \frac{1}{x} n^2 (n^2-1)^{1/2} \left[1 - \frac{c}{2nx} - \frac{2}{n} + \frac{\varepsilon_4}{n^2} \right] \\
&= \frac{1}{x} n^3 \left[1 - \frac{c}{2nx} - \frac{2}{n} + \frac{\varepsilon_5}{n^2} \right].
\end{aligned} \tag{44}
$$

Since, from (39)

$$
X_1 = -\frac{1}{x} n^3 (1 - 3/n + \varepsilon_6/n^2), \tag{45}
$$

it follows that, when x_1 is chosen so that $c/x_1 < 2$, X is positive for all $x \geq x_1$, and all sufficiently large n. We have thus shown that X is positive, so that $d_n(r)$ is strictly increasing, for all r, when n is sufficiently large. □

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