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ELIMINATION OF DEFECTS OF MEROMORPHIC MAPPINGS OF $\mathbf{C}^m$ INTO $\mathbf{P}^n(\mathbf{C})$

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Abstract. The Nevanlinna defect relation and other results on the Nevanlinna theory assert that each meromorphic mapping f of C^m into $\mathbf{P}^n(\mathbf{C})$ has few deficient hyperplanes in $\mathbf{P}^n(\mathbf{C})$. However, it seems to me that meromorphic mappings with a deficient hyperplane must be very few. In this paper, we show that for an arbitrary given transcendental meromorphic mapping f (which may be linearly degenerate), we can eliminate all deficient hyperplanes in the sense of Nevanlinna by a small deformation of f.

1. Introduction

For a nondegenerate meromorphic mapping f of $\mathbf{C}^m$ into the complex projective space $\mathbf{P}^n(\mathbf{C})$, Nevanlinna's defect relation

$$\sum_{j=1}^{N} \delta_f(H_j) \leq n+1$$

holds for hyperplanes $\{H_j\}_{j=1}^N \in \mathbf{P}^n(\mathbf{C})^*$ in a general position. Hence the set of hyperplanes with a positive Nevanlinna deficiency in a set $X \subset \mathbf{P}^n(\mathbf{C})^*$ of hyperplanes in a general position is at most countable. (Such a hyperplane is called a deficient hyperplane or a defect.) Sadullaev [5] proved that the set of hyperplanes with a positive Valiron deficiency is of capacity zero (or it is a locally pluripolar set). Furthermore, we observe that the set of Valiron deficient hyperplanes has projective logarithmic capacity zero in the sense of Molzon–Shiffman–Sibony [3]. These results assert that defects of a meromorphic mapping are very few, and sometimes the Nevanlinna theory is called the equidistribution theory. However, it seems to me that meromorphic mappings with deficient hyperplanes must be very few.

In this note, we deal with the problem of whether we can eliminate all defects for a given meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ by a small deformation of the mapping. In one dimensional case, for any finite-order transcendental entire function f, the inequality

$$\sum_{a \in \mathbf{C}} \delta_f(a) \leq \delta_{f'}(0)$$

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holds (see Hayman [2, p. 104]), and we also find $\alpha \in \mathbf{C}$ such that $\tilde{f}(z) = f(z) + \alpha z$ satisfies $\delta_{\tilde{f}'}(0) = 0$. Hence we see

$$\sum_{a \in \mathbf{C}} \delta_{\tilde{f}}(a) = 0,$$

that is, $\tilde{f}$ does not have a finite defect. In Section 3, we shall prove that for any transcendental meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, we can eliminate all Nevanlinna defects by a small deformation $\tilde{f}$ of f. Here a small deformation $\tilde{f}$ of f means that their Nevanlinna order functions satisfy

$$|T_f(r) - T_{\tilde{f}}(r)| \leq O(\log r), \qquad r \to +\infty.$$

2. Preliminaries

2.1. Notation and terminology. Let $z = (z_1, \dots, z_m)$ be the natural coordinate system in $\mathbf{C}^m$. Set

$$\langle z, \xi \rangle = \sum_{j=1}^{m} z_j \xi_j \quad \text{for} \qquad \xi = (\xi_1, \dots, \xi_m), \qquad \|z\|^2 = \langle z, \overline{z} \rangle,$$

$$B(r) = \{z \in \mathbf{C}^m \mid \|z\| < r\}, \qquad \partial B(r) = \{z \in \mathbf{C}^m \mid \|z\| = r\},$$

$$\psi = dd^c \log \|z\|^2 \qquad \text{and} \qquad \sigma = d^c \log \|z\|^2 \wedge \psi^{m-1},$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$, and $\psi^k = \psi \wedge \dots \wedge \psi$ (k-times).

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then f has a reduced representation $(f_0 : \dots : f_n)$, where $f_0, \dots, f_n$ are holomorphic functions on $\mathbf{C}^m$ with $\operatorname{codim}_{\mathbf{C}}\{z \in \mathbf{C}^m \mid f_0(z) = \dots = f_n(z) = 0\} \geq 2$. We write $f = (f_0, \dots, f_n)$ as the same letter of the meromorphic mapping f. Denote $D^\alpha f = (D^\alpha f_0, \dots, D^\alpha f_n)$ for a multi-index α, where $D^\alpha \phi = \partial^{|\alpha|}\phi / \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}$, $\alpha = (\alpha_1, \dots, \alpha_m)$, $|\alpha| = \alpha_1 + \dots + \alpha_m$ and a function ϕ.

Definition. (See Fujimoto [4, Section 4].) We define the generalized Wronskian of f by

$$W_{\alpha^0, \dots, \alpha^n}(f) = \det(D^{\alpha^k} f : 0 \leq k \leq n),$$

for $n+1$ multi-indices $\alpha^k = (\alpha_1^k, \dots, \alpha_m^k)$, $0 \leq k \leq n$.

By Fujimoto [4, Section 4], for every linearly nondegenerate meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, there are $n+1$ multi-indices $\alpha^0, \dots, \alpha^n$ such that $\{D^{\alpha^0} f, \dots, D^{\alpha^n} f\}$ is an admissible basis with $|\alpha^k| \leq n+1$. Then $W_{\alpha^0, \dots, \alpha^n}(\phi f) = \phi^{n+1} W_{\alpha^0, \dots, \alpha^n}(f) \not\equiv 0$ holds for any nonzero holomorphic function ϕ on $\mathbf{C}^m$, where $\phi f = (\phi f_0, \dots, \phi f_n)$.

For a nonconstant meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, the proximity function $m_f(r,H)$ and the counting function $N_f(r,H)$ of a hyperplane H in $\mathbf{P}^n(\mathbf{C})$ are given by

$$m_f(r,H) := \int_{\partial B(r)} \log \frac{\|f\|\|\mathbf{a}\|}{|\langle f, \mathbf{a}\rangle|}\sigma$$

and

$$N_f(r,H) := \int_{r_0}^{r} \frac{dt}{t} \int_{(f^*H)\cap B(t)} \psi^{m-1}$$

for some fixed $r_0 > 0$, where

$$H = \Big\{ w = (w_0, \dots, w_n) \in \mathbf{C}^{n+1} \setminus \{0\} \mid \sum_{j=0}^{n} a_j w_j = 0 \Big\},$$

$\mathbf{a} = (a_0, \dots, a_n) \in \mathbf{C}^{n+1} \setminus \{0\}$ and f^*H denotes the pullback of H under f. The Nevanlinna order function $T_f(r)$ of f is given by

$$T_f(r) := \int_{r_0}^{r} \frac{dt}{t} \int_{B(t)} f^*\omega \wedge \psi^{m-1},$$

where $\omega = \{\omega_\alpha\} = dd^c \log \sum_{j=0}^{n} (|w_j/w_\alpha|^2)$, in a neighborhood $U_\alpha := \{w_\alpha \neq 0\}$. We write

$$N\big(r, (\phi)\big) := \int_{r_0}^{r} \frac{dt}{t} \int_{(\phi)\cap B(t)} \psi^{m-1},$$

where (ϕ) denotes the divisor determined by a meromorphic function ϕ on $\mathbf{C}^m$. We note that

$$\log \|f\| = \log \sqrt{|f_0|^2 + \cdots + |f_n|^2} = \log(|f_0| + \cdots + |f_n|) + O(1).$$

Then we observe that

$$T_f(r) = \int_{\partial B(r)} \log \Big(\sum_{j=0}^{n} |f_j|^2 \Big)^{1/2} \sigma + O(1) = \int_{\partial B(r)} \log \sum_{j=0}^{n} |f_j| \sigma + O(1).$$

The Nevanlinna deficiency $\delta_f(H)$ and the Valiron deficiency $\Delta_f(H)$ of a hyperplane H for f are given by

$$\delta_f(H) := \liminf_{r\to+\infty} \frac{m_f(r,H)}{T_f(r)} = 1 - \limsup_{r\to+\infty} \frac{N_f(r,H)}{T_f(r)}$$

and

$$\Delta_f(H) := \limsup_{r\to+\infty} \frac{m_f(r,H)}{T_f(r)} = 1 - \liminf_{r\to+\infty} \frac{N_f(r,H)}{T_f(r)}.$$

We now define the projective logarithmic capacity of a set in the projective space $\mathbf{P}^n(\mathbf{C})$ (see Molzon–Shiffman–Sibony [3, p. 46]). Let E be a compact subset of $\mathbf{P}^n(\mathbf{C})$ and $\mathscr{P}(E)$ denote the set of probability measures supported on E. We define

$$V_\mu(x) := \int_{w\in\mathbf{P}^n(\mathbf{C})} \log\frac{\|x\|\|w\|}{|\langle x,w\rangle|}\,d\mu(w), \qquad \mu\in\mathscr{P}(E)$$

and

$$V(E) := \inf_{\mu\in\mathscr{P}(E)}\sup_{x\in\mathbf{P}^n(\mathbf{C})} V_\mu(x).$$

Define the projective logarithmic capacity of E by

$$C(E) := \frac{1}{V(E)}.$$

(This is a Frostman type capacity.) If $V(E)=+\infty$, we say that the set E is of projective logarithmic capacity zero. For an arbitrary subset K of $\mathbf{P}^n(\mathbf{C})$, we put

$$C(K) = \sup_{E\subset K} C(E),$$

where the supremum is taken over all the compact subset E of K. Note that there is a probability measure $\mu_0\in\mathscr{P}$ such that $V(E)=\sup_{x\in\mathbf{P}^n(\mathbf{C})} V_{\mu_0}(x)$.

2.2. Some results. A. Vitter [7] proved the following theorem:

Theorem A (Lemma on logarithmic derivatives). *Let $f=(f_0:f_1)$ be a reduced representation of a meromorphic mapping $f\colon \mathbf{C}^m\to\mathbf{P}^1(\mathbf{C})$. Set $F=f_1/f_0$. Then there exist positive constants a_1, a_2, a_3 such that*

$$\int_{\partial B(r)} \log^+\Big|\frac{F_{z_j}}{F}\Big|\sigma \le a_1 + a_2\log r + a_3\log T_f(r), \qquad j=1,\dots,m \ //.$$

Here the notation "$A(r)\le B(r)$ //" means that the inequality $A(r)\le B(r)$ holds for r outside a countable union of intervals I of finite Lebesgue measure.

Molzon–Shiffman–Sibony [3] proved the following result on the projective logarithmic capacity.

Theorem B ([3, p. 47]). *Let $\varphi\colon[0,1]\to\mathbf{P}^n(\mathbf{C})$ be a real smooth nondegenerate arc in $\mathbf{P}^n(\mathbf{C})$ and K a compact subset of $[0,1]$. Then the projective logarithmic capacity $C\big(\varphi(K)\big)$ is positive if and only if K has a positive logarithmic capacity in $\mathbf{C}$.*

Here "smooth nondegenerate arc φ" means that there exists a lift $\widetilde{\varphi}\colon[0,1]\to\mathbf{C}^{n+1}\setminus\{0\}$ such that the k-th derivatives $\{\widetilde{\varphi}^{(k)}(t)\}_{k\ge0}$ of $\widetilde{\varphi}(t)$ span $\mathbf{C}^{n+1}$ for every $t\in[0,1]$.

Furthermore, using a similar argument to Tsuji [6, p. 199, Theorem V.5], we have

Proposition 1. *Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that $\lim_{r\to+\infty} T_f(r) = +\infty$. Then there exist $r_1 < r_2 < \cdots < r_n \to +\infty$ and sets $E_n : E_{n+1} \subset E_n$, $n = 1, 2, \ldots$, in $\mathbf{P}^n(\mathbf{C})^*$ with*

$$V(E_n) \geq 2 \log T_f(r_n)$$

such that, if H does not belong to E_n,

$$m_f(r, H) \leq 4\sqrt{T_f(r)}\, \log T_f(r)$$

for $r > r_n$. Hence

$$\lim_{r\to+\infty} \frac{m_f(r, H)}{T_f(r)} = 0$$

outside a set $E \subset \mathbf{P}^n(\mathbf{C})^*$ of projective logarithmic capacity zero. Here $\mathbf{P}^n(\mathbf{C})^*$ denotes the dual projective space of $\mathbf{P}^n(\mathbf{C})$ which consists of all hyperplanes in $\mathbf{P}^n(\mathbf{C})$.

This is proved by a method similar to Tsuji [6], using the Frostman type capacity $C(E) = 1/V(E)$ and the de la Vallee Poussin type capacity

$$\widetilde{C}(E) := \sup_{\lambda\in\mathscr{M}} \left\{ \lambda(E) \mid U_\lambda(z) := \int_{w\in\mathbf{P}^n(\mathbf{C})} \log \frac{\|z\|\|w\|}{|\langle z, w\rangle|}\, d\lambda(w) \leq 1 \text{ for all } z \in E \right\},$$

where $\mathscr{M}$ denotes a set of Borel measures on E. Details are omitted here. The proof (Lemma B) of the inequality (7) in Tsuji [6, p. 199] for the projective logarithmic potential is the following. We include here the proofs of Lemma A and Lemma B, since Ninomiya's book is written in Japanese.

Lemma A (cf. Ninomiya [4]). *Let E be a Borel set in $\mathbf{P}^n(\mathbf{C})$. Then we have*

$$C(E) = \widetilde{C}(E).$$

Proof. For any $\varepsilon > 0$, we take a Borel measure $\mu \geq 0$ with $\mu(E) = \widetilde{C}(E) + \varepsilon$. Then the inequality

$$\sup_{z\in\mathbf{P}^n} \int_{w\in\mathbf{P}^n} \log \frac{\|z\|\|w\|}{|\langle z, w\rangle|}\, d\mu(w) > 1$$

holds, and there is a measure $\mu_0 \in \mathscr{P}(E)$ such that

$$V(E) = \sup_{z\in\mathbf{P}^n} \int_E \log \frac{\|z\|\|w\|}{|\langle z, w\rangle|}\, d\mu_0(w).$$

We put $\mu_1 = \big(\widetilde{C}(E)+\varepsilon\big)\mu_0$. Then we have

$$\sup_{z\in\mathbf{P}^n}\int_{w\in\mathbf{P}^n}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu_1(w) > 1,$$

or, equivalently,

$$\sup_{z\in\mathbf{P}^n}\int_{E}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu_0(w) > \frac{1}{\widetilde{C}(E)+\varepsilon}.$$

Hence we get $V(E) \geq 1/\big(\widetilde{C}(E)+\varepsilon\big)$, so $\widetilde{C}(E) \geq C(E)$. Next we shall show $\widetilde{C}(E) \leq C(E)$ under the assumption $\widetilde{C}(E) > 0$. For any $\varepsilon > 0$, there is a measure ν such that $\nu(E) = \widetilde{C}(E) - \varepsilon$ and

$$\sup_{z\in\mathbf{P}^n}\int_{w\in\mathbf{P}^n}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\nu(w) \leq 1.$$

Now we put $\mu_0 = \nu/\nu(E)$. Then we see $\mu_0(E) = 1$, so $\mu_0 \in \mathscr{P}(E)$, and we have

$$\begin{aligned}\frac{1}{C(E)} = V(E) &= \inf_{\mu\in\mathscr{P}}\sup_{z\in\mathbf{P}^n}\int_{w\in\mathbf{P}^n}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu(w)\\ &\leq \sup_{z\in\mathbf{P}^n}\int_{w\in\mathbf{P}^n}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu_0(w)\\ &\leq \sup_{z\in\mathbf{P}^n}\int_{w\in\mathbf{P}^n}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,\frac{d\nu(w)}{\nu(E)} \leq \frac{1}{\nu(E)} = \frac{1}{\widetilde{C}(E)-\varepsilon}.\end{aligned}$$

Thus we have $\widetilde{C}(E) - \varepsilon \leq C(E)$, and hence we have $\widetilde{C}(E) \leq C(E)$. Therefore we obtain $C(E) = \widetilde{C}(E)$. □

Lemma B (cf. Ninomiya [4]). *Let $E_1, E_2, \ldots, E_n, \ldots$ be Borel sets in $\mathbf{P}^n(\mathbf{C})$, and $E = \bigcup_{n=1}^{\infty} E_n$. Then*

$$\frac{1}{V(E)} \leq \frac{1}{V(E_1)} + \frac{1}{V(E_2)} + \cdots \frac{1}{V(E_n)} + \cdots.$$

Proof. For any $\varepsilon > 0$, there is a measure $\mu \in \mathscr{P}(E)$ such that

$$\sup_{z\in\mathbf{P}^n}\int_{E}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu(w) < V(E)+\varepsilon.$$

Put $\mu' = \mu/\big(V(E)+\varepsilon\big)$ (≥ 0). Then we have

$$\sup_{z\in\mathbf{P}^n}\int_{E}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu'(w) < 1.$$

Since $\log(\|z\|\|w\|/|\langle z,w\rangle|) \geq 0$, we have

$$\sup_{z\in\mathbf{P}^n}\int_{E_n}\log\frac{\|z\|\|w\|}{|\langle z,w\rangle|}\,d\mu'(w) < 1.$$

Hence $\mu'(E_n) \leq \widetilde{C}(E_n)$, and we obtain

$$\frac{1}{V(E)+\varepsilon} = \mu'(E) \leq \sum_{n=1}^{\infty}\mu'(E_n) \leq \sum_{n=1}^{\infty}\widetilde{C}(E_n) = \sum_{n=1}^{\infty}\frac{1}{V(E_n)}. \qquad \square$$

Proposition 2. *Consider the set*

$$\mathscr{A} = \left\{\left(1, a_1, \dots, a_n, a_1a_2, \dots, a_1a_n, a_2a_3, \dots, a_{n-1}a_n, a_1a_2a_3, \dots, \prod_{j=1}^{n} a_j\right) \;\middle|\; a_j \in \mathbf{C}\right\} \subset \mathbf{P}^N(\mathbf{C})^*, \tag{1}$$

where $N = 2^n - 1$. *After some rearrangement it contains vectors of the form* $(1, \alpha, \alpha^2, \dots, \alpha^N)$, $\alpha \in \mathbf{C}$.

Proof. Let $a_1 = \alpha^{k_1}$, $a_2 = \alpha^{k_2}$, $\dots$, $a_l = \alpha^{k_l}$, $\dots$, $a_n = \alpha^{k_n}$, where

$$k_1 = 1, \qquad k_m = \sum_{l=1}^{m-1} k_l + 1, \qquad m = 2, 3, \dots, n.$$

We shall prove this by induction. In the case where $n = 2$, we have $a_1 = \alpha$, $a_2 = \alpha^2$, so $a_1a_2 = \alpha^3$. Hence we obtain $(1, a_1, a_2, a_1a_2) = (1, \alpha, \alpha^2, \alpha^3) \in \mathbf{C}^{2^2} \setminus \{0\}$, so this case is proved. Supposing that it was proved until n, we shall prove the case where $n+1$. We set $a_{n+1} = \alpha^{k_{n+1}}$, where $k_{n+1} = \sum_{l=1}^{n} k_l + 1$. Then we observe that $k_{n+1} = 2^n$, since

$$k_{n+1} = \sum_{l=1}^{n} k_l + 1 = \sum_{l=1}^{n-1} k_l + k_n + 1 = 2k_n = 2\cdot 2^{n-1} = 2^n$$

when $k_n = 2^{n-1}$. By the assumption of induction, we can get $1, \alpha, \alpha^2, \dots, \alpha^{2^n-1}$ by using $a_1, \dots, a_n, a_1a_2, \dots, \prod_{j=1}^{n} a_j$. Hence we can get $\alpha^{2^n}, \alpha^{2^n+1}, \dots, \alpha^{2^{n+1}-1}$ by using $a_1a_{n+1}, \dots, a_na_{n+1}, a_1a_2a_{n+1}, \dots, \prod_{j=1}^{n} a_j a_{n+1}$. $\square$

3. Elimination of defects of meromorphic mappings

For a meromorphic mapping g of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, we can eliminate all defects by a small deformation of g. We say that meromorphic mapping g is transcendental if

$$\lim_{r\to+\infty}\frac{T_g(r)}{\log r}=+\infty.$$

Note that a meromorphic mapping g is rational if and only if

$$T_g(r)=O(\log r),\qquad r\to+\infty.$$

Lemma 1. *There are monomials $\zeta_1,\dots,\zeta_n$ in $z_1,\dots,z_m$ such that any n derivatives in $\{D^\alpha\zeta:=(D^\alpha\zeta_1,\dots,D^\alpha\zeta_n)\mid|\alpha|\le n+1\}$ are linearly independent over the field M of meromorphic functions on $\mathbf{C}^m$, where $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbf{Z}_{\ge0}$ is a multi-index and $D^\alpha\zeta_k=\partial^{|\alpha|}\zeta_k/\partial {z_1}^{\alpha_1}\cdots\partial {z_m}^{\alpha_m}$.*

Proof. We now take positive integers $k_1,\dots,k_{nm}$, inductively. First, we take some positive integer $k_1>n$, and then an integer k_j as large as

$$k_j>\prod_{r=1}^{j-1}(k_r!),\qquad j=2,\dots,nm.$$

We now put

$$\zeta=(\zeta_1,\zeta_2,\dots,\zeta_n)=(z_1^{k_1}\cdots z_m^{k_m},z_1^{k_{m+1}}\cdots z_m^{k_{2m}},\dots,z_1^{k_{(n-1)m+1}}\cdots z_m^{k_{nm}}).$$

Then, for a multi-index $\alpha=(\alpha_1,\dots,\alpha_m)$ with $|\alpha|\le n+1$, we have

$$\begin{aligned}D^\alpha(\zeta_s)&=D^\alpha(z_1^{k_{(s-1)m+1}}\cdots z_m^{k_{sm}})\\&=\frac{k_{(s-1)m+1}!}{(k_{(s-1)m+1}-\alpha_1)!}\cdots\frac{k_{sm}!}{(k_{sm}-\alpha_1)!}z_1^{k_{(s-1)m+1}-\alpha_1}\cdots z_m^{k_{sm}-\alpha_m}\\&\quad(=A_{s,\alpha}z_1^{k_{(s-1)m+1}-\alpha_1}\cdots z_m^{k_{sm}-\alpha_m},\ \text{say.})\end{aligned}$$

Hence we can write

$$D^\alpha\zeta=\phi_\alpha(A_{1,\alpha}\xi_1,\dots,A_{n,\alpha}\xi_n),$$

where $\phi_\alpha\in M$ and ξ_i is some monomial in $z_1,\dots,z_m$. Then for any n multi-indices $\alpha^{j_1},\dots,\alpha^{j_n}$, we observe that

$$\begin{vmatrix}A_{1,\alpha^{j_1}}&\dots&A_{n,\alpha^{j_1}}\\A_{1,\alpha^{j_2}}&\dots&A_{n,\alpha^{j_2}}\\\vdots&\ddots&\vdots\\A_{1,\alpha^{j_n}}&\dots&A_{n,\alpha^{j_n}}\end{vmatrix}\ne0.$$

Thus any n derivatives in $\{D^\alpha\zeta\mid|\alpha|\le n+1\}$ are linearly independent over the field M of meromorphic functions on $\mathbf{C}^m$. □

We note that we can take $k_1,\dots,k_{nm}$ arbitrarily large.

Lemma 2. *Let $h = (h_0 : h_1 : \cdots : h_n)$ be a reduced representation of a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and $\zeta_1, \ldots, \zeta_n$ linearly independent monomials in $z_1, \ldots, z_m$ as in Lemma 1. Then there exist $(\tilde{a}_1, \ldots, \tilde{a}_n)$ such that $\tilde{a}_j = \alpha^{k_j}$, $j = 1, \ldots, n$, with $k_1 = 1$, $k_m = \sum_{l=1}^{m-1} k_l + 1$, $m = 2, 3, \ldots, n$, $\alpha \in \mathbf{C}$, and*

$$f := (h_0 : h_1 + \tilde{a}_1\zeta_1 h_0 : h_2 + \tilde{a}_2\zeta_2 h_0 : \cdots : h_n + \tilde{a}_n\zeta_n h_0)$$

is a reduced representation of a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$.

Proof. Let

$$\begin{aligned}\Phi(c_j; a_j) &\equiv \Phi(c_0, c_1, \ldots, c_n; a_1, \ldots, a_n)\\ &:= c_0 h_0 + c_1(h_1 + a_1\zeta_1 h_0) + c_2(h_2 + a_2\zeta_2 h_0) + \cdots + c_n(h_n + a_n\zeta_n h_0)\\ &= (c_0 + a_1 c_1\zeta_1 + a_2 c_2\zeta_2 + \cdots + a_n c_n\zeta_n)h_0 + c_1 h_1 + \cdots + c_n h_n.\end{aligned}$$

Suppose that for $a_1, \ldots, a_n$, there are $c_0, \ldots, c_n$ such that $\Phi(c_j; a_j) \equiv 0$. Then

$$c_0 h_0 + c_1 h_1 + \cdots + c_n h_n = -(c_1 a_1\zeta_1 h_0 + c_2 a_2\zeta_2 h_0 + \cdots + c_n a_n\zeta_n h_0). \tag{2}$$

Now we expand h_j, $j = 0, \ldots, n$, to the Taylor series at the origin, and especially we write

$$h_0 = \sum_{|\beta|=0}^{\infty} d_\beta z_1^{\beta_1} \cdots z_m^{\beta_m} = \sum_{|\beta|=0}^{\infty} d_\beta z^\beta,$$

where $\beta = (\beta_1, \ldots, \beta_m)$ is a multi-index. We write β^0 as an index of one of the lowest terms of h_0. Then all $z^{\beta^0}, \zeta_1 z^{\beta^0}, \ldots, \zeta_n z^{\beta^0}$ are different from each other. Now we compare the coefficients of both sides of (2) on each $\zeta_i z^{\beta^0}$, $i = 1, \ldots, n$, and z^{β^0}. Then we have

$$\begin{cases} c_0 d_n^0 + c_1 d_n^1 + \cdots + c_n d_n^n = -(c_1 a_1 e_n^1 + \cdots + c_n a_n e_n^n),\\ c_0 d_{n-1}^0 + c_1 d_{n-1}^1 + \cdots + c_n d_{n-1}^n = -(c_1 a_1 e_{n-1}^1 + \cdots + c_{n-1} a_{n-1} e_{n-1}^{n-1}),\\ \cdots\\ c_0 d_1^0 + c_1 d_1^1 + \cdots + c_n d_1^n = -c_1 a_1 e_1^1,\\ c_0 d_0^0 + c_1 d_0^1 + \cdots + c_n d_0^n = 0.\end{cases}$$

Here d_i^j, e_i^j are determined by $h_0, \ldots, h_n$, $\zeta_1, \ldots, \zeta_n$. Hence we have

$$\begin{cases} c_0 d_n^0 + c_1(d_n^1 - a_1 e_n^1) + \cdots + c_n(d_n^n - a_n e_n^n) = 0,\\ c_0 d_{n-1}^0 + c_1(d_{n-1}^1 - a_1 e_{n-1}^1) + \cdots + c_{n-1}(d_{n-1}^{n-1} - a_{n-1} e_{n-1}^{n-1}) + c_n d_{n-1}^n = 0,\\ \cdots\\ c_0 d_1^0 + c_1(d_1^1 - a_1 e_1^1) + c_2 d_1^2 + \cdots + c_n d_1^n = 0,\\ c_0 d_0^0 + c_1 d_0^1 + \cdots + c_n d_0^n = 0,\end{cases} \tag{3}$$

where $d_0^0 = d_{\beta^0} \neq 0$. Then, if the homogeneous equations (3) have nontrivial solutions $c_0, \ldots, c_n$, the determinant of the matrix consisting of these coefficients of (3) must be identically zero. Namely,

$$(4) \qquad \begin{vmatrix} d_n^0 & d_n^1 - a_1 e_n^1 & * & \cdots & * & d_n^n - a_n e_n^n \\ d_{n-1}^0 & d_{n-1}^1 - a_1 e_{n-1}^1 & * & \cdots & d_{n-1}^{n-1} - a_{n-1} e_{n-1}^{n-1} & d_{n-1}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_1^0 & d_1^1 - a_1 e_1^1 & d_1^2 & \cdots & * & d_1^n \\ d_0^0 & d_0^1 & * & \cdots & * & d_0^n \end{vmatrix} = 0.$$

Here,

$$(d_n^n - a_n e_n^n)(d_{n-1}^{n-1} - a_{n-1} e_{n-1}^{n-1}) \cdots (d_1^1 - a_1 e_1^1) d_0^0 = d_{\beta^0}^{n+1} a_1 \cdots a_n + \cdots + \prod_{i=0}^{n} d_i^i$$

and $e_i^i = d_{\beta^0} = d_0^0 \neq 0$. Then the term including $a_1 a_2 \cdots a_n$ in (4) appears only in the above term, so (4) $\not\equiv 0$ as a function of $a_1, \ldots, a_n$. If we take $a_1 = a$, $a_l = a^{k_l}$, where $k_l = \sum_{j=1}^{l-1} k_j + 1$, $l = 2, \ldots, n$, then (4) is a polynomial in a of degree $\sum_{l=1}^{n} k_l + 1 = 2^n - 1$ ($\neq$ constant). Hence (4) has at most $2^n - 1$ solutions. Therefore, if we change $a \in \mathbf{C}$ continuously, there exists an $\tilde{a} \in \mathbf{C}$ such that (4) does not equal zero, that is, (4) does not have a nontrivial solution. Thus we can choose $\tilde{a}_1, \ldots, \tilde{a}_n$ such that $h_0, h_1 + \tilde{a}_1 \zeta_1 h_0, \ldots, h_n + \tilde{a}_n \zeta_n h_0$ are linearly independent over $\mathbf{C}$, so f is linearly nondegenerate. □

Lemma 3. *Let $f = (f_0 : \cdots : f_n)$ and $h = (h_0 : \cdots, h_n)$ be as in Lemma 2. Then we have*

$$|T_f(r) - T_h(r)| \leq O(\log r), \qquad r \to \infty.$$

Proof.

$$\begin{aligned} \log \|f\| &= \log\Big(|h_0| + \sum_{k=1}^{n} |h_k + a_k \zeta_k h_0| \Big) + O(1) \\ &\leq \log\Big(|h_0| + \sum_{k=1}^{n} (|h_k| + |a_k \zeta_k|\,|h_0|) \Big) + O(1) \\ &\leq \log\Big\{ \Big(1 + \sum_{k=1}^{n} (1 + |a_k \zeta_k|) \Big) (|h_0| + |h_k|) \Big\} + O(1) \\ &\leq \log\Big\{ \Big(1 + \sum_{k=1}^{n} (1 + |a_k \zeta_k|) \Big) \|h\| \Big\} + O(1) \\ &= \log 2 \Big(1 + \sum_{k=1}^{n} |a_k \zeta_k| \Big) + \log \|h\| + O(1). \end{aligned}$$

Hence we have

$$\begin{aligned}
T_f(r) &= \int_{\partial B(r)} \log\|f\|\sigma \\
&\le \int_{\partial B(r)} \log\|h\|\sigma + \int_{\partial B(r)} \log 2\left(1+\sum_{k=1}^{n}|a_k\zeta_k|\right)\sigma + O(1) \\
&= T_h(r) + O(\log r), \qquad r\to\infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\log\|h\| &= \log\sum_{k=0}^{n}|h_k| + O(1) \\
&\le \log\left(|h_0| + \sum_{k=1}^{n}(|h_k + a_k\zeta_k h_0| + |a_k\zeta_k h_0|)\right) + O(1) \\
&\le \log\left(\sum_{k=0}^{n}|f_k| + \sum_{k=1}^{n}|a_k\zeta_k|\,|h_0|\right) + O(1) \\
&\le \log\left(\|f\| + \sum_{k=1}^{n}|a_k\zeta_k|\,\|f\|\right) + O(1) \\
&\le \log\left\{\left(1+\sum_{k=1}^{n}|a_k\zeta_k|\right)(\|f\|)\right\} + O(1) \\
&\le \log\|f\| + \log\left(1+\sum_{k=1}^{n}|a_k\zeta_k|\right) + O(1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
T_h(r) &\le \int_{\partial B(r)} \log\|f\|\sigma + \int_{\partial B(r)} \log\left(1+\sum_{k=1}^{n}|a_k\zeta_k|\right)\sigma + O(1) \\
&= T_f(r) + O(\log r), \qquad r\to\infty. \ \square
\end{aligned}$$

We shall prove the main theorem.

Theorem. *Let $g\colon \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a given transcendental meromorphic mapping. Then there exists a regular matrix $L = (l_{ij})_{0\le i,j\le n}$ of the form $l_{ij} = c_{ij}\zeta_i + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbf{C} : 0 \le i,j \le n)$, such that $\det L \neq 0$ and $f := L\cdot g\colon \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is a meromorphic mapping without Nevanlinna deficient hyperplanes, where $\zeta_1,\ldots,\zeta_n$ are monomials in $z_1,\ldots,z_m$ and linearly independent over $\mathbf{C}$.*

Here the mapping $f := L \cdot g\colon \mathbf{C}^m \to \mathbf{P}^n(C)$ means a product of the matrix $L = (l_{ij})$ and a vector of a reduced representation $\tilde{g} =^t (g_0 : \ldots : g_n)$ of g which does not depend on the choice of $\tilde{g}$, and a Nevanlinna deficient hyperplane H for f means a hyperplane with $\delta_f(H) > 0$.

Remark 1. For the mappings as in the theorem, the inequality $|T_f(r) - T_g(r)| \le O(\log r)$, $r \to +\infty$ holds, and also the mapping g may be linearly degenerate or of infinite order.

Remark 2. A rational mapping g always has a Nevanlinna defect hyperplane if $m = 1$ or there is a regular linear change L_0 such that $L_0 \cdot g$ has a reduced representation which consists of polynomials including different degrees. But otherwise g does not have Nevanlinna deficient hyperplanes.

Remark 3. If g is of finite order, we can replace "Nevanlinna deficiency" with "Valiron deficiency" in the conclusion of the theorem.

Remark 4. If $m = 1$, we can take $\zeta_k = z^k$, $k = 1, \ldots, n$.

Proof of the theorem. There is a regular linear change L_1 such that

$$h := L_1 \cdot g = (h_0 : \cdots : h_n)\colon \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$$

is a reduced representation of the meromorphic mapping h and

$$N\bigl(r, (h_j)\bigr) = \bigl(1 - o(1)\bigr) T_h(r), \qquad r \to +\infty,\ j = 0, 1, \ldots, n. \tag{5}$$

Now we choose $\zeta_1, \ldots, \zeta_n$ as in Lemma 1. (We take larger gap powers if necessary.) Set

$$\mathscr{A} = \biggl\{ \biggl(1, a_1, \ldots, a_n, a_1 a_2, \ldots, a_1 a_n, \ldots, \prod_{j=1}^{n} a_j \biggr) \mid a_j \in \mathbf{C} \biggr\}.$$

Then the set $\mathscr{A}$ from Proposition 2 contains $E = \{(1, \alpha, \alpha^2, \ldots, \alpha^{2^n - 1}) \mid \alpha \in \mathbf{C}\}$. Hence $\mathscr{A}$ contains a nondegenerate smooth arc $\sigma([0,1])$ of $\sigma\colon [0,1] \to \mathbf{P}^N(\mathbf{C})^*$. Therefore, by Theorem B, $\mathscr{A}$ is of positive projective logarithmic capacity. Hence by Lemma 2 the set of vectors $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{C}^n \setminus \{0\}$ such that $f^{\mathbf{a}} = (h_0, h_1 + a_1 \zeta_1 h_0, \cdots, h_n + a_n \zeta_n h_0)$ is linearly nondegenerate, has a positive projective logarithmic capacity. On the other hand, the number of couples of $n+1$ multi-indices $\alpha^{j_0}, \ldots, \alpha^{j_n}$ with $|\alpha^{j_k}| \le n + 1$ is at most finite. Hence, there are multi-indices $\beta^{j_0}, \ldots, \beta^{j_n}$ and a subset $\mathscr{B}$ of $\mathscr{A}$ with a positive projective logarithmic capacity such that for any vector $\mathbf{a} \in \mathscr{B}$, $\{D^{\beta^{j_0}} f^{\mathbf{a}}, \ldots, D^{\beta^{j_n}} f^{\mathbf{a}}\}$ is an admissible basis. We write these multi-indices $\beta^0, \ldots, \beta^n$ instead of $\beta^{j_0}, \ldots, \beta^{j_n}$. Then for these multi-indices $\beta^0, \ldots, \beta^n$ we consider the generalized Wronskian of

$f := f^{\mathbf{a}}, (\mathbf{a} = (a_1, \dots, a_n) \in \mathscr{B})$. Thus we can write

$$\begin{aligned}\mathbf{W}_\beta &:= W_{\beta^0,\dots,\beta^n}(f) = W_{\beta^0,\dots,\beta^n}(h_0, h_1 + a_1\zeta_1 h_0, \dots, h_n + a_n\zeta_n h_0)\\ &= h_0^{n+1} W_{\beta^0,\dots,\beta^n}\left(1, \frac{h_1}{h_0} + a_1\zeta_1, \dots, \frac{h_n}{h_0} + a_n\zeta_n\right)\\ &= h_0^{n+1} W_{\beta^0,\dots,\beta^n}(1, \mathbf{h}_1 + a_1\zeta_1, \dots, \mathbf{h}_n + a_n\zeta_n)\\ &= h_0^{n+1}\left(W_0 + a_1 W_1 + \cdots + \prod_{i=1}^n a_i W_N\right) \not\equiv 0,\end{aligned}$$

where $\mathbf{h_j} = h_j/h_0$, $j = 1, \dots, n$, and W_k is a generalized Wronskian of some $1, \mathbf{h}_1, a_1\zeta_1, \dots, \mathbf{h}_n, a_n\zeta_n$, $0 \le k \le N = 2^n - 1$, by the multi-linearity of a determinant. Next we consider the meromorphic mapping F of the form

$$F := (W_0/d : W_1/d : \cdots : W_N/d)\colon \mathbf{C}^m \to \mathbf{P}^N(\mathbf{C}),$$

where $d = d(z)$ is a meromorphic function consisting of common factors among $W_0, \dots, W_N$ such that $W_0/d, \dots, W_N/d$ are holomorphic functions without common factors up to unit. This $d(z)$ exists, because $\mathbf{C}^m$ is a Cousin domain. Then there is an i_0 such that $\mathbf{h}_{i_0}$ is transcendental and $W_{\beta^0,\dots,\beta^n}(\zeta_1, \dots, \mathbf{h}_{i_0}, \dots, \zeta_n)$ is not identically zero, since h is transcendental. Hence it is not proportionate to $W_N = W_{\beta^0,\dots,\beta^n}(\zeta_1, \dots, \zeta_n) \not\equiv 0$. Thus the meromorphic mapping F is not constant. Therefore there exists an $\mathbf{a}_0 = \left(1, \tilde a_1, \dots, \tilde a_n, \tilde a_1\tilde a_2, \dots, \prod_{j=1}^n \tilde a_j\right) \in \mathscr{B}$ such that

$$\limsup_{r\to\infty} \frac{m_F(r, H_{\mathbf{a}_0})}{T_F(r)} = 0, \tag{6}$$

since the set of Valiron deficient hyperplanes of a nonconstant meromorphic mapping is of projective logarithmic capacity zero, that is,

$$N_F\big(r, (\langle F, \mathbf{a}_0\rangle)\big) = N_F(r, H_{\mathbf{a}_0}) = \big(1 - o(1)\big) T_F(r), \qquad r \to +\infty,$$

where

$$H_{\mathbf{a}_0} = \{\xi = (\xi_0, \dots, \xi_N) \mid \langle \xi, \mathbf{a}_0\rangle = 0\}$$

and

$$\langle F, \mathbf{a}_0\rangle = \left\{W_0 + \tilde a_1 W_1 + \cdots + \prod_{k=1}^n \tilde a_k W_N\right\}/d.$$

For this $\tilde a_1, \dots, \tilde a_n$, we consider the meromorphic mapping given by the following reduced representation:

$$f := L_2 \cdot h = (f_0 : \cdots : f_n)\colon \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C}),$$

where

$$L_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tilde{a}_1\zeta_1 & 1 & \cdots & 0 \\ \tilde{a}_2\zeta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_n\zeta_n & 0 & \cdots & 1 \end{pmatrix}, \qquad \det L_2 = 1 \neq 0;$$

hence

$$f_0 = h_0 \qquad \text{and} \qquad f_k = h_k + \tilde{a}_k\zeta_k h_0, \qquad k = 1, \dots, n.$$

Then from Lemma 3 we obtain

$$T_f(r) = T_g(r) + O(\log r) = \big(1 + o(1)\big)T_g(r), \qquad r \to +\infty,$$

if g is not rational.

Lemma 4. *Let F and f be as above. Then there exists a positive number K such that*

$$T_F(r) \le KT_f(r).$$

Proof.

$$\begin{aligned} T_F(r) &= \int_{\partial B(r)} \log(|W_0| + |W_1| + \cdots + |W_N|)\frac{1}{|d|}\sigma \\ &= \int_{\partial B(r)} \log(|W_0| + |W_1| + \cdots + |W_N|)\sigma - \int_{\partial B(r)} \log|d|\sigma \\ &\le \sum_{j=1}^{N} \int_{\partial B(r)} \log^+ |W_j|\sigma + K_1 N\big(r, (h_0)\big) + O(1) \\ &\le \sum_{j=1}^{N} \bigg\{ \int_{\partial B(r)} \log^+ \frac{|W_j|}{|\mathbf{h}_{j_1}\cdots \mathbf{h}_{j_s}\zeta_{j_{s+1}}\cdots\zeta_{j_n}|}\sigma \\ &\quad + \int_{\partial B(r)} \log^+ |\mathbf{h}_{j_1}\cdots \mathbf{h}_{j_s}\zeta_{j_{s+1}}\cdots\zeta_{j_n}|\sigma \bigg\} + K_1 N\big(r, (h_0)\big) + O(1) \\ &= o\big(T_h(r)\big) + K_2 T_h(r) + O(1) \le KT_f(r) \end{aligned}$$

for some positive constants $K \ge K_2 \ge K_1 > 0$, using Theorem A. □

We now continue the proof of the theorem. We take an arbitrary vector $\mathbf{b} = (b_0, \dots, b_n) \in \mathbf{C}^{n+1} \setminus \{0\}$, which determines the hyperplane $H = \{w \in \mathbf{C}^{n+1} \setminus \{0\} \mid \langle w, \mathbf{b}\rangle = 0\}$ in $\mathbf{P}^n(\mathbf{C})$. We may assume $b_n \neq 0$. Set

$$A = \langle f, \mathbf{b}\rangle = \sum_{k=0}^{n} b_k f_k.$$

We note that

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ b_1 & b_2 & \cdots & b_{n-1} & b_n \end{vmatrix} = b_n \neq 0;$$

hence we observe that $f_0, f_1, \ldots, f_{n-1}, A$ are linearly independent over $\mathbf{C}$. Thus we have

$$\begin{aligned}
m_f(r, H_{\mathbf{b}}) &= \int_{\partial B(r)} \log \frac{\|f\|}{|A|}\sigma \\
&= \int_{\partial B(r)} \log \frac{|W_{\beta^0,\ldots,\beta^n}(f_0,\ldots,f_n)|}{|A|\,|f_0|\cdots|f_{n-1}|}\sigma \\
&\quad + \int_{\partial B(r)} \log \frac{\|f\|\,|f_0|\cdots|f_{n-1}|}{|W_{\beta^0,\ldots,\beta^n}(f_0,\ldots,f_n)|}\sigma \\
&= \int_{\partial B(r)} \log \frac{|b_n^{-1}|\,|W_{\beta^0,\ldots,\beta^n}(f_0,\ldots,f_{n-1},A)|}{|A|\,|f_0|\cdots|f_{n-1}|}\sigma \\
&\quad + \int_{\partial B(r)} \log \frac{\|f\|\,|f_0|\cdots|f_{n-1}|}{|f_0|^{n+1}}\sigma \\
&\quad + \int_{\partial B(r)} \log \frac{|f_0|^{n+1}}{|W_{\beta^0,\ldots,\beta^n}(f_0,\ldots,f_n)|}\sigma + O(1), \\
&\leq \int_{\partial B(r)} \log^+ \frac{|W_{\beta^0,\ldots,\beta^n}(f_0,\ldots,f_{n-1},A)|}{|A|\,|f_0|\cdots|f_{n-1}|}\sigma + \int_{\partial B(r)} \log \frac{\|f\|^{n+1}}{|f_0|^{n+1}}\sigma \\
&\quad + \int_{\partial B(r)} \log \frac{1}{\left|W_0 + a_1 W_1 + \cdots + \prod_{j=1}^n a_j W_N\right|}\sigma + O(1) \\
&\leq o\big(T_f(r)\big) + (n+1) m_f(r, H_{(1,0,\ldots,0)}) \\
&\quad + \int_{\partial B(r)} \log \frac{1}{\left|W_0 + a_1 W_1 + \cdots + \prod_{j=1}^n a_j W_N\right|}\sigma \\
&\quad + \int_{\partial B(r)} \log(|W_0| + |W_1| + \cdots + |W_N|)\sigma + O(1) \\
&\leq o\big(T_f(r)\big) + (n+1) m_f(r, H_{(1,0,\ldots,0)}) \\
&\quad + \int_{\partial B(r)} \log \frac{(|W_0| + |W_1| + \cdots + |W_N|)(1/|d|)}{\left|W_0 + a_1 W_1 + \cdots + \prod_{j=1}^n a_j W_N\right|(1/|d|)}\sigma + O(1) \\
&= o\big(T_f(r)\big) + o\big(T_f(r)\big) + \int_{\partial B(r)} \log \frac{\|F\|}{|\langle F, \mathbf{a}_0\rangle|}\sigma + O(1) \qquad \text{(by (5))} \\
&= o\big(T_f(r)\big) + o\big(T_F(r)\big) = o\big(T_f(r)\big), \quad // \qquad \text{(by (6))},
\end{aligned}$$

since $T_F(r) \leq KT_f(r)$, for some $K > 0$ by Lemma 4, and using Theorem A. Therefore we obtain

$$\delta_f(H_{\mathbf{b}}) = \liminf_{r \to +\infty} \frac{m_f(r, H_{\mathbf{b}})}{T_f(r)} = 0,$$

that is, $\delta_f(H) = 0$ for any $H \in \mathbf{P}^n(\mathbf{C})^*$. This completes the proof of the theorem. □

Problem 1. Is the conclusion of the theorem true if "Nevanlinna deficiency" is replaced with "Valiron deficiency"?

It seems to me that the set N of meromorphic mappings with Nevanlinna defects is small in the space M of all meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$.

Problem 2. Something can probably be said in terms of Baire categories. What if one considers plurisubharmonic functions in M? Is N then pluripolar?

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