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# THE OUTSIDE OF THE TEICHMÜLLER SPACE OF PUNCTURED TORI IN MASKIT'S EMBEDDING

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**Abstract.** We show that for each cusp on the boundary of Maskit's embedding $\mathscr{M} \subset \mathbf{H}$ of the Teichmüller space of punctured tori there is a sequence of parameters in the complement of $\overline{\mathscr{M}}$ converging to the cusp such that the parameters correspond to discrete groups with elliptic elements. Using Tukia's version of Marden's isomorphism theorem we identify them as cusps on the boundary of certain deformation spaces of Koebe groups with a non-simply connected invariant component.

## 1. Introduction

A *Kleinian group* $G$ is a discrete group of Möbius transformations, that is, of conformal bijections of the extended complex plane $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We use matrix notation for Möbius transformations, identifying the transformation

$$(1.1) \qquad z \mapsto \frac{az+b}{cz+d}$$

with the matrix

$$(1.2) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{C}).$$

The extended complex plane is divided into two disjoint sets according to the type of the action of $G$ on these sets: the *set of discontinuity*, or the *ordinary set* $\Omega(G)$ where the group $G$ acts discontinuously, and the *limit set* $\Lambda(G) = \widehat{\mathbf{C}} \setminus \Omega(G)$. For material on Kleinian groups, we refer to Maskit [21].

Let $G$ be a Kleinian group. If $X \subset \widehat{\mathbf{C}}$, then

$$(1.3) \qquad \mathrm{Stab}(X) = \mathrm{Stab}_G(X) = \{g \in G : g(X) = X\}.$$

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is the *stabilizer* of $X$ in $G$. A component of the ordinary set such that $\mathrm{Stab}_G(\Omega_0) = G$ is called an *invariant component* of $G$. $G$ is a *Fuchsian group* of the first kind if $\Omega(G)$ consists of two $G$-invariant round disks. $G$ is called a *terminal b-group* of type $(g,n)$ if it has a simply connected invariant component $\Omega_0(G) \subset \Omega(G)$ such that $\Omega_0/G$ is a Riemann surface of genus $g$ with $n$ punctures, and if $(\Omega \setminus \Omega_0)/G$ is a collection of $2g-2+n$ thrice punctured spheres.

The (*quasiconformal*) *deformation space* or *Teichmüller space* $\mathscr{T}(G)$ of a Kleinian group $G$ is

(1.4)
$$\mathscr{T}(G) = \bigl\{w\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}} \text{ quasiconformal} : w \circ g \circ w^{-1} \in \mathrm{PSL}(2,\mathbf{C}) \text{ for all } g \in G\bigr\}/\sim,$$

where $w_1 \sim w_2$ if there is a Möbius transformation $A \in \mathrm{PSL}(2,\mathbf{C})$ such that

$$w_1 \circ g \circ w_1^{-1} = A \circ w_2 \circ g \circ w_2^{-1} \circ A^{-1} \qquad \text{for all } g \in G. \tag{1.5}$$

If $G$ is finitely generated, then $\mathscr{T}(G)$ is a finite dimensional complex manifold. If $G$ is a torsion-free terminal $b$-group such that the quotient Riemann surface $\Omega_0(G)/G$ has genus $g$ and $n$ punctures, then $\mathscr{T}(G)$ is naturally isomorphic to $\mathscr{T}(g,n)$, the Teichmüller space of Riemann surfaces with finite area, genus $g$, and $n$ punctures. The complex dimension of this space is $3g-3+n$. See [2], [14], [17] and the references given there for more details on these spaces and the identifications between the various related spaces.

The *Maskit embedding* [19], [15] of the Teichmüller space $\mathscr{T}(g,n)$ is defined by first identifying $\mathscr{T}(g,n)$ with $\mathscr{T}(G)$, where $G$ is a torsion-free terminal $b$-group as above. $\mathscr{T}(G)$ is then analytically embedded in $\mathbf{C}^{3g-3+n}$ using traces and/or fixed points of a suitable collection of group elements.

In this paper we study the complex one-dimensional case of once-punctured tori, $(g,n) = (1,1)$. This is essentially the only one-dimensional case; $\mathscr{T}(0,4)$ is canonically biholomorphic to $\mathscr{T}(1,1)$. In fact, the conformal map identifying $\mathscr{T}(1,1)$ with $\mathscr{T}(0,4)$ has a particularly simple form if one uses Maskit's embedding for both spaces. For more details on this we refer to Kra [15, Section 6].

Maskit's embedding of $\mathscr{T}(1,1)$ can be represented as the set $\mathscr{M}$ of parameters $\mu \in \mathbf{H}$ for which the group $G[\mu]$ generated by two transformations

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad S(z) = z + 2, \tag{1.6}$$

and

$$T[\mu] = \begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix} \qquad \text{or} \qquad T[\mu](z) = \frac{1}{z} + \mu, \tag{1.7}$$

is a terminal $b$-group representing a punctured torus on its invariant component $\Omega_0(G)$. Let

$$\widetilde{S} = T[\mu]^{-1} S^{-1} T[\mu] = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{1.8}$$

From (1.6)–(1.8) we see that for any $\mu \in \mathbf{C}$, $G[\mu]$ contains a discrete Fuchsian subgroup

$$\Gamma_0 = \langle S, \widetilde{S} \rangle, \tag{1.9}$$

the level 2 principal congruence subgroup of $\mathrm{PSL}(2,\mathbf{Z})$, that keeps both $\mathbf{H}$ and $\mathbf{H}^*$ fixed. Moreover, $\Lambda(\Gamma_0) = \mathbf{R} \cup \{\infty\}$. For parameters $\mu \in \mathscr{M}$ it is useful to think of $G = G[\mu]$ as the HNN extension of $\langle S, \widetilde{S} \rangle$ by $T[\mu]$. It follows from Maskit's second combination theorem, [21, Theorem VII.E.5], that this is the case for all parameters $\mu$ such that $\operatorname{Im} \mu > 2$, implying $\mathbf{H} + 2i \subset \mathscr{M}$.

Let $\Omega_0(G[\mu]) = \Omega_0[\mu]$ be the invariant component of $G[\mu]$. The quotient surface $\Omega(G[\mu])/G[\mu]$ consists of the disjoint union of a punctured torus $\Omega_0(G[\mu])/G[\mu]$ and a thrice punctured sphere. The punctures of the thrice punctured sphere $\mathbf{H}^*/\operatorname{Stab}(\mathbf{H}^*)$ correspond to the three parabolic conjugacy classes of $S$, $\widetilde{S}$ and

$$K = \widetilde{S}S = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \tag{1.10}$$

in $\langle S, \widetilde{S} \rangle$. Note that for $\mu \in \mathscr{M}$ there are only two parabolic conjugacy classes in $G[\mu]$: $S$ and $\widetilde{S}$ are conjugate in $G[\mu]$ by (1.8), and all parabolic elements of $G[\mu]$ are conjugate to parabolics of $\langle S, \widetilde{S} \rangle$ by Maskit's second combination theorem, [21, Theorem VII.E.5]. The parabolic transformation $K = \widetilde{S}S$ corresponds to the puncture on the torus component of the quotient Riemann surface.

The set $\mathscr{M}$ and its boundary have been studied in detail by Keen and Series in [10] and by Wright in the unpublished manuscript [31]. In this note we consider discrete groups generated by $S$ and $T[\mu]$ for parameters $\mu \in \mathbf{C} \setminus \overline{\mathscr{M}}$. We can restrict our attention to the case $\mu \in \overline{\mathbf{H}} \setminus \mathscr{M}$, as a simple calculation shows that for all $\mu \in \mathbf{C}$

$$G[\mu] = EG[-\mu]E, \tag{1.11}$$

for $E(z) = -z$, $E^2 = 1$. In other words, $G[-\mu]$ is discrete if and only if $G[\mu]$ is discrete.

We consider the following questions:

(1) *For which parameters $\mu \in \mathbf{C} \setminus \overline{\mathscr{M}}$ is the group $G[\mu]$ discrete?*
(2) *If $G[\mu]$ is discrete, what is the geometry of the quotient spaces $\Omega(G[\mu])/G[\mu]$ and $\mathbf{H}^3/G[\mu]$?*

If $\mu \in \mathscr{M}$, the geometry of the quotient space $\mathbf{H}^3/G[\mu]$ is well understood. The boundary of the convex core (See Section 5 for the terminology used here) is a pleated surface in the sense of Thurston [29] with two components, a totally geodesic sphere with three punctures, and a punctured torus that is planar outside a measured geodesic lamination. Keen and Series [10] studied the associated pleating structure on $\mathscr{M}$ in detail. In particular, they showed that the *pleating ray*, the locus of points in $\mathscr{M}$ with a fixed pleating lamination (fixed as a projective measured lamination) is homeomorphic to a line, with exactly one endpoint on $\partial\mathscr{M}$. If the pleating lamination is a simple closed geodesic, the associated pleating ray is said to be *rational*. Each rational pleating ray $\mathscr{P}_{p/q}$ ends at a *cusp* $\mu_{p/q} \in \mathscr{M}$, such that $G[\mu_{p/q}]$ is a geometrically finite Kleinian group that represents the disjoint union of two thrice punctured spheres.

Keen and Series showed that each rational pleating ray is contained in the real locus of a polynomial in $\mu$, namely the trace of an element $W_{p/q}$ of $G[\mu]$ representing the pleating locus on $\Omega(G[\mu])/G[\mu]$. Using a slight modification the circle chain method of Keen and Series [10] and Wright [31] (see Sections 6 and 8), a result of McShane, Parker and Redfern [23] on cusp groups, and Maskit's second combination theorem [22] we prove that if we only consider parameters in the real locus associated with a pleating ray, we can determine which parameters close to the cusp correspond to a discrete group.

**9.4. Theorem.** *On the extension $\mathscr{P}^+_{p/q}$ of each rational pleating ray there is an open neighborhood $\mathscr{U}$ of the cusp $\mu_{p/q}$ on $\mathscr{P}^+_{p/q}$ such that if $\mu \in \mathscr{P}^+_{p/q} \cap \mathscr{U}$ and $|\operatorname{tr} W_{p/q}| = 2\cos(\pi/n)$ for some $n \in \mathbf{N}$, $n \geq 3$, then*

$$G[\mu] = F *_{W_{r/s}[S,T[\mu]]} . \tag{1.12}$$

*For these values of $\mu$, $G[\mu]$ is a Kleinian group representing a thrice punctured sphere and a sphere with a puncture and two branch points of order $n$ on its ordinary set.*

*If $\mu \in \big(\mathscr{P}^+_{p/q} \cap \mathscr{U}\big) \setminus \bar{\mathscr{M}}$ is not of this form, then $G[\mu]$ is not discrete.*

A similar observation for the Bers embedding of Teichmüller space is made by Kra and Maskit in [16], and for the so-called Riley slice of Schottky space by Riley in the introduction to [27].

We do not have an estimate for the size of the neighborhood $\mathscr{U}$ or how big $n$ has to be for a fixed rational number $p/q$ to guarantee that the group is discrete. Computer experiments suggest that the groups should be discrete for all $n \geq 2$. See Remark 9.7 for a special case.

We also study groups that are HNN extensions of triangle groups that uniformize a pair of spheres with a puncture and two cone points of equal orders. In Sections 2–5 we introduce a pleating structure on the deformation spaces $\mathscr{M}_n$ of groups of this form, analogous to [10], concentrating on rational pleating rays. The motivation for studying these groups is that we can show that the discrete groups found in Theorem 9.4 are conjugate to groups that correspond to cusps on the boundaries of the spaces $\mathscr{M}_n$. The deformation spaces $\mathscr{M}_n$ are treated in [1] and [26]. For details on the pleating structure on the Riley slice see [11] and [13].

**9.6. Corollary.** *Let $p/q \in \mathbf{Q}$, $n \in \mathbf{N}$, $n \geq 3$, and $p' \in \mathbf{Z}$ such that $pp' = 1 \bmod q$. Let $\mu_{p'/q,n}$ be the boundary point of $\mathscr{M}_n$ on the ray $\mathscr{P}^n_{p'/q}$. If the group $G[\mu_{p/q}(n)]$ is discrete, then it is conjugate to $G_n[\mu_{p'/q,n}]$ by a Möbius transformation.*

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Finally, I would like to express my gratitude to the referee for their valuable comments.

## 2. Deformation spaces of Koebe groups of type $(1,1,\infty)$

In this section we consider Kleinian groups that can be obtained from hyperbolic triangle groups by an HNN construction conjugating two maximal cyclic subgroups. These groups include torsion-free terminal $b$-groups (see [15]) and a class of Koebe groups (see [26]) where one simple closed geodesic on the corresponding punctured torus is represented, respectively, by a parabolic or an elliptic transformation.

**2.1. Definition.** A Kleinian group $G$ that has an invariant component $\Omega_0 \subset \Omega(G)$ is called a *Koebe group* if any other component $\Delta' \subset \Omega(G) \setminus \Omega_0$ is a round disk. If the stabilizers of the disk components are hyperbolic triangle groups, the Koebe group is *terminal.* If the invariant component $\Omega_0$ is simply connected, the group is called a *$b$-group.* In this paper Koebe group usually means a terminal Koebe group that is not a $b$-group.

In this paper a *presentation* of a Kleinian group contains, in addition to the usual presentation of a group, a list of conjugacy classes that are assumed to consist of parabolic transformations. Sometimes the group may contain additional parabolic elements not included in this list.

A Fuchsian group $F$ acting on the hyperbolic plane $\mathbf{H}$ is called a (*hyperbolic*) *triangle group* if $\mathbf{H}/F$ is a sphere with $p$ punctures and and $e$ projections of fixed points of elliptic elements of $F$ such that $p+e=3$. We refer to the projections of elliptic fixed points on $\Omega(G)/G$ as *cone points*. The *signature* of a triangle group is a triple $(n_1,n_2,n_3)$, $n_i\in\{2,3,\dots,\infty\}$ that records the orders of the transformations of $F$ corresponding to the cone points and punctures. In particular, a puncture on $\mathbf{H}/F$ is denoted by $\infty$ in the signature.

Let $G$ be a Kleinian group. An elliptic Möbius transformation $g\in G$ is a *primitive* element of $G$, if it is conjugate to a rotation by $2\pi/n$, $n\in\mathbf{Z}$, and if any elliptic element in $G$ with the same fixed points as $g$ is a power of $g$. Let $x\in\widehat{\mathbf{C}}$ be the fixed point of a parabolic transformation of $G$. A parabolic transformation $g\in G$ is *primitive* if it is not a positive power of any other transformation of $G$.

**2.2. Definition.** Let $F_n$, $3\le n\le\infty$, be a triangle group of signature $(n,n,\infty)$. If $3\le n<\infty$, two primitive elliptic transformations $A$ and $B$ are *canonical generators* for $F_n$, if $F_n$ has the presentation

$$F_n=\langle A_n,B_n : A_n^n=B_n^n=\mathrm{id},\ K_n=B_nA_n \text{ parabolic}\rangle. \tag{2.1}$$

If $n=\infty$, two primitive parabolic transformations $A_\infty$ and $B_\infty$ in $F_\infty$ are *canonical generators* if $F_\infty$ has the presentation

$$F_\infty=\langle A_\infty,B_\infty : A_\infty,B_\infty,\ \text{and } K_\infty=B_\infty A_\infty \text{ parabolic}\rangle. \tag{2.2}$$

Let $F_n$ be as above. We wish to construct a new Kleinian group $G=\langle F_n,C\rangle$ that represents a punctured torus on an invariant component of its set of discontinuity. This amounts to the following gluing construction. Let $X$ be a component of the quotient $\Omega(F_n)/F_n$. We cut out disks around the elliptic special points (for $n$ finite) or punctures (for $n=\infty$) on $X$ corresponding to the canonical generators $A_n$ and $B_n$, and glue the boundaries of the remaining surface together, thus producing a punctured torus. This can be achieved by adding a new generator $C_n[\tau]$ satisfying

$$C_n^{-1}A_nC_n=B_n^{-1} \tag{2.3}$$

such that the group $G_n=\langle F_n,C\rangle$ is discrete and $G_n$ has the presentation

$$\begin{aligned}G_n=\langle A_n,B_n,C_n : A_n^n=B_n^n=\mathrm{id},\ &C_n^{-1}A_nC_n=B_n^{-1},\\ &K_n=B_nA_n \text{ parabolic}\rangle,\end{aligned} \tag{2.4}$$

and similarly for $n=\infty$; see [15] or [26] for details. If $G_n$ has the presentation (2.4), then it is the *HNN extension* of $F_n$ by $C_n$, denoted

$$G_n=F_n *_{C_n}. \tag{2.5}$$

There is a one complex parameter family of transformations $C_n[\mu]$ for each $n=3,4,\dots,\infty$ satisfying condition (2.3). We describe the two cases briefly, and refer to [15] and [26] for details.

$n=\infty$. It is convenient to normalize the torsion-free triangle group $F_\infty$ so that it acts discontinuously on the union of the upper and lower half planes. We use a normalization where the generators are

$$S = A_\infty = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{S} = B_\infty = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{2.6}$$

The HNN extensions of $F_\infty$ are groups of the form

$$G[\mu] = G_\infty[\mu] = \langle S, \widetilde{S}\,\rangle *_{T[\mu]} = \langle S, T[\mu]\rangle, \tag{2.7}$$

where

$$T[\mu] = \begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix}, \tag{2.8}$$

satisfying

$$\widetilde{S} = T[\mu]^{-1} S^{-1} T[\mu]. \tag{2.9}$$

It was shown in [15] that for $\operatorname{Im}\mu > 2$, $G[\mu]$ is a terminal $b$-group uniformizing a punctured torus. The *Maskit embedding* of the Teichmüller space of once-punctured tori, $\mathscr{M}$ is the open set of parameters $\mu \in \mathbf{H}$, containing $\{\mu \in \mathbf{H} \mid \operatorname{Im}\mu > 2\}$ for which $G[\mu]$ is a terminal $b$-group. For these parameters

$$G[\mu] = F_\infty *_{T[\mu]}\,. \tag{2.10}$$

$3 \le n < \infty$. After normalization we can assume that $F_n$ is generated by two elliptic transformations

$$A_n = \begin{pmatrix} e^{-i\pi/n} & 0 \\ 0 & e^{i\pi/n} \end{pmatrix}, \tag{2.11}$$

and

$$\begin{aligned} B_n &= \begin{pmatrix} i\sin(\pi/n)\cosh d_n - \cos(\pi/n) & -i\sin(\pi/n)\sinh d_n \\ i\sin(\pi/n)\sinh d_n & -i\sin(\pi/n)\cosh d_n - \cos(\pi/n) \end{pmatrix} \\ &= \begin{pmatrix} 2i/\sin(\pi/n) - e^{i\pi/n} & -2i\cot(\pi/n) \\ 2i\cot(\pi/n) & -2i/\sin(\pi/n) - e^{-i\pi/n} \end{pmatrix}, \end{aligned} \tag{2.12}$$

where

$$d_n = \operatorname{arcosh}\frac{\cos^2\pi/n + 1}{\sin^2\pi/n} = 2\operatorname{arcosh}\frac{1}{\sin(\pi/n)} \tag{2.13}$$

is the hyperbolic distance between the fixed points of $A$ and $B$ in $\mathbf{D}$.

The general form of the transformation $C_n$ satisfying (2.3) is

$$C_n[\mu] = \begin{pmatrix} \sqrt{\mu}\,\sinh(d_n/2) & -\sqrt{\mu}\,\cosh(d_n/2) \\ \sqrt{\mu}^{-1}\cosh(d_n/2) & -\sqrt{\mu}^{-1}\sinh(d_n/2) \end{pmatrix} = \begin{pmatrix} \sqrt{\mu}\,\cot(\pi/n) & -\sqrt{\mu}\,/\sin(\pi/n) \\ 1/\bigl(\sqrt{\mu}\,\sin(\pi/n)\bigr) & -\cot(\pi/n)/\sqrt{\mu} \end{pmatrix}. \tag{2.14}$$

Clearly the choice of the branch of the square root does not change $C[\mu]$ as a Möbius transformation. It was shown in [26] that for

$$|\mu| > \coth^2(d_n/4) = \left(\frac{1+\sin(\pi/n)}{\cos(\pi/n)}\right)^2, \tag{2.15}$$

the group

$$G_n[\mu] = \langle F_n, C_n[\mu]\rangle \tag{2.16}$$

is a Koebe group that represents a punctured torus on its invariant component, and that the map $G_n[\mu] \mapsto \mu$ is a global coordinate for the deformation space of any fixed Koebe group $G_n[\mu_0]$. We denote the image of the deformation space of $G_n[\mu_0]$ under this map by $\mathscr{M}_n$.

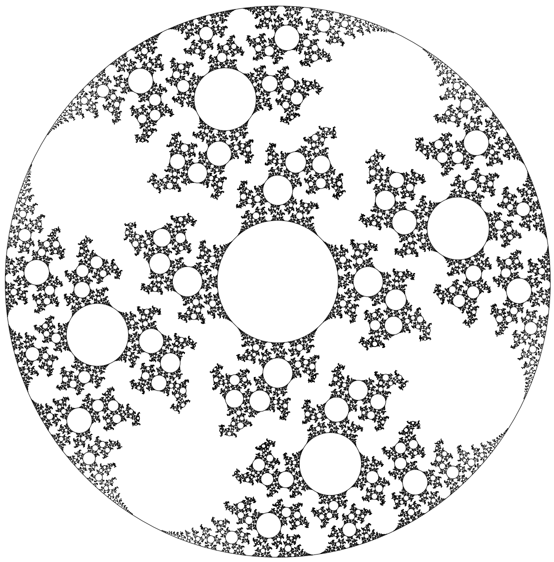

Figure 1. The limit set of a Kleinian group from $\mathscr{M}_4$ corresponding to the parameter $\mu = 4.32 + 1.26i \approx 4.5\,e^{0.28}$.

If $\mu \in \mathscr{M}_n$, $n$ finite, the invariant component $\Omega_0$ of $G_n[\mu]$ is infinitely connected, see Figure 1. There are homotopically nontrivial simple closed curves in $\Omega_0$ that are stabilized by conjugates of the finite cyclic group generated by $A_n$. These loops project $n$-to-1 to a non-dividing simple closed geodesic $\alpha$ on the punctured torus $\Omega_0/G_n[\mu]$.

The *mapping class group* of a Riemann surface $Y$ is the group of orientation preserving homeomorphisms of $Y$ modulo those homotopic to the identity map. The deformation space $\mathscr{T}(G_n[\mu])$ is $\mathscr{T}(1,1)$ factored by the action of the subgroup of the mapping class group generated by the $n$th power of the Dehn twist about this special curve. Complex analytically $\mathscr{T}(G_n[\mu])$, $3 \le n < \infty$, is a punctured disk, in $\mathscr{M}_n$ the puncture is $\infty$. More detailed treatments of these deformation spaces can be found in [1] and [26].

The quotient space $M = \mathbf{H}^3/G_n$ is a three-orbifold whose topology can be described as follows: Let $X = \Omega_0(G_n)/G_n$ be a punctured torus, and let $\alpha \subset X$ be the simple closed curve on $X$ corresponding to the elliptic element $A_n$. The orbifold $M$ is obtained from $X \times [0,1]$ by adding a singular two-handle $\mathbf{D}_n \times [0,1]$. Here $\mathbf{D}_n$ is a disk with a metric cone singularity of order $n$ in the origin, that is, $\mathbf{D}_n$ is the quotient of the unit disk $\mathbf{D}$ by the action of a finite cyclic group generated by a rotation $z \mapsto e^{2\pi i/n} z$. The curve $\partial\,\mathbf{D}_n \times \frac{1}{2}$ is identified with $\alpha \times \{0\}$, and $\partial\,\mathbf{D}_n \times [0,1]$ is identified with an annular neighborhood of $\alpha$ in $X \times \{0\}$. The line $\{0\} \times [0,1] \subset M$ is the *singular locus* of $M$, a line that connects the two cone points of order $n$ in $\Omega(G_n)/G_n$.

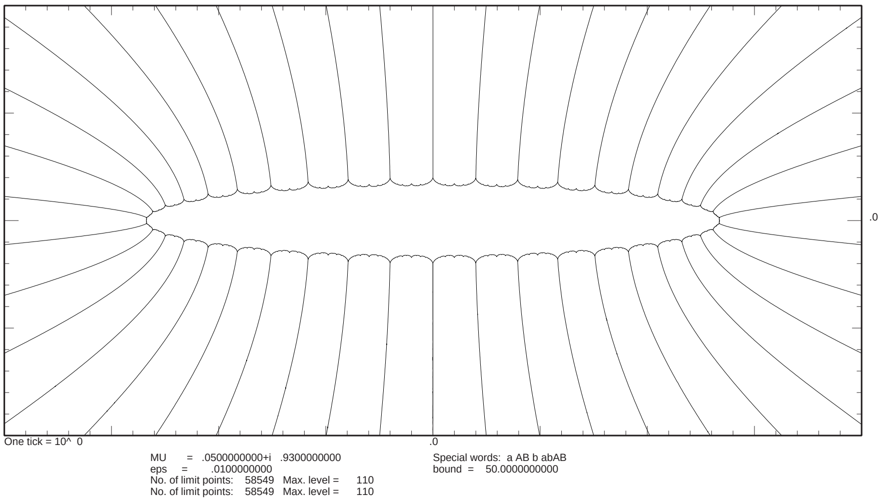


Figure 2. This picture shows a 2 to 1 covering of $\mathscr{M}_{21}$ by "the trace plane" $\widetilde{\mathscr{M}}_n$. The plotted parameter is $i\,\mathrm{tr}\,C_n[\tau] = i\cot(\pi/21)(\tau - 1/\tau)$. The lifts of integral pleating rays $\mathscr{P}^{21}_m$, $m = 0, 1, \dots, 20$ are shown. See Lemma 2.3 for more details on the covering, and Section 5 for the definition of pleating rays.

Although a parameter $\mu \in \mathscr{M}_n$ defines the transformation $C_n[\mu]$ uniquely as a Möbius transformation, the trace of $C_n$ is not a single-valued analytical function on $\mathscr{M}_n$. The expression of $C_n$ involves $\tau = \sqrt{\mu}$ in an essential way, and the choice of the branch of the square root causes an ambiguity. Therefore, it is sometimes convenient to work in a $2-1$ covering space. This is the parameterization used in Figure 2.

**2.3. Lemma.** *Let*

$$\widetilde{\mathscr{M}_n} = \{\tilde{\mu} \in \mathbf{C} : \exists \mu \in \mathscr{M}_n \text{ such that } \tilde{\mu} = \operatorname{tr} C[\mu]\}. \tag{2.17}$$

*The map $\tilde{\mu} \mapsto \mu$ is a $2-1$ analytical covering map $\widetilde{\mathscr{M}_n} \to \mathscr{M}$.*

*Proof.* Let $\mu = \tau^2$. By $C_n[\tau]$ we mean the expression of $C_n[\mu]$ with a fixed choice of the square root. With this notation,

$$\operatorname{tr} C[\tau] = \cot(\pi/n)(\tau - 1/\tau). \tag{2.18}$$

$\mathscr{M}_n$ is connected, and clearly no parameter from the unit circle can be contained in $\mathscr{M}_n$. Thus, $\mathscr{M}_n \subset \{\mu = \tau^2 : |\tau| > 1\}$. The map $\tau \mapsto \tau - (1/\tau)$ maps the outside of the unit disk injectively onto the complement of the interval $[-2i, 2i] \subset \mathbf{C}$.

(2.19)
$$\begin{array}{l}
\left\{\tau \in \mathbf{C} : |\tau| > 1\right\} \xrightarrow[1-1]{\tau \mapsto \tau - (1/\tau)} \mathbf{C} \setminus \left[-2i \cot\dfrac{\pi}{n}, 2i \cot\dfrac{\pi}{n}\right] \supset \left\{\operatorname{tr} C[\tau] : \tau^2 \in \mathscr{M}_n\right\} \\
\qquad {\scriptstyle z \mapsto z^2} \Big\downarrow {\scriptstyle 2-1} \\
\left\{\tau : |\tau| > 1\right\} \supset \mathscr{M}_n.
\end{array}$$

Combining these observations, it is clear that the map $\operatorname{tr} C[\tau] \mapsto \mu$ is 2 to 1 from $\widetilde{\mathscr{M}_n} = \left\{\operatorname{tr} C[\tau] : \mu \in \mathscr{M}_n\right\}$. □

## 3. Spaces of measured laminations

Keen and Series studied the Maskit embedding $\mathscr{M}$ of $\mathscr{T}(1,1)$ in [10]. They related the parameters in $\mathscr{M}$ to the geometry of the quotient of $\mathbf{H}^3$ by the action of the group $G_\infty[\mu]$. In order to do the same kind of investigation for $\mathscr{M}_n$ we study the space of measured geodesic laminations on the punctured torus, and define the appropriate space of measured laminations associated with $\mathscr{M}_n$, $n$ finite. The main difference to $\mathscr{M}$ comes from the fact that there is a simple closed geodesic $\gamma$ on $\Omega(G_n)/G_n$ that is represented by an elliptic element of $G_n$. On the orbifold $M = \mathbf{H}/G_n$, $\gamma$ is a loop that winds once around the singular locus of $M$.

**3.1. Definition.** Let $Y$ be a hyperbolic Riemann surface. A *geodesic lamination* on $Y$ is a closed subset of $Y$ that is the disjoint union of a collection of simple (not necessarily closed) geodesics on $Y$. A *transverse measure* $\sigma$ on a geodesic lamination $\lambda$ assigns to any embedded image $I$ of $[0,1]$ transversal to $\lambda$, a Borel measure with finite total mass such that the measure is supported on $I \cap \lambda$, and the measure is invariant under isotopies that preserve the lamination $\lambda$. The pair $(\lambda, \sigma)$ is a *measured geodesic lamination.* The lamination $\lambda$ is the *support* of $\sigma$.

We denote the space of measured geodesic laminations on a Riemann surface $Y$ by $\mathscr{ML}(Y)$. We use the weak topology for measures on the space $\mathscr{ML}(Y)$: A sequence $(\lambda_n, \sigma_n) \in \mathscr{ML}(Y)$ converges (weakly) to $(\lambda, \sigma) \in \mathscr{ML}(Y)$ if

$$\lim_{n\to\infty} \int_I \varphi \, d\sigma_n = \int_I \varphi \, d\sigma \tag{3.1}$$

for any interval $I$ transversal to $\lambda$ and $\lambda_n$ for all $n$, and for all compactly supported continuous functions $\varphi$. The space of *projective measured laminations* on $Y$ is

$$\mathscr{PML}(Y) = \mathscr{ML}(Y) \setminus \{0\}/(\gamma, \sigma) \sim (\gamma, r\,\sigma), \tag{3.2}$$

where $r > 0$. Using the topology induced from $\mathscr{ML}(Y)$ by the above projection, $\mathscr{PML}(Y)$ is homeomorphic to $\mathbf{S}^{6g-7+2p}$, where $g$ is the genus of $Y$ and $p$ is the number of punctures on $Y$.

The structure of those geodesic laminations that support a transverse measure is more restricted than that of a general geodesic lamination.

**3.2. Proposition.** *Let $X$ be a punctured torus. Let $\lambda \subset X$ be the support of a nonzero measured geodesic lamination. Then either*

(1) *$\lambda$ is a simple closed geodesic and $X \setminus \lambda$ is a sphere with a puncture and two geodesic boundary components of equal length , or*

(2) *$\lambda$ is the closure of an infinite simple geodesic, and $X \setminus \lambda$ is a* punctured bigon.

*Proof.* [29, Proposition 9.5.2]. □

For the general case we refer the reader to Thurston [29, Section 9] and to Otal [25, Section A.3]. For the rest of the paper we will only be concerned with the case of a punctured torus. If $Y$ and $Y'$ are two homeomorphic Riemann surfaces of finite area, the associated spaces of measured laminations are canonically isomorphic. We denote the space of measured laminations on a punctured torus by $\mathscr{ML}(1,1)$, and the corresponding projectivized space by $\mathscr{PML}(1,1)$.

The space $\mathscr{PML}(1,1)$ is naturally identified with $\widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$: a geodesic lamination on a punctured torus can be identified with a geodesic on the square

Euclidean torus, and the geodesics on the Euclidean torus are in 1 to 1 correspondence with their slopes $r \in \mathbf{R} \cup \{\infty\}$ in the standard uniformization of the square torus as $\mathbf{C}/\mathbf{Z}^2$. This identifies simple closed geodesics on the punctured torus with $\widehat{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\}$. We will call a simple closed geodesic on the punctured torus that is identified with $r \in \mathbf{Q}$ the *$r$-curve*. For more details on this identification see e.g. Birman and Series [3] Appendix, or Series [28].

## 4. $G_n$ and simple closed curves on punctured tori

The set of simple closed curves on a Riemann surface $Y$ can be embedded into $\mathscr{PML}(Y)$, by associating a Dirac mass to each intersection of the simple closed curve with a transversal. With this identification, simple closed curves form a dense countable subset in $\mathscr{PML}(Y)$. In this section we describe methods from combinatorial group theory that will be used for bookkeeping of group elements that correspond to simple closed curves in the groups $G_n$. In the following definition the "relation" $a_2^\infty = 1$ means that the element $a_2 \in G$ has infinite order.

**4.1. Definition.** Let

$$G = \langle a_1, a_2 : a_2^n = 1 \rangle =: \langle a_1 \rangle * \langle a_2 : a_2^n = 1 \rangle, \tag{4.1}$$

$n = 2, 3, \dots, \infty$. A word

$$W = a_{\nu_1}^{\alpha_1} a_{\nu_2}^{\alpha_2} \cdots a_{\nu_p}^{\alpha_p}, \tag{4.2}$$

where $p \in \mathbf{N}$, $\alpha_i \in \mathbf{Z}$, $\nu_1 = \pm 1$, is *cyclically reduced* in $G$ (*with generators* $a_1$, $a_2$) if

(1) $\nu_i \neq \nu_{i+1}$ for $1 \le i \le p-1$, $\nu_1 \neq \nu_p$, and
(2) $1 \le \alpha_i \le n$ if $\nu_i = 2$.

It is easy to see from the presentation (2.4) that for parameters $\mu \in \mathscr{M}_n$,

$$G_n[\mu] = \langle A_n \rangle * \langle C_n[\mu] \rangle. \tag{4.3}$$

We will make use of the following fact:

**4.2. Theorem.** *Two cyclically reduced words in the group $G_n[\mu]$ for $\mu \in \mathscr{M}_n$ are conjugate if and only if they are cyclic permutations of each other.*

*Proof.* [18, Theorem 1.4].

Let us first consider the case of terminal $b$-groups. A simple closed curve on the punctured torus corresponds to a conjugacy class of transformations of $G[\mu]$. As described in Section 3, the free homotopy classes of simple closed curves on a punctured torus that are not parallel to the puncture, can be enumerated by $\mathbf{Q}\cup\{\infty\}$. We will sketch a method of producing a representative

$$W_{p/q} = W_{p/q}[\mu] = W_{p/q}\big[S, T[\mu]\big] \tag{4.4}$$

called the $p/q$-*word* that is cyclically reduced in $G[\mu]$ with the generators $S^{-1}$ and $T[\mu]$ such that $W_{p/q}$ represents the $p/q$-curve on $\Omega_0[\mu]/G[\mu]$. See e.g. [10], [24], or [4] for proofs.

The special words can be defined inductively using the Farey sequence as follows. Two rational numbers $p/q$ and $p'/q'$ (where $q,q'>0$ and $(p,q)$, $(p',q')$ are relatively prime) are called (*Farey*) *neighbors* if

$$pq' - p'q = \pm 1. \tag{4.5}$$

In the construction of the $p/q$-words, we first set

$$W_{1/0} = W_\infty = S^{-1} \qquad \text{and} \qquad W_{0/1} = W_0 = T. \tag{4.6}$$

If $a/b < c/d$ (with the convention $1/0 = \infty > r$ for all $r\in\mathbf{Q}$) are neighbors, we require

$$W_{(a+c)/(b+d)} = W_{c/d}W_{a/b}. \tag{4.7}$$

Note that we get the words for negative integers $n=-1,-2,\dots$ by solving the equations

$$W_{n+1} = W_\infty W_n. \tag{4.8}$$

This process assigns a unique word $W_{p/q}$ to each rational number $p/q\in\mathbf{Q}$. It is easy to see from the construction of the words that for $p\ge 0$, $p$ is the number of $S^{-1}$'s and $q$ is the number of $T$'s in $W_{p/q}$. For $p<0$, $-p$ is the number of $S$'s in the word $W_{p/q}$.

**4.3. Proposition.** *Let $p/q$ and $r/s$ be Farey neighbors. Then*

(1) $G[\mu] = \langle W_{p/q}, W_{r/s}\rangle$ *, and*
(2) *If $p/q < r/s$, then*

$$W_{r/s}^{-1}W_{p/q}^{-1}W_{r/s}W_{p/q} = T^{-1}S^{-1}TS = K, \tag{4.9}$$

*where*

$$K = \widetilde{S}S = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}. \tag{4.10}$$

*Proof.* See [10].

Let us now consider the case of finite $n$. There is a canonical surjective homomorphism

$$\Phi_n\colon \langle a\rangle * \langle b\rangle \to \langle a\rangle * \langle b : b^n = 1\rangle. \tag{4.11}$$

Clearly some of the different words project to the same group element under $\Phi_n$. In particular, this means that we do not get a 1-1 correspondence between the elements of $G_n$ represented by the words $W_{p/q}$ and free homotopy classes of simple closed curves on the punctured torus. However, the identifications take place in a controlled manner.

**4.4. Proposition.** *Let $n \in \{3, 4, \dots, \infty\}$, and $\mu \in \mathscr{M}_n$. The transformations $W_{p/q}\bigl[A_n, C[\mu]\bigr]$ and $W_{p'/q'}\bigl[A_n, C[\mu]\bigr]$ are conjugate in $G_n[\mu]$ if and only if $p'/q' = p/q + kn$ for some $k \in \mathbf{Z}$. $W_{p/q}[A_n, C\bigl[\mu]\bigr]$ is not conjugate with any $W^{-1}_{p'/q'}\bigl[A_n, C[\mu]\bigr]$, $p'/q' \in \mathbf{Q} \cup \{\infty\}$.*

*Proof.* Let $[p/q]$ be the integral part of $p/q$. If we write $W_{p/q}\bigl[A_n, C[\mu]\bigr]$ in terms of $W_{[p/q]}\bigl[A_n, C[\mu]\bigr]$ and $W_{[p/q]+1}\bigl[A_n, C[\mu]\bigr]$ as a word $V_{p/q}\bigl[W_{[p/q]}, W_{[p/q]+1}\bigr]$, then clearly $V_{p/q}[W_{0/1}, W_{1/1}] = W_{p/q-[p/q]}\bigl[A_n, C[\mu]\bigr]$. As $W_{m+kn} = W_m$ for all $k \in \mathbf{Z}$ as elements of $G_n$, the words $W_{p/q}\bigl[A_n, C[\mu]\bigr]$ and $W_{kn+p/q}\bigl[A_n, C[\mu]\bigr]$ represent the same element in $G_n$. We claim there are no further conjugacies among the special words.

The group $G_n$ is isomorphic to the free product $\langle A_n : A_n^n = 1\rangle * \langle C\rangle$. Clearly we can restrict our attention to the case $0 \le p/q < n$. The words $W_{p/q}$, $p/q \in \mathbf{Q} \cap [0, n)$ are cyclically reduced in $G_n$. By Theorem 4.2, two elements of the group $G_n = \langle A_n : A_n^n = 1\rangle * \langle C\rangle$ are conjugate if their cyclically reduced presentations are cyclic permutations of one another. For a word $W_{p/q}$, $p$ is the number of $C^{-1}$'s and $q$ is the number of $A_n$'s. Both of these quantities are invariant under cyclic permutation, implying that $W_{p/q}$ is conjugate with $W_{p'/q'}$, with $0 \le p/q$, $p'/q' < n$, only if $p/q = p'/q'$.

To prove the second statement, we observe that we get a cyclically reduced word $\widetilde{W}^{-1}_{p'/q'}$ from $W^{-1}_{p'/q'}$, without changing the group element that the words represent in $G_n$, by formally inverting $W_{p'/q'}$, and then replacing each $A^{-j}$, $1 \le j \le n-1$, by $A^{n-j}$. $W_{p/q}$ has only negative powers of $C$, and $\widetilde{W}^{-1}_{p'/q'}$ has only positive powers of $C$, so by the same reasoning as above, no $W_{p/q}$ can be conjugate to any $W^{-1}_{p'/q'}$. □

The mapping class group has a natural action on $\mathscr{ML}(Y)$ and on $\mathscr{PML}(Y)$. On $\mathscr{PML}(1,1)$ this action is the same as the action of $\mathrm{PSL}(2, \mathbf{Z})$ on $\widehat{\mathbf{R}}$ if we use the identification of $\mathscr{PML}(1,1)$ with $\widehat{\mathbf{R}}$ indicated in Section 3. In particular, the Dehn twist $\mathrm{Dehn}_\alpha$ about the simple closed curve on $\Omega_0/G[\mu]$ that corresponds to the conjugacy class of the parabolic transformation $S$, acts on $\widehat{\mathbf{R}}$ as a translation by 1. Proposition 4.4 means that if two simple closed curves on the punctured torus $\Omega_0/G_n$ differ by a power of $\mathrm{Dehn}_\alpha^n$, they correspond to the same group element in $G_n$. The action of $\langle \mathrm{Dehn}_\alpha^n \rangle$ fixes $\infty$, which corresponds to the projective measured lamination with support $\alpha$, and it is discontinuous on $\mathbf{R} = \mathscr{PML}(1,1) \setminus \{\alpha\}$. The quotient space

$$\mathscr{PML}_n(1,1) = \mathscr{PML}(1,1)/\langle \mathrm{Dehn}_\alpha^n \rangle. \tag{4.12}$$

is the natural space of laminations associated with the deformation spaces $\mathscr{M}_n$.

## 5. Pleating structure for $\mathscr{M}_n$

Let $G_n[\mu]$, $n \in \{3,4,\dots,\infty\}$, $\mu \in \mathscr{M}_n$, be a Koebe group uniformizing a punctured torus as in Section 2. Let $\mathscr{C}\big(\Lambda(G_n[\mu])\big) \subset \mathbf{H}^3$ be the *hyperbolic convex hull* of the limit set of $G_n[\mu]$. $\mathscr{C}\big(\Lambda(G_n[\mu])\big)$ is invariant under the action of $G_n[\mu]$ in $\mathbf{H}^3$, and the quotient

$$\mathscr{C}M[\mu] = \mathscr{C}\big(\Lambda(G_n[\mu])\big)/G_n[\mu] \tag{5.1}$$

is the *convex core* of the orbifold $M[\mu] = \mathbf{H}^3/G_n[\mu]$.

The boundary of $\mathscr{C}M[\mu]$ is homeomorphic to $\partial M[\mu]$. Let $\partial\mathscr{C}_0 = \partial\mathscr{C}_0\big(\Lambda(G_n)\big) \subset \mathbf{H}^3$ be the component of the boundary of the convex hull of the limit set that lies above the invariant component $\Omega_0(G_n)$. The quotient surface $\mathscr{S} = \partial\mathscr{C}_0/G_n$ is a punctured torus. $\mathscr{S}$ inherits a geometric structure from the embedding of $\partial\mathscr{C}_0$ in $\mathbf{H}^3$, in which it is planar outside a geodesic lamination. We say that $\mathscr{S}[\mu]$ is *pleated along* this lamination, called the *pleating locus* on $\mathscr{S}[\mu]$. The pleating locus carries a natural transverse measure, the *bending measure* that measures the bending along any interval transversal to the pleating locus. A good reference for this material is [5].

The Teichmüller space $\mathscr{T}(G_n)$ is naturally isomorphic with $\mathscr{T}(1,1)/\langle \mathrm{Dehn}_\alpha^n \rangle$, and the natural space of laminations associated with $\mathscr{T}(G_n)$ is $\mathscr{PML}_n(1,1)$ as defined at the end of Section 4. The *pleating map* associated with the family $\{G_n[\mu]\}_{\mu\in\mathscr{M}_n}$ is

$$\mathrm{pl}\colon \mathscr{M}_n \to \mathscr{PML}_n(1,1), \tag{5.2}$$

where $\mathrm{pl}(\mu)$ is the projective class in $\mathscr{PML}_n(1,1)$ of the pleating lamination of $\mathscr{M}_n[\mu]$. Keen and Series [12] showed that the pleating map is continuous.

**5.1. Remark.** For the groups $G_n[\mu]$, $\mu \in \mathscr{M}_n$, the convex core $\mathscr{C}M[\mu]$ has a singular locus homeomorphic to an interval, that connects the two cone points on the sphere component of $\partial\mathscr{C}M[\mu]$. Two simple closed curves are considered to be homotopic on $M[\mu]$ (or on $\mathscr{C}M[\mu]$) if they can be mapped to each other by a sequence of operations consisting of homotopies of the underlying space of $M$

fixing the singular locus of $M$, and of "tightening" an arc that winds around the singular line exactly $n$ times in a small neighborhood. See Figure 3. We define $\mathscr{PML}_n(1,1)$ to be the space of geodesic measured laminations on $\partial\,(\mathbf{H}^3/G_n[\mu])$ with this definition of homotopy on the orbifold. Another way to describe the simple closed curves in this space is to note that if the regular covering space of a surface (in our case the invariant component $\Omega_0(G[\mu])$) is not simply connected, it is natural to consider as equivalent all the simple closed curves that are represented by the same covering transformation.

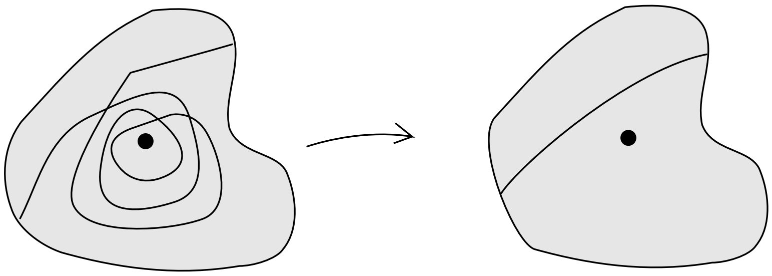

Figure 3. Tightening an arc that winds three times around a singular line where the local structure is the product of an interval with $\mathbf{D}/z\mapsto z^3$. The figure shows a projection where the singular line is represented by the dot in the middle.

**5.2. Definition.** A geodesic measured lamination is *rational* if it is supported on a simple closed curve. In particular, the laminations in $\mathscr{PML}(G_n)$, $n=3,4,\dots,\infty$, that correspond to words $W_{m/1}$, $m\in\mathbf{Z}$ are called *integral laminations.*

**5.3. Definition.** Let $n=3,4,\dots,\infty$, and let $\lambda\in\mathscr{PML}(G_n)$. The set

$$\mathscr{P}^n_\lambda=\{\mu\in\mathscr{M}_n\mid \mathrm{pl}(\mu)=\lambda\} \tag{5.3}$$

is the $\lambda$-*pleating ray* in $\mathscr{M}_n$. Let $p/q\in\mathbf{Q}$, and let $\gamma_{p/q}\in\mathscr{PML}_n$ be the projective equivalence class represented by the transformation $W_{p/q}[A_n,C]\in G_n[\mu]$.

$$\mathscr{P}^n_{p/q}=\mathscr{P}^n_{\gamma_{p/q}} \tag{5.4}$$

is a *rational pleating ray.*

The trace of the transformation $W_{p/q}\big[A_n,C[\mu]\big]$ is not uniquely defined as an analytical function of $\mu\in\mathscr{M}_n$ if $n$ is finite. However, the ambiguity of the choice of the branch of the square root does not affect the Möbius transformation, in particular, whether the transformation is hyperbolic or not. See Figure 2 for a collection of pleating rays.

**5.4. Definition.** Let $n=3,4,\dots,\infty$. The *real locus of* $W_{p/q}\big[A_n,C[\mu]\big]$ is

$$\mathscr{V}^n_{p/q}=\big\{\mu\in\mathscr{M}_n:\mathrm{tr}\,W_{p/q}\big[A_n,C[\mu]\big]\in\mathbf{R}\big\}. \tag{5.5}$$

**5.5. Proposition.** *$\mathscr{P}^n_{p/q}$ is a union of connected components of $\mathscr{V}^n_{p/q}\cap\mathscr{M}_n$.*

*Proof.* For $n=\infty$ this is Proposition 5.4 of [10]. The same proof applies for finite $n$; one needs to observe that the proofs of Lemma 4.6 and Lemma A.2 of [10] do not use their assumption that the groups under consideration in [10] are free. □

In Section 6 we observe, analogously to the case $n=\infty$, that $\mathscr{P}^n_{p/q}$ is a single component of $\mathscr{V}^n_{p/q}$.

## 6. Circle chains

In this section we generalize the definition of circle chains used in [10] and [31] for groups with elliptic elements. For $n=\infty$ the chains defined here are the same as in [10] and [31]. For finite values of $n$ they retain most of the useful properties of the case $n=\infty$, the difference being the fact that the chains will consist of a finite number of disks. We will sometimes refer to these cases as *finite chains*. In order to simplify notation, we will use the following convention: the notation $i \in \mathbf{Z} \bmod nq$ should be read $i \in \mathbf{Z}$ if $n=\infty$.

**6.1. Definition.** Let $n=3,4,\dots,\infty$, and $\mu \in \mathbf{C}$. Let $G[\mu]=\langle F_n, C_n[\mu]\rangle$ be a group of Möbius transformations (not necessarily discrete), where $F_n=\langle A_n, B_n\rangle$ as earlier, and $C$ is a transformation satisfying

$$C^{-1}A_nC=B_n^{-1} \tag{6.1}$$

as in Section 2. Let $W_{p/q}=W_{p/q}[A_n,C]$ be the $p/q$-word defined in Section 4. A collection $\{\delta_i\}$, $i\in\mathbf{Z} \bmod nq$ of closed, round disks is a *combinatorial $p/q$-chain for $G_n$ (with generators $A_n$ and $C$)* if it satisfies the following conditions:

(1) $\delta_0$ is tangent to $\Lambda(F)$, the limit set of $F$, at the fixed point of the parabolic element

$$K_n=W_{r/s}^{-1}W_{p/q}^{-1}W_{r/s}W_{p/q}, \tag{6.2}$$

(2) $W_{p/q}\delta_0=\delta_0$,
(3) $C(\delta_j)=\delta_{j+p}$ for all $j=0,\dots,q$, and
(4) $A(\delta_j)=\delta_{j+q}$ for all $j\in\mathbf{Z}$.

If a circle chain $\{\delta_i\}$ satisfies (1)–(4), we say $\{\delta_i\}$ *connects* $\operatorname{fix}K$ to $A(\operatorname{fix}K)$. The chain is *tangent* if $\delta_i$ and $\delta_{i+1}$ are tangent and the interiors $\operatorname{int}\delta_i$ and $\operatorname{int}\delta_{i+1}$ are disjoint for every $i\in\mathbf{Z}$. If we know that the group is discrete, we say that the chain is *proper* if

(1) the interiors of the disks $\delta_i$ are contained in $\Omega(G)$ for all $i$,
(2) the interiors of adjacent disks $\delta_i$ and $\delta_{i+1}$ intersect for all $i$, and
(3) $\operatorname{int}\delta_i\cap\operatorname{int}\delta_j=\emptyset$ for $|i-j|>1$.

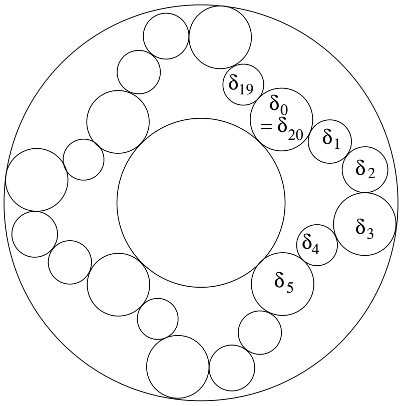


Figure 4. A tangent $\frac{3}{5}$ circle chain for $G_4$.

**6.2. Remarks.** (1) The combinatorial $p/q$ chain is associated to a group with a fixed pair of generators. For a different choice of generators there may be a chain with different combinatorics. We use this fact in the proof of Theorem 9.4, see also McShane, Parker and Redfern [23].

(2) The disks $\delta_0,\dots,\delta_{p+q-1}$ form a combinatorially interesting set: this part of the chain can be generated starting from $\delta_0$ by the following process:

(i) if $0 \le i < q$, set $\delta_{i+p} = C(\delta_i)$,
(ii) if $q \le i < p+q$, set $\delta_{i-q} = A^{-1}(\delta_i)$.

(3) Unlike Keen and Series [10] we do not require that the group $G$ is discrete. In fact, in Section 9, we use the existence of special circle chains with some additional properties to prove that the group $G[\mu]$ defined in Section 2 is discrete for a number of the parameters $\mu$ found in Lemma 7.3.

(4) The dependence on $n$ of the definition of the circle chain is a result of the fact that $A_n^n = \mathrm{id}$. This implies that the chain is actually finite, consisting of $nq$ circles. See Figure 4.

The results in this section are analogs of some of the results of [10, Sections 4 and 5]. The proofs are essentially the same. We only give some of the arguments and calculations for the case of finite $n$, and refer to [10] for the details.

**6.3. Proposition.** *Let $n = 3,4,\dots,\infty$, and $\mu \in \mathbf{C}$. The group $G_n[\mu]$ has a proper $p/q$ chain if and only if $\mu \in \mathscr{P}^n_{p/q}$.*

*Proof.* [10, Proposition 4.11].

**6.4. Lemma.** *Let $0 \le m < n$. The integral pleating ray $\mathscr{P}^n_{m/1}$ is the radial half line*

$$\mathscr{P}^n_{m/1} = \mathscr{M}_n \cap \{\mu = t^2 e^{-i2\pi m/n} : t > 0\} \tag{6.3}$$

*in $\mathscr{T}(G_n[\mu])$.*

*Proof.* It is easy to check that

$$W_{m/1} = \begin{pmatrix} e^{i\pi m/n}\sinh(d/2)\sqrt{\mu} & * \\ * & -e^{-i\pi m/n}\sinh(d/2)/\sqrt{\mu} \end{pmatrix}, \tag{6.4}$$

so if we set $\mu = t^2 e^{2i\varphi}$, we get

$$\operatorname{Im}\operatorname{tr} W_{m/1} = \left(t + \frac{1}{t}\right)\sinh(d/2)\sin\left(\varphi + \frac{\pi m}{n}\right). \tag{6.5}$$

Thus

$$\mathscr{V}^n_{m/1} \subset \{\mu \in \mathscr{M}_n : \arg(\mu) = -2\pi m/n\}. \tag{6.6}$$

It remains to show the existence of an $m/1$ chain. Here we can follow the argument of [10, Lemma 5.2]. □

**6.5. Proposition.** *Let $n \in \{3, 4, \ldots\}$, and $p/q \in \mathbf{Q} \setminus \mathbf{Z}$. Then*

(1) *$\mathscr{P}^n_{p/q} \neq \emptyset$ for all $p/q \in \mathbf{Q} \bmod n\mathbf{Z}$.*

(2) *The pleating ray $\mathscr{P}^n_{p/q}$ is contained in the sector*

$$\{\mu \in \mathscr{M}_n : -2\pi([p/q]+1)/n \le \arg\mu \le -2\pi[p/q]/n\} \tag{6.7}$$

*bounded by the integral rays $\mathscr{P}^n_{[p/q]}$ and $\mathscr{P}^n_{[p/q]+1}$, where $[p/q]$ is the integral part of $p/q$.*

(3) *$\mathscr{P}^n_{p/q}$ is the component of $\mathscr{V}^n_{p/q} \cap \mathscr{M}_n$ asymptotic to the line*

$$\{\mu \in \mathscr{M}_n : \arg\mu = -2\pi p/(qn)\}. \tag{6.8}$$

*Proof.* (1): By Lemma 6.4, the half-lines $\{\mu \in \mathscr{M}_n : \arg\mu = 2\pi k/n\}$, $k \in \mathbf{Z}$, are mapped to the integral values in $\mathbf{Q}/\mathbf{Z}$, and no other points are mapped to these values. Let $S$ be a large circle in $\mathscr{M}_n$ centered at $0$. The restriction of the pleating map, $\mathrm{pl}|_S$ is a continuous map of degree $1$ of a circle to a circle, in particular, a surjective map.

(2) and (3): The fact that the radial lines (6.3) contain all points of $\mathscr{M}_n$ that are mapped to integral values implies that the nonintegral rays are contained in the sectors (6.7), because we assumed $n \ge 3$.

Let $\mu = \tau^2$. An easy induction argument as in [10, Proposition 3.1] shows that

$$W_{p/q}[\mu] = \begin{pmatrix} e^{p\ i\pi/n}\tau^q\sinh(d/2)^q + O(|\tau^{q-1}|) & O(|\tau|^q) \\ O(|\tau|^{q-2}) & O(|\tau|^{q-2}) \end{pmatrix} \tag{6.9}$$

as $|\tau| \to \infty$. When $|\tau| \to \infty$ the condition $\operatorname{tr} W_{p/q} \in \mathbf{R}$, that defines $\mathscr{V}^n_{p/q}$, approaches

$$\arg\mu = \arg\tau^2 = -2\pi\left(\frac{p}{nq} + \frac{k}{q}\right), \qquad k \in \mathbf{Z}. \tag{6.10}$$

For large values of $n$ there are several branches in the sector (6.7) that qualify as candidates for points in $\mathscr{P}^n_{p/q}$. Clearly it is enough to consider the case $0 < p/q < 1$. To rule out all but one of the branches of $\mathscr{V}^n_{p/q}$ we do an induction argument on $q$:

(a) Let $p/q = 1/2$. Then the branch of $\mathscr{V}^n_{1/2}$ asymptotic to the radial line with argument $-\pi/n$ is the only one asymptotically contained in the sector between $\mathscr{P}^n_0$ and $\mathscr{P}^n_1$. By Proposition 5.5 this implies that $\mathscr{P}^n_{1/2}$ equals this component.

(b) Assume that $\mathscr{P}^n_{k/q}$, $k = 0, 1, \dots, q$ satisfy the claims (2) and (3) of the theorem. Let $p \in \{1, 2, \dots, n-1\}$ such that $p$ and $q+1$ are relatively prime. Then, (6.10) and elementary properties of the rationals imply that there is exactly one component of $\mathscr{V}^n_{p/(q+1)} \cap \mathscr{M}_n$ contained between the correct rays of denominator $q$. Continuity of the pleating map now implies, as above, that this component is exactly the pleating ray. □

Combining these observations with the analysis of Keen and Series [10, Section 5] we get the following theorem. Here we use the notation $\mu_{p/q,\infty} = \mu_{p/q}$.

**6.6. Theorem.** *Let $n \in \{3, 4, \dots, \infty\}$, and $p/q \in \mathbf{Q}$. If $n$ is finite, the pleating ray $\mathscr{P}^n_{p/q}$ coincides with the branch of $\mathscr{V}^n_{p/q}$ asymptotic with the half line of argument $\arg\mu = -2\pi i p/(qn)$. $\mathscr{P}^\infty_{p/q}$ coincides with the vertical component of $\mathscr{V}^\infty_{p/q} \cap \mathscr{M}_\infty$ asymptotic to the line $\{\mu \in \mathbf{C} : \operatorname{Im}\mu = 2p/q\}$. For all $n \in \{3, 4, \dots, \infty\}$ the pleating ray $\mathscr{P}^n_{p/q}$ is homeomorphic to $\mathbf{R}$, and $\overline{\mathscr{P}}^n_{p/q} \cap \partial\mathscr{M}_n$ consists of a single point $\mu_{p/q,n}$ such that the transformation $W_{p/q}\big[A_n, C[\mu_{p/q,n}]\big]$ is parabolic. Furthermore, the group $G_n[\mu_{p/q,n}]$ is geometrically finite, and it has a tangent circle chain connecting $\operatorname{fix} K_n$ to $A_n \operatorname{fix} K_n$.*

*Proof.* One can copy the arguments of [10, Theorem 5.1], with only insignificant changes resulting from the fact that the trace function in [10] is a polynomial in $\mu$, and for finite $n$ it is a polynomial in $\sqrt{\mu}$ and $1/\sqrt{\mu}$. We refer to [10] for details.

## 7. Extensions of pleating rays and disk-preserving subgroups

In this section we consider an extension of a rational pleating ray across the boundary of $\mathscr{M}$. We will use the fact that the rational pleating rays coincide with the asymptotically vertical components of the real locus of the trace of the corresponding special word in $\mathscr{M}$.

**7.1. Definition.** $\mathscr{P}^+_{p/q}$, the connected component of

$$\overline{\mathscr{P}_{p/q}} \cup \{\mu \in \mathscr{V}_{p/q} \cap (\mathbf{H} \setminus \overline{\mathscr{M}})\} \tag{7.1}$$

containing $\mathscr{P}_{p/q}$ is the *extended $p/q$ ray.*

**7.2. Remarks.** (1) The trace function, $\operatorname{tr} W_{p/q}\colon \mathbf{C} \to \mathbf{C}$, may have a critical point close to the boundary, even at $\mu_{p/q}$, that is, $\mathscr{P}^+_{p/q}$ might have a branching point. The somewhat awkward definition of the extended ray is used in this form so that we only get the correct ray corresponding to the pleating ray *inside* $\mathscr{M}$, and we avoid discussing the issue of critical points. In particular, the extended ray is not necessarily homeomorphic to $\mathbf{R}$.

(2) Similarly, one could define extended rays for $\mathscr{M}_n$, $n = 3, 4, \ldots$. However, the methods used in Sections 8 and 9 to study the discreteness of groups in $\mathbf{H} \setminus \overline{\mathscr{M}}$ cannot be directly applied to treat the corresponding cases for finite $n$.

Figure 5. Parts of the real loci $\mathscr{V}_{0/1}$ and $\mathscr{V}_{2/7}$. The cusped, nonsmooth curve in the figure is an approximation part of the boundary of $\mathscr{M}$, $\mathscr{M}$ is the region above this boundary. $\mathscr{P}^+_{0/1} = \mathscr{V}_{0/1} = \{\mu \in \mathbf{H} : \operatorname{Re}\mu = 0\}$, and the curve passing through the point labeled $2/7$ is contained in the extended ray $\mathscr{P}^+_{2/7}$. The third smooth curve in the figure is part of $\mathscr{V}_{2/7} \setminus \mathscr{P}^+_{2/7}$.

**7.3. Lemma.** *There are points $\mu_{p/q}(n) \in \mathscr{P}^+_{p/q}$ for all large values of $n$ such that $W_{p/q}[\mu_{p/q}(n)]$ is elliptic of order $n$, and*

$$\lim_{n\to\infty} \mu_{p/q}(n) = \mu_{p/q}. \tag{7.2}$$

*Proof.* The function $\mu \mapsto \operatorname{tr} W_{p/q}[\mu]$ is holomorphic. Thus, the image of a small neighborhood of $\mu_{p/q}$ covers a small neighborhood of $\operatorname{tr} W_{p/q} = \pm 2$, thus for large $n$ the values $\operatorname{tr} W_{p/q}[\mu] = \pm 2\cos(\pi/n)$ are attained. $\square$

Let $p/q$ and $r/s > p/q$ be Farey neighbors, $\mu \in \mathbf{C}$. Let

$$F = F_{p/q}[\mu] = \langle W_{p/q}, K\rangle = \langle W_{p/q}, W_{r/s}^{-1} W_{p/q}^{-1} W_{r/s}\rangle. \tag{7.3}$$

In most of what follows we aim at proving that for big $n$ the groups $G[\mu_{p/q}(n)]$ are discrete. The method is based on a 'reconstruction' of $G[\mu]$ starting from a Fuchsian subgroup. In Section 2 we presented $G[\mu]$ as the HNN-extension of the torsion-free Fuchsian group $\langle S,\widetilde{S}\,\rangle$. In Section 9 we will use a different set of generators, in terms of which the group is the HNN-extension of $F_{p/q}$ for $\mu \in \mathscr{P}^+_{p/q}$.

Keen and Series, [10, Proposition A.1], show that if $\mu \in \mathscr{P}_{p/q}$, then $F_{p/q}$ is a Fuchsian group of the second kind, that represents a punctured cylinder with boundary geodesics of equal length. For parameters in the extension of the $p/q$ ray we have:

**7.4. Lemma.** *If $\mu \in \mathscr{P}^+_{p/q}$ and $\operatorname{tr} W_{p/q} = 2\cos(\pi/n)$ for some $n \in \mathbf{N}$, $n \geq 3$, then $F = F_{p/q}[\mu]$ is a triangle group of signature $(n, n, \infty)$.*

*Proof.* The elements $W_{p/q}^{-1}$, $W_{r/s}^{-1} W_{p/q} W_{r/s}$ and their product

$$W_{r/s}^{-1} W_{p/q} W_{r/s} W_{p/q}^{-1} = K \tag{7.4}$$

all have real traces. It follows from [6, Lemma 5.23] that $F$ preserves a disk $D$. The quadrilateral with vertices at the fixed points of $W_{p/q}$, $W_{r/s}^{-1} W_{p/q} W_{r/s}$, $K$ and

$$\widetilde{K} = \widetilde{K}_{p/q} = W_{p/q}^{-1} K W_{p/q} = W_{p/q}^{-1} W_{r/s}^{-1} W_{p/q} W_{r/s} \tag{7.5}$$

in $\overline{D}$ satisfies the conditions of Poincaré's theorem. □

**7.5. Lemma.** *If $\mu \in \mathscr{P}^+_{p/q}$ and $\operatorname{tr} W_{p/q} \in (-2,2) \setminus \{2\cos(\pi/n) : n \in \mathbf{N},\ n \geq 2\}$, then $F = F_{p/q}[\mu]$ is not discrete.*

*Proof.* If $W_{p/q}$ is an infinite order elliptic, the group is trivially non-discrete, so we can assume $W_{p/q}$ is a non-primitive elliptic. Let $W_{\mathrm{prim}}$ be a primitive generator of $\langle W_{p/q}\rangle$. $F$ has parabolic elements, so any fundamental polygon has cusps extending to the circle at infinity. The commutator relation implies that the isometric circles of $W_{p/q}$ and $W_{r/s}^{-1} W_{p/q}^{-1} W_{r/s}$ are tangent at the fixed points of $K$ and $\overline{K}$. Clearly the isometric circles of $W_{\mathrm{prim}}^{\pm 1}$ and $W_{r/s}^{-1} W_{\mathrm{prim}}^{\mp 1} W_{r/s}$ intersect inside $D$. The Ford fundamental region of $F$ would thus be bounded away from the circle at infinity, which is impossible because there are parabolic elements in the group. □

Lemma 7.5 has the following immediate corollary.

**7.6. Proposition.** *Let $\mu \in \mathscr{P}^+_{p/q} \setminus \bar{\mathscr{M}}$. If $G[\mu]$ is discrete, then $|\operatorname{tr} W_{p/q}| = 2\cos(\pi/n)$ for some $n = 2, 3, \ldots$.*

This argument uses the disk-preserving subgroup $F_{p/q}$ in an essential way. We can only find a subgroup like this for rational rays, and do not know how to treat groups corresponding to parameters that are not contained in an extended rational ray. However, we have the following result as an application of Jørgensen's inequality [7] which implies that the discrete groups we can find in $\mathbf{H} \setminus \mathscr{M}$ are isolated. A Kleinian group is *elementary* if its limit set has fewer than two points.

**7.7. Proposition.** *For each $\mu_{p/q}(n)$ there is an open neighborhood $U_{p/q}(n) \subset \mathbf{C}$ such that if $\mu \in U_{p/q}(n) \setminus \{\mu_{p/q}(n)\}$, then $G[\mu]$ is not discrete.*

*Proof.* Consider the subgroup

$$H_{p/q}(n) = \langle K, W^n_{p/q} \rangle, \tag{7.6}$$

where

$$K = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}. \tag{7.7}$$

By Jørgensen's inequality, [7, Lemma 1], we know that if

$$\tau(\mu) = |\operatorname{tr}(K W_{p/q}[\mu]^n K^{-1} W_{p/q}[\mu]^{-n} - 2| < 1, \tag{7.8}$$

then either $H_{p/q}(n)$ is elementary or it is not discrete. If $\mu = \mu_{p/q}(n)$, then $K W_{p/q}[\mu]^n K^{-1} W_{p/q}[\mu]^{-n}$ is the identity, $\tau(\mu) = 0$, and $H_{p/q}(n) = \langle K \rangle$ is discrete. If $W_{p/q}[\mu]$ is loxodromic, $H_{p/q}(n)$ cannot be elementary. Also, if $W_{p/q}[\mu]$ is elliptic and $H_{p/q}(n)$ is elementary, then it has to be of order $2$, $3$, $4$, or $6$ by the classification of elementary groups (see [21, Section V.D]). This can only happen for a finite number of values of $\mu$. The combination of these observations gives the desired neighborhood. ▫

## 8. Tangent circle chains and discreteness

In this section we prove Theorem 8.3, a technical result used in the proof of Theorem 9.4. We show how to use the existence of a tangent circle chain to establish the discreteness of a group of the form

$$G_n = \langle F_n, C \rangle, \tag{8.1}$$

where $F_n = \langle A, B \rangle$ is a triangle group of signature $(n, n, \infty)$, and $C$ conjugates the canonical generators $A$ and $B$ of $F_n$ as in Section 2. A tangent circle chain with an additional property (8.3) will be used to construct the closed topological disks required in Maskit's fourth version of his second combination theorem, [22, Theorem II], Theorem 8.1 below. The full statement of the combination theorem is quite long and general, and we will only state a much weaker version that will be sufficient for our application. Condition (B) in the theorem below is a restatement of the condition "cyclic stabilizers are parabolic" of [22, Theorem II]. When we replace the original condition by (B), we restrict to the case when new parabolics are created by the combination of $F$ and $C$. This is exactly the situation we have in Section 9.

Let $H$ be a nonelementary Kleinian group. The set of discontinuity $\Omega(H)$ has a natural hyperbolic metric of curvature $-1$. We will denote the hyperbolic area of $\Omega(H)/H$ by $\operatorname{area}(H)$.

**8.1. Theorem.** *Let $F = \langle A_n, B_n\rangle$ be a Fuchsian group of signature $(n,n,\infty)$, $n \geq 3$, with canonical generators $A_n$ and $B_n$, and let $J_1 = \langle A_n\rangle$ and $J_2 = \langle B_n\rangle$ be finite cyclic subgroups of $F$. Let $W_{p/q}$ be the word defined in Section 4. Assume the following*

(A) *For $m = 1,2$, there is a $J_m$-invariant closed topological disk $B_m$, with boundary loop $W_m \subset \Omega(F)$; there is a finite $J_m$-invariant set of points $\Theta_m \subset W_m$ such that $W_m$ is locally circular at points of $\Theta_m$; and there is a Möbius transformation, $C$, mapping the exterior of $B_1$ onto the interior of $B_2$, so that*

(A1) *for all $g \in F \setminus J_m$ and all points $x \in W_m \cap g(W_m)$, both $x$ and $g^{-1}(x)$ are points of $\Theta_m$,*

(A2) *if there is an $x \in B_1 \cap g(B_2)$ for some $g \in F$ then $x \in \Theta_1 \cap g(\Theta_2)$, and*

(A3) *$C$ conjugates $J_1$ onto $J_2$ and $C(\Theta_1) = \Theta_2$.*

(B) *$W_{p/q}[A_n, C]$ is parabolic, and*

(C) *$\widehat{\mathbf{C}} \setminus (B_1 \cup B_2) \neq \emptyset$.*

*Then,*

(1) *$G = \langle F, C\rangle = F*_C$,*

(2) *$G$ is discrete,*

(3) *every element of $G$ that is not a conjugate of an element of $F$ and is not a conjugate of $W_{p/q}[A_n, C]$ is loxodromic,*

(4) *if $D_0$ is a fundamental set of $F$ such that $D_0 \cap B_m$ is a fundamental set for the action of $J_m$ in $B_m$, for $m = 1,2$, then the set*

$$D = D_0 \setminus \bigl(\operatorname{int} B_1 \cup B_2 \cup G(\operatorname{fix} W_{p/q})\bigr) \tag{8.2}$$

*is a fundamental set of $G$, and*

(5) $\operatorname{area}(G) = \operatorname{area}(F) + 4\pi/n$.

Clearly, there can be non-discrete groups $G_n$ that possess tangent circle chains. We will work with tangent circle chains that satisfy the additional property

$$\delta_i \cap \delta_j = \emptyset \qquad \text{if } |i-j| > 1 \bmod nq. \tag{8.3}$$

In an infinite tangent circle chain satisfying (8.3), only the circle $\delta_i$ intersects the two circles $\delta_{i-1}$ and $\delta_{i+1}$, and for finite chains the same holds with the modifications caused by the fact that $\delta_j = \delta_{j+nq}$ for all $j \in \mathbf{Z}$.

In Proposition 9.2 we show that the circle chains of cusp groups $G[\mu_{p/q}]$ of $\mathscr{M}$ satisfy (8.3). In fact, they have two disjoint circle chains with closely related combinatorics, see Proposition 9.1 and [23]. In the proof of Theorem 9.4 we produce circle chains satisfying (8.3) for parameters on $\mathscr{P}^+_{p/q}$ by perturbing one of the chains of $G[\mu_{p/q}]$.

Let $n = 3, 4, \ldots$, and let $\{\delta_i\}$, $i \in \mathbf{Z}$ be a tangent $p/q$ circle chain for $G_n = \langle A_n, C\rangle$ satisfying (8.3). We will now construct two closed topological disks $\mathscr{D}_B$ and $\mathscr{D}_A$ that will be the disks $B_1$ and $B_2$ required to apply Theorem 8.1. Let $\gamma_i \subset \delta_i$ be the unique circular arc orthogonal to $\partial\,\delta_i$ connecting the points of tangency of $\delta_i$ with $\delta_{i-1}$ and $\delta_{i+1}$, $i \in \mathbf{Z} \bmod nq$. Let

$$\mathscr{W}_{A,0} = \bigcup_{i\in\mathbf{Z}} \gamma_i, \tag{8.4}$$

and

$$\mathscr{W}_{B,0} = C^{-1}(\mathscr{W}_{A,0}). \tag{8.5}$$

See Figure 6 for the construction. If $p/q \notin \mathbf{Z}$, then

$$\mathscr{W}_{A,0} \cap \mathscr{W}_{B,0} = \overline{\bigcup_{i=1}^{q-1}} \gamma_i. \tag{8.6}$$

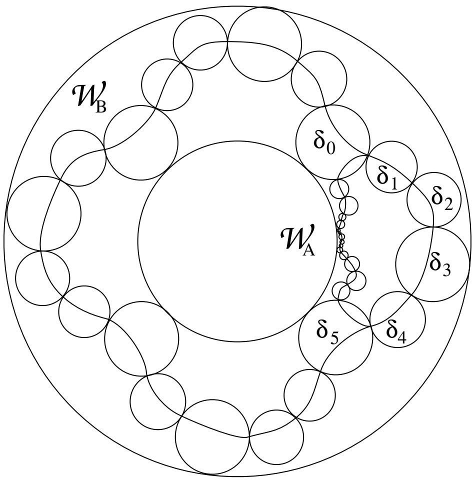


Figure 6. The construction of the Jordan curves $\mathscr{W}_{A,0}$ and $\mathscr{W}_{B,0}$ from a circle chain for $G_4$ satisfying (8.3).

Condition (8.3) implies that $\mathscr{W}_{A,0}$ and $\mathscr{W}_{B,0}$ are Jordan curves. Next we will modify the Jordan curves so that they meet the requirements of Theorem 8.1. The boundaries of the disks in (8.1) are only allowed to meet at the points of tangency of the disks $\delta_0, \dots, \delta_q$, not along a union of circular arcs as above. If $p/q \in \mathbf{Z}$, then $\mathscr{W}_{A,0}$ and $\mathscr{W}_{B,0}$ are round disks, tangent at exactly one point, and we do not change them.

Let $p/q \in \mathbf{Q} \setminus \mathbf{Z}$. In our application of Theorem 8.1, we are mainly interested in the discreteness of the group in question, so we can assume after a change of generators that $0 < p/q < 1$.

Let $\gamma$ be a directed geodesic in $\mathbf{D}$ connecting the points $z_1$ and $z_2$ on $\partial\mathbf{D}$, $z$ a point on $\gamma$, and $\varepsilon > 0$. Let $z(\varepsilon)$ be the point on the geodesic perpendicular to $\gamma$ at $z$ that lies the distance $\varepsilon$ to the right from $z$. We denote by $\gamma(\varepsilon)$ the union of the geodesic rays connecting $z_1$ to $z(\varepsilon)$ and $z(\varepsilon)$ to $z_2$, including the end points $z_1, z_2 \in \partial\mathbf{D}$.

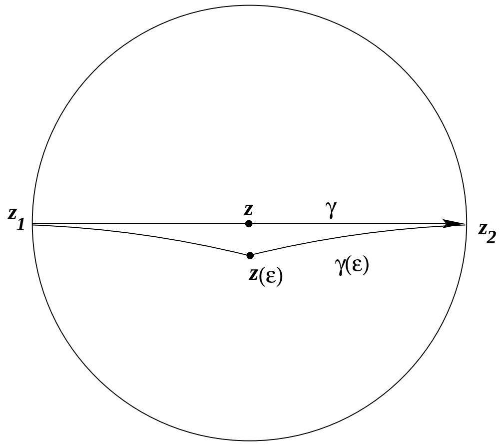


Figure 7. The modification of $\mathscr{W}_A$ inside a disk of the circle chain.

Let $i \in \{1, 2, \dots, q\}$, and define $V_i$ to be the word in $A$ and $C$ constructed as in 6.2(2) such that $V_i(\delta_0) = \delta_i$. Let $k(i)$ be the number of $C$'s in the word $V_i$. Set

$$
\begin{aligned}
&\gamma_i^A = \gamma_i\bigl(k(i) - 1)\varepsilon\bigr), && \text{for } i = 1, 2, \dots, q, \\
(8.7)\qquad &\mathscr{W}_A = \bigcup_{j \in \mathbf{Z}} \bigcup_{i=1}^{q} A_n^j(\gamma_i^A), && \text{and} \\
&\mathscr{W}_B = C^{-1}(\mathscr{W}_A).
\end{aligned}
$$

Let $\mathscr{D}_A$ be the component of $\widehat{\mathbf{C}} \setminus \mathscr{W}_A$ contained in the same component of $\widehat{\mathbf{C}} \setminus \Lambda(F)$ as $\mathscr{W}_A$. Let $\mathscr{D}_B$ be the component of $\widehat{\mathbf{C}} \setminus \mathscr{W}_B$ contained in the same component of $\widehat{\mathbf{C}} \setminus \Lambda(F)$ as $\mathscr{W}_B$. Similarly, $\mathscr{D}_{A,0}$ and $\mathscr{D}_{B,0}$ denote the corresponding components of $\widehat{\mathbf{C}} \setminus \mathscr{W}_{A,0}$ and $\widehat{\mathbf{C}} \setminus \mathscr{W}_{B,0}$. For use in Lemma 8.2 we extend $k$ to $\mathbf{Z}$ as a $q$-periodic function.

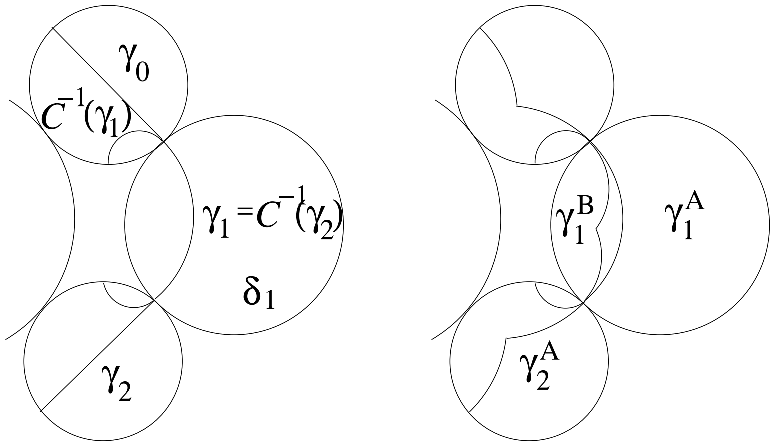


Figure 8. An example of the modification of $\mathscr{W}_A$ and $\mathscr{W}_B$ in the case $p/q = 1/2$.

**8.2. Lemma.** *If* $\operatorname{int} \mathscr{D}_{A,0} \cap \operatorname{int} \mathscr{D}_{B,0} = \emptyset$, *then* $\overline{\mathscr{D}_A} \cap \overline{\mathscr{D}_B} = \bigcup_{i=1}^{q} \delta_i \cap \delta_{i-1}$.

*Proof.* The only thing to check is that the modification has separated $\overline{\mathscr{D}_A}$ and $\overline{\mathscr{D}_B}$ from each other inside the disks $\delta_i$, $i = 1, 2, \dots, q-1$. Let us denote by $\gamma_i^B$ the part of $\mathscr{W}_B$ contained in $\delta_i$. By definition,

$$\begin{aligned} \gamma_i^A &= \gamma\big((k(i)-1)\varepsilon\big) \qquad \text{and,} \\ \gamma_i^B &= C^{-1}(\gamma_{i+p}^A) = \gamma\big((k(i+p)-1)\varepsilon\big). \end{aligned} \tag{8.8}$$

Clearly, $k(i) = k(i+p) - 1$, and this gives the claim. □

**8.3. Theorem.** *Let* $G_n = \langle F, C \rangle$, *where* $F = \langle A_n, B_n \rangle$ *is a triangle group of signature* $(n, n, \infty)$ $(n \in \mathbf{N} \cup \{\infty\},\ n \geq 3)$ *with canonical generators* $A_n$ *and* $B_n$, *and let* $C$ *be a transformation satisfying*

$$C^{-1} A_n C = B_n^{-1}. \tag{8.9}$$

*If* $G_n$ *has a tangent* $p/q$*-chain satisfying* (8.3) *such that*

(i) *the interiors of the disks* $\mathscr{D}_A$ *and* $\mathscr{D}_B$, *as constructed above, are disjoint, and*
(ii) *the transformations* $W_{p/q}[A_n, C]$ *are parabolic,*

*then*

(1) $G_n = F *_C$,
(2) $G_n$ *is discrete,*
(3) $G_n$ *is geometrically finite,*
(4) $\Omega(G_n)/G_n = S_1 \cup S_2$, *where* $S_1$ *has signature* $(0, 3; n, n, \infty)$, *and* $S_2$ *has signature* $(0, 3; \infty, \infty, \infty)$.

To prove this theorem we need to check that the existence of a chain satisfying (8.3) implies that the assumptions of Theorem 8.1 are satisfied. We use the following terminology in the proof:

**8.4. Definition.** Let $G$ and $H \subset G$ be Kleinian groups. A set $E \subset \widehat{\mathbf{C}}$ is *precisely invariant* under $H$ in $G$, or *precisely $H$-invariant*, if $h(E) = E$ for all $h \in H$ and if $g(E) \cap E = \emptyset$ for all $g \in G \setminus H$.

*Proof of Theorem* 8.3. Without loss of generality we can assume that $F$ is normalized as in Section 2. Set

$$\Theta_A = \bigcup_{i \in \mathbf{Z}} A^i(\mathscr{W}_A \cup \mathscr{W}_B), \qquad \Theta_B = \bigcup_{i \in \mathbf{Z}} B^i(\mathscr{W}_A \cup \mathscr{W}_B). \tag{8.10}$$

Note that the curves $\mathscr{W}_A$ and $\mathscr{W}_B$ are not locally circular at $\Theta_A$ and $\Theta_B$, respectively. However, as remarked in [22, Section 0.2.3], this requirement can be weakened considerably. In our situation (see Figures 7 and 8) the curves $\mathscr{W}_A$ and $\mathscr{W}_B$ consist of circular arcs that are perpendicular to the boundaries of the disks $\delta_i$ and $C^{-1}(\delta_i)$ at the points of $\Theta_A$ and $\Theta_B$. Also, we can make a small local perturbation of the curves $\mathscr{W}_A$ and $\mathscr{W}_B$ at their points of tangency such that the curves are locally circular at these points without introducing new points of tangency, or changing the mapping properties (8.7) of the disks $\mathscr{D}_A$ and $\mathscr{D}_B$.

(B) of Theorem 8.1 is assumption (ii). Also, $\mathbf{D} \subset \widehat{\mathbf{C}} \setminus (\mathscr{D}_A \cup \mathscr{D}_B)$, and thus (C) holds.

Let us check that assumption (A) of Theorem 8.1 is satisfied. The disks $\mathscr{D}_A$ and $\mathscr{D}_B$ are by definition invariant under the cyclic groups generated by $A$ and $B$, respectively. The same applies to $\mathscr{W}_A$, $\mathscr{W}_B$, $\Theta_A$ and $\Theta_B$. The transformation $C$ maps $\mathscr{W}_B$ onto $\mathscr{W}_A$. Also, $C$ maps the fixed points of $B$ onto the fixed points of $A$. The conjugation relation (8.9) is chosen in such a way that the fixed point of $B$ contained in $\mathbf{D}$ is mapped to $\infty \in \mathscr{D}_A$. Thus,

$$C\bigl(\operatorname{int}(\widehat{\mathbf{C}} \setminus \mathscr{D}_B)\bigr) = \operatorname{int} \mathscr{D}_A, \tag{8.11}$$

as required.

Let us now consider the conditions (A1)–(A3) of Theorem 8.1. (A3) is clear from (8.9) and (8.10). (A1) holds if and only if the set

$$\mathscr{D}'_A = \mathscr{D}_A \setminus \Theta_A \tag{8.12}$$

is precisely invariant under $\langle A \rangle$, and

$$\mathscr{D}'_B = \mathscr{D}_B \setminus \Theta_B \tag{8.13}$$

has the the corresponding property. (A2) is true if we can show that for all $g \in F$,

$$g(\mathscr{D}'_A) \subset \widehat{\mathbf{C}} \setminus \mathscr{D}'_B. \tag{8.14}$$

If, in addition to (8.12)–(8.14), we have for all $g \in F$,

$$g(\mathscr{D}'_B) \subset \widehat{\mathbf{C}} \setminus \mathscr{D}'_A, \tag{8.15}$$

and

$$C(\widehat{\mathbf{C}} \setminus \mathscr{D}_B) \subset \mathscr{D}'_A \tag{8.16}$$

and

$$C^{-1}(\widehat{\mathbf{C}} \setminus \mathscr{D}_A) \subset \mathscr{D}'_B, \tag{8.17}$$

then the triple $\big(\mathscr{D}'_B, \mathscr{D}'_A, \widehat{\mathbf{C}} \setminus (\mathscr{D}_A \cup \mathscr{D}_B)\big)$ is called an *interactive triple* ([21, VII.D.7]). We will show that this is the case.

By Maskit [21, Propositions VII.E.3 and VII.E.4], it is enough to show that

(i) $\mathscr{D}'_A$ is precisely $\langle A \rangle$-invariant,
(ii) $\mathscr{D}'_B$ is precisely $\langle B \rangle$-invariant in $F$, and
(iii) $C$ maps $\widehat{\mathbf{C}} \setminus \mathscr{D}'_B$ onto $\operatorname{int} \mathscr{D}_A \cup \Theta_A$.

Clearly, (iii) follows from (8.11) and the fact that $C(\Theta_B) = \Theta_A$. By [21, Proposition II.I.3], (i) and (ii) are equivalent with the existence of a fundamental set $P$ for $F$ such that $P \cap \mathscr{D}'_A$ is a fundamental set for the action of $\langle A \rangle$ in $\mathscr{D}'_A$, and $P \cap \mathscr{D}'_B$ is a fundamental set for the action of $\langle B \rangle$ in $\mathscr{D}'_B$.

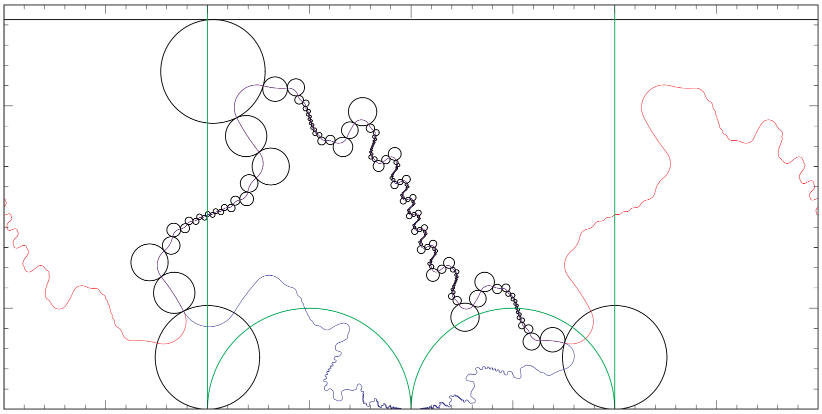

Figure 9. The disks $\delta_0, \ldots, \delta_{229}$ of the circle chain for $G_\infty[\mu_{21/229}]$. $\delta_{21} = C(\delta_0)$ is the disk tangent to the line $\operatorname{Im} z = \operatorname{Im} \mu_{21} \approx 1.92$. The curve $\mathscr{W}_A$ intersects the disk $\{|z - \frac{1}{2}| < \frac{1}{2}\}$. One obtains the fundamental set $P$ by adding the set $\mathscr{D}_A \cap \{|z - \frac{1}{2}| < \frac{1}{2}\}$ to the standard fundamental polygon $P_0 = \{z \in \mathbf{C} : -1 \le \operatorname{Im} z < 1,\ |z + \frac{1}{2}| \ge \frac{1}{2},\ |z - \frac{1}{2}| > \frac{1}{2}\}$ and by removing its image $B(\mathscr{D}_A \cap \{|z - \frac{1}{2}| < \frac{1}{2}\})$.

We now construct this fundamental set. The property (8.11) implies

$$C(\mathscr{D}_A) \subset \mathscr{D}_A. \tag{8.18}$$

By the $A$-invariance of $\mathscr{D}_A$

$$A^{-1}C(\mathscr{D}_A) \subset \mathscr{D}_A. \tag{8.19}$$

The basic combinatorial properties of circle chains (see Definition 6.1) imply

$$A^{-1}C(\delta_{q-1}) = \delta_{p-1}. \tag{8.20}$$

Thus, $C^{-1}A^{-1}C(\delta_{q-1})$ is a disk that meets $\mathscr{W}_B$, but not $\mathscr{W}_A$. Similarly, we see that for all $i = 1, \dots, q-1$ the part of the boundary of $\mathscr{D}_A$ inside $\delta_i$, namely $\gamma_i^A$, is mapped by $B$ into the part of $\mathscr{W}_B$ not touching $\mathscr{W}_A$. For $B^{-1}$ we have the corresponding result by the same kind of reasoning: for $i = 1, \dots, q-1$, we have

$$AC(\delta_i) = \delta_{p+q+i}. \tag{8.21}$$

As $i > 0$, the combinatorial properties of the $p/q$ chain imply that the part of $\mathscr{W}_A$ inside $\delta_i$ is mapped into the part of $\mathscr{W}_B$ not touching $\mathscr{W}_A$. Thus, the images $B^{\pm 1}(\operatorname{int}\mathscr{D}_A)$ are disjoint from $\mathscr{D}_A$. We can now modify the 'standard' Ford fundamental polygon of the triangle group $F$ to obtain the required fundamental set (see Figure 6 for an illustration, and Figure 9 for a complicated case for $n = \infty$).

Now, claims (1)–(3) follow directly from Theorem 8.1.

*Proof of claim* (4). Theorem 8.1 gives

$$\operatorname{area}(G) = 2\cdot 2\pi\left(1-\frac{2}{n}\right) + 4\pi/n = 2\pi\left(1-\frac{2}{n}\right) + 2\pi. \tag{8.22}$$

By (1) we know that the parabolics $A$, $AB$, and $W_{p/q}$ define different conjugacy classes in $G$. Theorem 8.1(4) implies that the action of the stabilizer of $\mathbf{D}$ in $G$ is the same as the action of $F$ on $\mathbf{D}$. Thus, the quotient $\Omega(G)/G$ contains one sphere with a puncture and two cone points of order $n$, contributing $2\pi(1-2/n)$ to the area (8.22). Furthermore, the fundamental set $D$ given by Theorem 8.1(4) shows that $AB$ corresponds to one puncture, and $W_{p/q}$ to two distinct punctures on the remainder of the quotient. The area of a surface with genus $g$ and $n$ punctures is $2\pi(2g-2+n)$. There are at least three punctures in $\big(\Omega(G)/G\big) \setminus \big(\mathbf{D}/\operatorname{Stab}(\mathbf{D})\big)$. (8.22) now implies that the quotient is as in claim (4). □

## 9. The discreteness and conjugacy theorems

In this section we study a tangent combinatorial circle chain $\{\Delta_i[\mu]\}_{i\in\mathbf{Z}}$ for $G_\infty[\mu]$, such that the disks in the chain are translates of $\mathbf{H}^*$. Keen and Series [10] and Wright [31] showed that there is an infinite $p/q$-combinatorial circle chain $\{\delta_i[\mu]\}_{i\in\mathbf{Z}}$ for $G[\mu]$, $\mu \in \overline{\mathscr{P}_{p/q}} \subset \overline{\mathscr{M}}$, with $\delta_0[\mu] \subset \mathbf{H}$ one of the invariant disks of

$$F_{p/q}[\mu] = \langle W_{p/q}[\mu], \widetilde{W}_{p/q}[\mu]\rangle. \tag{9.1}$$

Here $p/q, r/s \in \mathbf{Q}$ are Farey neighbors and we use the notation

$$\widetilde{W}_{p/q} = W_{r/s}^{-1} W_{p/q}^{-1} W_{r/s}. \tag{9.2}$$

If $\mu \in \mathscr{P}_{p/q}$, then the chain $\{\delta_i[\mu]\}$ is proper, and for the cusp parameter $\mu_{p/q}$ it is a tangent chain. The chains $\{\Delta_i\}$ are "dual" to the chains $\{\delta_i\}_{i\in\mathbf{Z}}$ in a sense that is made precise in Proposition 9.1.

In this section the roles of the subgroups $F_{p/q}$ and $\Gamma_0 = \operatorname{Stab}\mathbf{H}^*$ are interchanged and we view the group $G$ as an extension of $F_{p/q}$ by $W_{p/q}$. This has the following advantage: the chains $\{\Delta_i[\mu]\}_{i\in\mathbf{Z}}$ constructed below are tangent for all $\mu \in \mathbf{C}$ (see Proposition 9.2), and for large values of $n$ we can use these chains and Theorem 8.3 to prove the discreteness of the groups $G[\mu_{p/q}(n)]$.

Recall that at the cusp $\mu_{p/q}$, $F_{p/q}[\mu_{p/q}]$ is a torsion-free triangle group. The quotient $\Omega(G[\mu_{p/q}])/G[\mu_{p/q}]$ is the union of two thrice punctured spheres, one corresponding to $\mathbf{H}^*/\langle S, \widetilde{S}\rangle$, the other to $\delta_0/F_{p/q}$. McShane, Parker and Redfern [23] made the following useful observation:

**9.1. Proposition.** *$G[\mu_{p/q}]$ is the HNN extension of the torsion-free triangle group $F_{p/q}$ by $W_{r/s}^{-1}$. In particular, if $pp' = 1 \bmod q$, then $G[\mu_{p/q}]$ is the $p'/q$ cusp in the deformation space of terminal $b$-groups of the form $F_{p/q} *_{T'}$ representing a punctured torus on their invariant component.*

*Proof.* [23, Proposition 7.1].

Proposition 9.1 involves writing the group element $S$ (that is parabolic for all $\mu$) in terms of the new generators $W_{p/q}$ and $U = W_{r/s}^{-1}$. Note that the conjugates of $W_{p'/q}[W_{p/w}, U]$ in $G[\mu]$, being conjugates of $S$, are parabolic for all $\mu$. This turns out to be quite useful for our purposes.

Proposition 9.1 implies that, in addition to a tangent $p/q$-chain $\{\delta_i\}_{i\in\mathbf{Z}}$ for the generators $S$ and $T$, there is an infinite tangent $p'/q$ combinatorial circle chain $\{\Delta_i\}_{i\in\mathbf{Z}}$ for $\langle W_{p/q}[\mu_{p/q}], U[\mu_{p/q}]\rangle$, where $\Delta_0 = \mathbf{H}^*$. The two chains $\{\delta_i\}$ and $\{\Delta_i\}$ do not intersect. Thus,

$$\Delta_1 = T(\mathbf{H}^*) = \mathbf{H} + \mu, \tag{9.3}$$

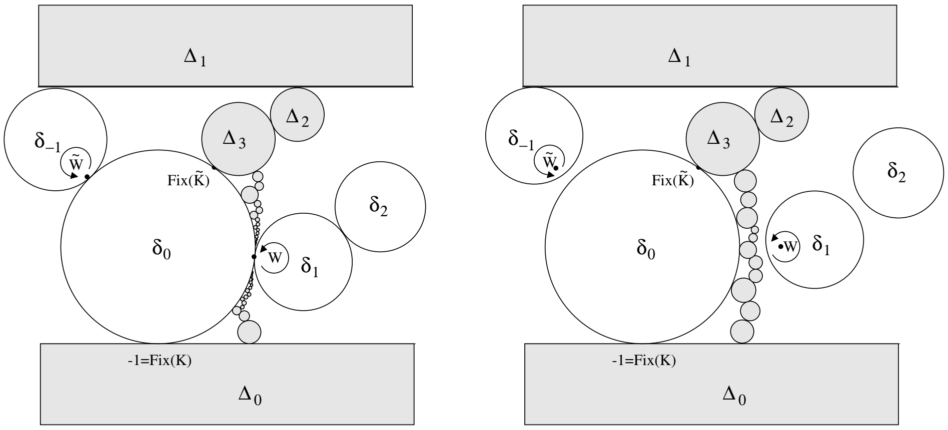


Figure 10. A schematic picture of how the two circle chains fit together for a cusp group (on the left) and for a parameter $\mu_{p/q}(5)$. For the cusp group, the grey disks form part of the chain $\{\Delta_i\}$ that accumulates at the fixed point of $W_{p/q}$.

see Figure 10. When the parameter $\mu$ moves on $\mathscr{P}^+_{p/q} \setminus \mathscr{M}$ away from the cusp, Proposition 9.2 below shows that the chain $\{\Delta_i\}$ remains tangent. On the other hand, the transformation $W_{p/q}[\mu]$ is elliptic for these parameters, and it cannot stabilize two tangent circles. Thus, the circle chain $\{\delta_i\}$ separates, as in Figure 10. We will use the chain $\{\Delta_i\}$ and Theorem 8.3 to prove Theorem 9.4.

Let $\Gamma_i$ be the stabilizer of $\Delta_i$ in $G[\mu_{p/q}]$. The map $\mu \mapsto \langle S, T[\mu]\rangle = G[\mu]$ induces a homomorphism from $G[\mu_{p/q}]$ (isomorphic to the free group on two generators) onto any $G[\mu]$, $\mu \in \mathbf{C}$. Let us denote by $\Gamma_i[\mu]$ the image of $\Gamma_i = \Gamma_i[\mu_{p/q}]$ by this homomorphism. As the group

$$\Gamma_0[\mu] = \mathrm{Stab}_{G[\mu]}\,\mathbf{H}^* = \langle S, \widetilde{S}\,\rangle = \Gamma_0 \tag{9.4}$$

does not depend on the parameter $\mu$, the groups $\Gamma_i[\mu]$, as conjugates in $G[\mu]$ of $\Gamma_0$, are discrete for all $\mu \in \mathbf{C}$. Thus, there is a naturally defined disk $\Delta_i[\mu]$ for each $\mu \in \mathbf{H}$, bounded by the limit set of $\Gamma_i[\mu]$. In particular,

$$\Delta_i[\mu_{p/q}] = \Delta_i. \tag{9.5}$$

This defines a $p'/q$-combinatorial circle chain $\{\Delta_i[\mu]\}_{i\in\mathbf{Z}}$ for $\mu \in \mathscr{P}^+_{p/q}$ as a perturbation of the chain $\{\Delta_i\}_{i\in\mathbf{Z}}$.

**9.2. Proposition.** (1) *$\Delta_i[\mu]$ and $\Delta_{i+1}[\mu]$ are tangent for any $\mu \in \mathbf{H}$.*

(2) *If $|i-j| > 1$, then $\Delta_i[\mu_{p/q}] \cap \Delta_j[\mu_{p/q}] = \emptyset$, that is, the circle chain $\{\Delta_i\}_{i\in\mathbf{Z}}$ satisfies* (8.3).

*Proof.* (1) $\Delta_i[\mu]$ and $\Delta_{i+1}[\mu]$ are stabilized by the same parabolic transformation conjugate in $G[\mu]$ to $W_{p'/q}[W_{p/q}, U]$, which is conjugate to $S^{\pm 1}$ in $G[\mu]$, and therefore parabolic for all $\mu$. Thus the disks are tangent at the fixed point of this element.

(2) Assume $j > i+1$ and that there is a point $x \in \Delta_i[\mu_{p/q}] \cap \Delta_j[\mu_{p/q}]$. By Maskit [20, Theorem 3]:

$$\Lambda(\Gamma_i) \cap \Lambda(\Gamma_j) = \Lambda(\Gamma_i \cap \Gamma_j). \tag{9.6}$$

$\Delta_i[\mu_{p/q}]$ and $\Delta_j[\mu_{p/q}]$ are round disk components of $G[\mu_{p/q}]$, so they can intersect in at most one point. Thus $\Gamma_i \cap \Gamma_j = \langle P \rangle$, where $P$ is a parabolic element fixing $x$. There are exactly three conjugacy classes of parabolic elements in $G[\mu_{p/q}]$. Thus $P$ has to be conjugate to one of $W_{p/q}$, $K$ or $S$. All of these elements correspond to punctures on $\mathbf{H}^*/\Gamma_q$ or $\delta_0/F_{p/q}$. Two translates of $\mathbf{H}^* = \Delta_0$ meet at $x \in \widehat{\mathbf{C}}$, the fixed point of $P$, implying that the transformation $P$ can only correspond to punctures on $\Delta_i/\Gamma_i$. This implies that $K$ and $W_{p/q}$ are not possible because they correspond to at least one puncture on $\delta_0/F_{p/q}$. The only remaining possibility is that the disks $\Delta_i[\mu_{p/q}]$ and $\ldots \Delta_j[\mu_{p/q}]$ are tangent at the fixed point of a parabolic $P[\mu_{p/q}]$, where $P[\mu_{p/q}]$ is conjugate in $G[\mu_{p/q}]$ to $S$. The homomorphism induced by $\mu \mapsto G[\mu]$ induces an isomorphism for parameters $\mu \in \overline{\mathscr{M}}$. Thus, there is a uniquely determined parabolic transformation

$$P[\mu] \in \Gamma_i[\mu] \cap \Gamma_j[\mu] \subset G[\mu] \tag{9.7}$$

for all $\mu \in \overline{\mathscr{M}}$. But then, because the groups $\Gamma_i[\mu] = \mathrm{Stab}(\Delta_i[\mu])$ are Fuchsian groups of the first kind for all $\mu \in \mathbf{C}$, there is a closed curve

$$\mathscr{C} \subset \bigcup \partial\Delta_i[\mu] \subset \Lambda(G[\mu]) \tag{9.8}$$

that divides $\widehat{\mathbf{C}} \backslash \mathscr{C}$ into two disjoint open sets $E$ and $E'$ such that $\Lambda(G[\mu]) \cap E \neq \emptyset$, and $\Lambda(G[\mu]) \cap E' \neq \emptyset$. But this is a contradiction; if $\mu \in \mathscr{M}$, then $G[\mu]$ has a simply connected invariant component $\Omega_0$, and $\partial\Omega_0 = \Lambda(G[\mu])$. □

We need to show that for $\mu \in \mathscr{P}^+_{p/q} \setminus \mathscr{M}$ close to the cusp $\mu_{p/q}$, the disks $\Delta_0[\mu], \ldots, \Delta_q[\mu]$ still satisfy (8.3). This will be a consequence of the following lemma. If $K$ and $K'$ are compact sets, we denote by $d(K, K')$ the *Hausdorff distance* of $K$ and $K'$, that is,

$$d(K, K') = \inf\{r > 0 : K \subset K'_r \text{ and } K' \subset K_r\}, \tag{9.9}$$

where

$$K_r = \{z \in \mathbf{C} : |z - z_0| < r \text{ for some } z_0 \in K\}. \tag{9.10}$$

**9.3. Lemma.** *Let $p/q \in \mathbf{Q}$, and $\varepsilon > 0$. There is an $\eta > 0$ such that if $|\mu - \mu_{p/q}| < \eta$, then for all $i = 1, 2, \ldots, q$,*

$$d\big(\Lambda\big(\Gamma_i[\mu_{p/q}]\big), \Lambda\big(\Gamma_i[\mu]\big)\big) < \varepsilon. \tag{9.11}$$

*Proof.* Let $p = p(\mu_{p/q}) \in \Lambda(\Gamma_i[\mu_{p/q}])$ be a fixed point of a hyperbolic transformation of $\Gamma_i[\mu_{p/q}]$, and let $c = c(\mu_{p/q})$ be the center of $\Lambda(\Gamma_i[\mu_{p/q}])$. $c(\mu)$ can be expressed as a continuous function of three distinct points on $\Lambda(\Gamma_i[\mu])$. Thus, there is a neighborhood $\mathscr{U} \subset \mathbf{C}$ of $\mu_{p/q}$ such that the maps $p\colon \mathscr{U} \to \mathbf{C}$ and $c\colon \mathscr{U} \to \mathbf{C}$ induced by the map $\mu \mapsto G[\mu]$, are continuous. This implies that there is an $\eta > 0$ such that for $|\mu - \mu_{p/q}| < \eta$, $|p(\mu) - p(\mu_{p/q})| < \frac{1}{3}\varepsilon$, and $|c(\mu) - c(\mu_{p/q})| < \frac{1}{3}\varepsilon$. Now, using the fact that $\Lambda(\Gamma_i[\mu])$ is a circle, we have

$$
\begin{aligned}
|z - c(\mu_{p/q})| &\le |z - c(\mu)| + |c(\mu) - c(\mu_{p/q})| \\
&= |p(\mu) - c(\mu)| + |c(\mu) - c(\mu_{p/q})| \\
&= |p(\mu) - p(\mu_{p/q})| + |p(\mu_{p/q}) - c\mu_{p/q})| + 2|c(\mu) - c(\mu_{p/q})| \\
&< |p_1(\mu_0) - c(\mu_0)| + \varepsilon
\end{aligned} \tag{9.12}
$$

for any point $z \in \Lambda(\Gamma_i[\mu])$. The claim follows from this by another easy application of the triangle inequality, and by observing that the same holds for the difference $|z - c(\mu)|$ for points $z \in \Lambda(\Gamma_i[\mu_{p/q}])$. □

**9.4. Theorem.** *On the extension $\mathscr{P}^+_{p/q}$ of each rational pleating ray there is an open neighborhood $\mathscr{U}$ of the cusp $\mu_{p/q}$ on $\mathscr{P}^+_{p/q}$ such that if $\mu \in \mathscr{P}^+_{p/q} \cap \mathscr{U}$ and $|\operatorname{tr} W_{p/q}| = 2\cos(\pi/n)$ for some $n \in \mathbf{N}$, $n \ge 3$, then*

$$G[\mu] = F *_{W_{r/s}[S,T[\mu]]} . \tag{9.13}$$

*For these values of $\mu$, $G[\mu]$ is a Kleinian group representing a thrice punctured sphere and a sphere with a puncture and two branch points of order $n$ on its ordinary set.*

*If $\mu \in \left(\mathscr{P}^+_{p/q} \cap \mathscr{U}\right) \setminus \bar{\mathscr{M}}$ is not of this form, then $G[\mu]$ is not discrete.*

*Proof.* The idea of the proof is to show that we can apply Theorem 8.3 in our situation. The finite tangent circle chains are obtained by perturbing the dual chain $\{\Delta_i\}_{i\in\mathbf{Z}}$. At the cusp $\mu_{p/q} \in \partial\mathscr{M}$ the tangent combinatorial $(p'/q)$-chain $(\Delta_i)_{i\in\mathbf{Z}}$ can be written as the collection

$$\{W^i_{p/q}(\Delta_j) : i \in \mathbf{Z},\ 0 \le j \le q-1\}. \tag{9.14}$$

By Proposition 9.2 we have

$$\Delta_i \cap \Delta_j = \emptyset \tag{9.14}$$

if $|i-j| > 1$. By Lemma 9.3 the finite piece $\Delta_0, \dots, \Delta_{q-1}$ is deformed continuously in $\mu$, and no new intersections of the $q$ first disks $\Delta_0[\mu], \dots, \Delta_{q-1}[\mu]$ are produced if $|\mu_{p/q} - \mu|$ is small. Thus, we can change $\mu$ in a small open set $\mathscr{U}$ containing $\mu_{p/q}$ such that

$$\Delta_i \cap \Delta_j = \emptyset \tag{9.16}$$

holds for all $\mu \in \mathscr{U}$ and all $i = 0, \ldots, q-1$. Let $\mu_{p/q}(n)$ be a parameter on $\mathscr{P}^+_{p/q}$ such that $W_{p/q}$ is primitive elliptic as in Lemma 7.3. If $\mu_{p/q}(n) \in \mathscr{U} \cap \mathscr{P}^+_{p/q}$, then the disks

$$\left\{W_{p/q}[\mu_{p/q}(n)]^i\big(\Delta_j[\mu_{p/q}(n)]\big) : i \in \mathbf{Z} \bmod n,\ 0 \le j \le q-1\right\} \tag{9.17}$$

form a finite combinatorial $(p'/q)$-chain. Also, after possibly making $\mathscr{U}$ slightly smaller, the interiors of the topological disks $\mathscr{D}_A$ and $\mathscr{D}_B$, as constructed in Section 8, are disjoint.

In order to apply apply Theorem 8.3 on discreteness and circle chains we need that the word $W_{p'/q}[W_{p/q}[S, T[\mu]], W_{r/s}\big[S, T[\mu]\big]\big]$ is parabolic for $\mu = \mu_{p/q}(n)$. But this is so, because it is a conjugate of $S$. ▫

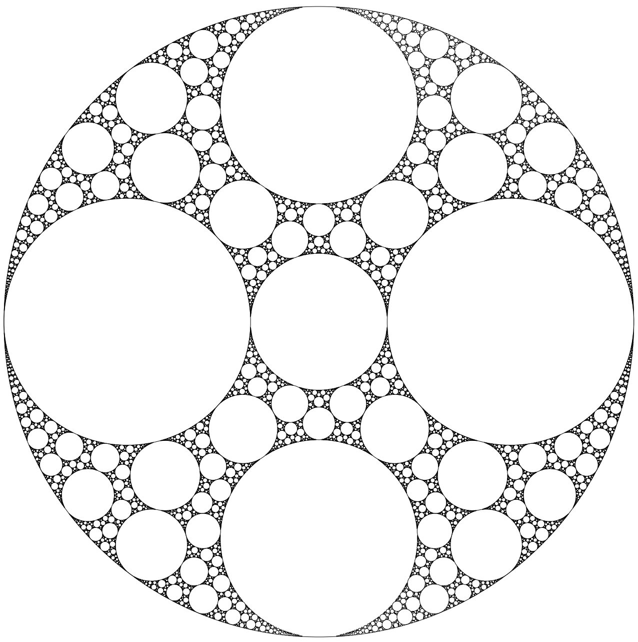

Figure 11. The limit set the Kleinian group on the extended ray $\mathscr{P}^+_{1/2}$ for the parameter $\mu$ such that $W_{1/2}$ is an elliptic of order $4$.

The proof of Corollary 9.6 is similar to that of Theorem III of Keen, Maskit and Series [8] on the uniqueness of maximally parabolic groups.

**9.5. Theorem.** *Let $G$ and $G'$ be geometrically finite Kleinian groups. If $f\colon \Omega(G) \to \Omega(G')$ is a conformal homeomorphism that induces an isomorphism $\varphi\colon G \to G'$, then $f$ is a Möbius transformation.*

*Proof.* Tukia [30, Theorem 4.2].

**9.6. Corollary.** *Let $\mu_{p'/q,n}$ be the boundary point of $\mathscr{M}_n$ on the ray $\mathscr{P}^n_{p'/q}$. Then the group $G[\mu_{p/q}(n)]$ is conjugate to $G_n[\mu_{p'/q,n}]$ by a Möbius transformation.*

*Proof.* Let $n \in \{3,4,\dots\}$ be such that Theorem 9.4 applies, and the group $G[\mu_{p/q}]$ is discrete. By Theorem 9.4, $G[\mu_{p/q}]$ is the HNN-extension of a triangle group of signature $(n,n,\infty)$. Thus it is of the form $G_n[\mu_0]$ for some $\mu_0 \in \mathbf{C} \setminus \overline{\mathbf{D}}$. Furthermore, $G_n[\mu_0]$ has a tangent $p'/q$-circle chain. Using this fact, we can construct a conformal map $P_n\colon \Omega(G_n[\mu_0]) \to \Omega(G_n[\mu_{p'/q}])$ that induces an isomorphism between these groups. By Theorem 9.5, $P_n$ is the restriction of a Möbius transformation. □

**9.7. Remark.** It is easy to check using (2.14) and (2.15) that for all $n = 3,4,\dots$, the end points of the integral rays in $\mathscr{M}_n$ are of the form

$$\mu_{m,n} = e^{2\pi i/n}\left(\frac{1+\sin(\pi/n)}{\cos(\pi/n)}\right)^2 = e^{2\pi i/n}\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)}. \tag{9.18}$$

Now one can check Theorem 9.4 for integral values of $p/q$ by conjugating the groups $G_n[\mu_{m,n}]$ in such a way that $C[\mu_{m,n}]$ is conjugated to $S$ and $A_n$ is conjugated to $T[\mu_0(n)]$, where $A_n$, $C$, $S$ and $T$ are as in Section 2 and

$$\mu_0(n) = 2\cos(\pi/n) \tag{9.19}$$

is as in Lemma 7.3.

$G[\mu_0(n)]$ is discrete for all $n \geq 2$: $T[\mu_0(2)](z) = G[0](z) = 1/z$ is an elliptic of order 2. Thus, $G[0] \subset \mathrm{PSL}(2,\mathbf{Z}[i])$ is a discrete group. It is easy to see that $\Omega(G[0])/G[0]$ consists of a single sphere with three punctures.

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