Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 465–488

UPPER SETS AND QUASISYMMETRIC MAPS

D.A. Trotsenko[1] **and J. Väisälä**

Academy of Sciences, Institute of Mathematics
Universitetskij prospekt 4, 630090 Novosibirsk, Russia; trotsenk@math.nsc.ru
University of Helsinki, Department of Mathematics
PL 4, Yliopistonkatu 5, FIN-00014 Helsinki, Finland; jvaisala@cc.helsinki.fi

Abstract. The upper set $\tilde{A}$ of a metric space A is a subset of $A \times (0, \infty)$, consisting of all pairs $(x, |x-y|)$ with $x, y \in A$, $x \neq y$. We consider various properties of $\tilde{A}$ and a metric of $\tilde{A}$, called the broken hyperbolic metric. The theory is applied to study basic properties of quasisymmetric maps.

1. Introduction

1.1. Let A be a metric space, where the distance between points a, b is written as $|a-b|$. The *upper set* of A is the subset

$$\tilde{A} = \{(x, |x-y|) : x \in A,\ y \in A \setminus \{x\}\}$$

of $A \times \mathsf{R}_+$, $\mathsf{R}_+ = (0, \infty)$. We assume that A contains at least two points in order that $\tilde{A}$ be nonempty. If $A \subset \mathsf{R}^n$, then $\tilde{A} \subset \mathsf{R}^{n+1}_+ = \mathsf{R}^n \times \mathsf{R}_+$, and we can consider the hyperbolic metric ϱ_h of R^{n+1}_+ in $\tilde{A}$. However, we find it convenient to replace ϱ_h by another metric ϱ, which is bilipschitz equivalent to ϱ_h, and which is easy to consider also in the case where A is an arbitrary metric space. The precise definition of ϱ is given in 2.2.

For each $\lambda > 0$ we partition $\tilde{A}$ into λ-components, where points z and z' belong to the same λ-component if they can be joined by a finite sequence $z = z_0, \dots, z_N = z'$ in $\tilde{A}$ with $\varrho(z_{j-1}, z_j) \le \lambda$ for all j. The family $\Gamma = \Gamma_\lambda(A)$ of all λ-components of $\tilde{A}$ is the main object of study in this paper. The case $\lambda \ge 1$ is the most interesting. There is a natural *ordering* of Γ, which gives Γ a structure of a treelike *graph*.

If A is connected, then also $\tilde{A}$ is connected, and Γ is the trivial graph consisting of one vertex. More generally, Γ is trivial if A is c-uniformly perfect with $c < e^\lambda$; see 4.12 for the definition of uniform perfectness. In the general case we can roughly say that the structure of Γ describes in which way A is disconnected.

1991 Mathematics Subject Classification: Primary 30C65.

[1] The research described in this paper was supported by the Russian Fund for Fundamental Research (project 96-01-01558), by the University of Helsinki, and by the Finnish Academy of Sciences.

In Section 6 we apply the theory of upper sets to quasisymmetric maps. Let A and B be metric spaces and let $f\colon A \to B$ be η-quasisymmetric; see 6.2 for the definition of quasisymmetry. It is well known that if A is uniformly perfect, then η can be chosen to be of the form $\eta(t) = C(t^\alpha \vee t^{1/\alpha})$; see [TV], [AT]. We show that this is true for all η-quasisymmetric maps of A *if and only if* there is λ such that $\Gamma_\lambda(A)$ is trivial. This property can also be expressed in terms of relative connectedness (Theorem 4.9).

The second author wants to make it clear that an early version of the theory of upper sets was created and developed by the first author D.A. Trotsenko in the late eighties and considered in the conference proceedings [Tr]. Also the application to quasisymmetric maps was proved by him. The main part of this paper was written during the visits of the first author at the University of Helsinki in 1997. The second author has "polished up" the theory by writing and simplifying details of proofs, introducing some terminology and auxiliary concepts, etc.

1.2. *Notation.* Throughout the paper, A will denote a metric space with distance between points x, y written as $|x-y|$. For nonempty sets $E, F \subset A$, we let $d(E)$ and $d(E,F)$ denote the diameter of E and the distance between E and F. For $r>0$, $B(E,r)$ is the r-neighborhood $\{x : d(x,E) < r\}$ of E. Moreover, $B(x,r)$ is the open ball with radius r centered at x; the closed ball is $\overline{B}(x,r)$. We let R and N denote the set of real numbers and the set of positive integers, respectively, and we write $\mathsf{N}_0 = \mathsf{N} \cup \{0\}$. For real numbers s, t we use the notation

$$s \wedge t = \min(s,t), \quad s \vee t = \max(s,t).$$

We make the convention that λ will always denote a real number with $\lambda \ge 1$.

Acknowledgement. We thank Pekka Alestalo for careful reading of the manuscript and for several valuable comments.

2. Broken hyperbolic metric

2.1. *Summary.* In this section we introduce the broken hyperbolic metric ϱ of the space $A \times \mathsf{R}_+$ and prove some properties of it.

2.2. *Definitions.* The ordinary hyperbolic metric ϱ_h of the half space $R^{n+1}_+ = \mathsf{R}^n \times \mathsf{R}_+$ is defined by the element of length

$$d\varrho_h(x,r) = \frac{(|dx|^2 + dr^2)^{1/2}}{r},$$

where $x \in \mathsf{R}^n$, $r \in \mathsf{R}_+$, $n \ge 1$. We find it convenient to replace ϱ_h by the metric ϱ defined by the element of length

$$d\varrho(x,r) = \frac{|dx| + |dr|}{r}.$$

Thus for $z, z' \in \mathsf{R}^{n+1}_+$ we have

$$\varrho(z, z') = \inf_\gamma \int_\gamma \frac{|dx| + |dr|}{r}$$

over all rectifiable arcs γ joining z and z' in R^{n+1}_+. Since $d\varrho_h \le d\varrho \le \sqrt{2}\, d\varrho_h$, we have

$$\varrho_h(z, z') \le \varrho(z, z') \le \sqrt{2}\, \varrho_h(z, z')$$

for all $z, z' \in \mathsf{R}^{n+1}_+$. The metric ϱ is the *broken hyperbolic metric* of R^{n+1}_+.

More generally, let A be an arbitrary metric space, and let $z = (x, r)$, $z' = (x', r')$ be points in $A \times \mathsf{R}_+$. Choose points $y, y' \in \mathsf{R}^n$ with $|y - y'| = |x - x'|$. Then the number

$$\varrho(z, z') = \varrho\big((y, r), (y', r')\big)$$

is independent of the choice of y and y', and it gives the broken hyperbolic metric ϱ of $A \times \mathsf{R}_+$. Since each triple of A can be isometrically embedded into R^2, we see that ϱ indeed is a metric in $A \times \mathsf{R}_+$.

Alternatively, the broken hyperbolic metric ϱ of $A \times \mathsf{R}_+$ can be defined as follows. Let

$$\pi\colon A \times \mathsf{R}_+ \to A, \quad \pi_2\colon A \times \mathsf{R}_+ \to \mathsf{R}_+$$

be the projections. By a *step* we mean a pair of points in $A \times \mathsf{R}_+$. A step (z, z') is *vertical* if $\pi(z) = \pi(z')$, and (z, z') is *horizontal* if $\pi_2(z) = \pi_2(z')$. The hyperbolic length of a vertical step (z, z') is

$$l_h(z, z') = \left| \log \frac{\pi_2(z)}{\pi_2(z')} \right|,$$

and the hyperbolic length of a horizontal step (z, z') is

$$l_h(z, z') = \frac{|\pi(z) - \pi(z')|}{\pi_2(z)}.$$

A *step path* in $A \times \mathsf{R}_+$ is a finite sequence $\bar{z} = (z_0, \ldots, z_N)$ of points in $A \times \mathsf{R}_+$ such that the step (z_{j-1}, z_j) is either horizontal or vertical for all $1 \le j \le N$. The hyperbolic length of $\bar{z}$ is the number

$$l_h(\bar{z}) = \sum_{j=1}^{N} l_h(z_{j-1}, z_j).$$

In the special case $A = \mathsf{R}^n$, $l_h(\bar{z})$ is the ordinary hyperbolic length of the path consisting of the segmental paths $[z_{j-1}, z_j]$, $1 \le j \le N$. The *broken hyperbolic distance* between points $z, z' \in A \times \mathsf{R}_+$ is defined as

$$\varrho(z, z') = \inf_{\bar{z}} l_h(\bar{z})$$

over all step paths $\bar{z}$ from z to z'. It is not difficult to show that the two definitions for ϱ are equivalent. However, in the sequel we shall use the second definition; the first one was given to illustrate the connection between ϱ and ϱ_h in R^{n+1}_+.

2.3. *Geodesics.* Let $z = (x, r)$ and $z' = (x', r')$ be points in $A \times \mathsf{R}_+$. We want to find a geodesic from z to z', that is, a step path $\bar{z} = (z_0, \dots, z_N)$ such that $z_0 = z$, $z_N = z'$, and $l_h(\bar{z}) = \varrho(z, z')$.

Assume that $r \le r'$ and that $\bar{z}$ is a step path from z to z'. Let $t = \max\{\pi_2(z_j) : 0 \le j \le N\}$ be the maximal height of $\bar{z}$. Then the sum of the hyperbolic lengths of all vertical steps of $\bar{z}$ is at least $\log(t/r) + \log(t/r')$. The corresponding sum for the horizontal steps is at least $|x - x'|/t$. Hence $l_h(\bar{z}) \ge l_h(\bar{y})$, where $\bar{y}$ is the step path (z, y_1, y_2, z') with $y_1 = (x, t)$, $y_2 = (x', t)$. We have

$$l_h(\bar{y}) = \frac{|x - x'|}{t} + 2\log t - \log(rr').$$

Using elementary calculus we see that the right-hand side attains its minimum at $t = |x - x'|/2$. We obtain the following result:

2.4. **Theorem.** *Let $z = (x, r)$ and $z' = (x', r')$ be points in $A \times \mathsf{R}_+$ with $r \le r'$ and $|x - x'| = s$. If $r' \le s/2$, then the geodesic in the metric ϱ from z to z' is the step path $\big(z, (x, s/2), (x', s/2), z'\big)$, and*

$$\varrho(z, z') = 2 + \log \frac{s^2}{4rr'}.$$

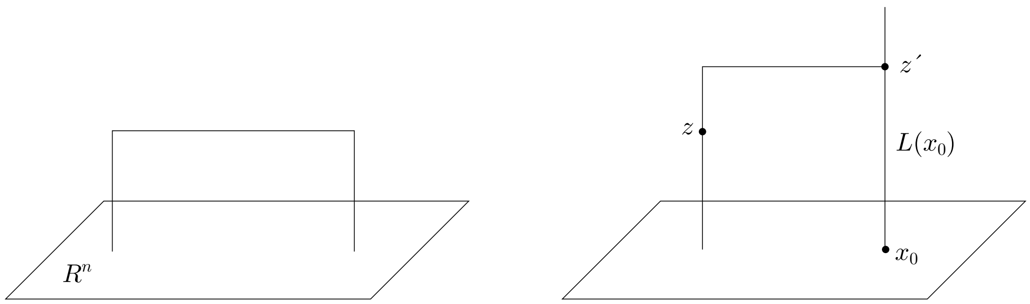
If $r' \ge s/2$, then the geodesic is the step path $\big(z, (x, r'), z'\big)$, and

$$\varrho(z, z') = \frac{s}{r'} + \log \frac{r'}{r}.$$

In particular, if $r = r' \ge s/2$ or if $x = x'$, then the geodesic reduces to the single step (z, z'). □

2.5. *Remarks.* 1. Observe that the first case of Theorem 2.4 can occur only if $\varrho(z, z') > 2$.

2. In the case $A = \mathsf{R}^n$, the geodesic from z to z' can be considered as an ordinary arc $\gamma \subset \mathsf{R}^{n+1}_+$. This arc consists of one, two or three line segments, and it lies on the boundary of the unique square Q such that Q and a pair of its sides are perpendicular to R^n, and the center of Q is in R^n. See Figure 1.



Geodesic of the metric ϱ in R^{n+1}_+ The point z' of $L(x_0)$ is closest to z

Figure 1.

2.6. *Notation.* For $x \in A$ we let $L(x)$ denote the ray $\{x\} \times \mathsf{R}_+ = \pi^{-1}\{x\} \subset A \times \mathsf{R}_+$.

2.7. **Lemma.** *Let $z = (x, r) \in A \times \mathsf{R}_+$ and $x_0 \in A$. If $|x - x_0| \ge r$, then*

$$\varrho\big(z, L(x_0)\big) = \varrho\big(z, (x_0, |x - x_0|)\big) = 1 + \log(|x - x_0|/r).$$

If $|x - x_0| \le r$, then

$$\varrho\big(z, L(x_0)\big) = \varrho\big(z, (x_0, r)\big) = |x - x_0|/r.$$

Proof. The lemma follows from 2.4 by elementary calculus. Alternatively, we can argue as follows. Without loss we may assume that $A = \mathsf{R}$ and that $x_0 = 0$. Let $z' \in L(x_0)$ be the point closest to z. Then z' must lie on the geodesic between z and $(-x, r)$, and the lemma follows from 2.4. □

2.8. *Remark.* If $A \subset \mathsf{R}^n$, then Lemma 2.7 means that the point z' of $L(x_0)$ closest to z is such that z lies on the boundary of a square with adjacent vertices $(x_0, 0)$ and z'. See Figure 1.

2.9. **Lemma.** *If $z = (x, r)$, $z' = (x', r') \in A \times \mathsf{R}_+$, then $|x - x'| \le r e^{\varrho(z,z') - 1}$.*

Proof. Set $s = |x - x'|$. Since $1 + \log(s/r) \le s/r$, Lemma 2.7 implies that

$$1 + \log\frac{s}{r} \le \varrho\big(z, L(x')\big) \le \varrho(z, z'),$$

and the lemma follows. □

2.10. *Remark.* In a recent manuscript [BS], M. Bonk and O. Schramm consider a metric ϱ' in $A \times \mathsf{R}$, defined by the explicit formula

$$\varrho'\big((x, s), (y, t)\big) = 2\log\frac{|x - y| + s \vee t}{\sqrt{st}}.$$

It is not difficult to show that this metric is bilipschitz equivalent to our metric ϱ. In fact, $\varrho'/2 \le \varrho \le 2\sqrt{2}\,\varrho'$.

3. Upper sets and λ-components

3.1. *Summary.* In this section we develop the basic theory of the upper set $\tilde{A}$ of a metric space A. In particular, we study the properties of the family $\Gamma_\lambda(A)$ of the λ-components of $\tilde{A}$, $\lambda \ge 1$.

3.2. *Definitions.* We recall from the introduction that the *upper set* of a metric space A is

$$\tilde{A} = \big\{(x, |x - y|) : x \in A,\ y \in A \setminus \{x\}\big\} \subset A \times \mathsf{R}_+.$$

Recall also that we always assume that A contains at least 2 points, and hence $\tilde{A} \neq \varnothing$. The broken hyperbolic metric of $A \times \mathsf{R}_+$ defines a metric ϱ in $\tilde{A}$.

Recall from 1.2 that we always assume that $\lambda \geq 1$. By a λ-*sequence* we mean a finite sequence $\bar{z} = (z_0, \ldots, z_N)$ in $\tilde{A}$ such that $\varrho(z_{j-1}, z_j) \leq \lambda$ for all $1 \leq j \leq N$. Two points in $\tilde{A}$ belong to the same λ-*component* of $\tilde{A}$ if they can be joined by a λ-sequence in $\tilde{A}$. If there is only one λ-component, $\tilde{A}$ is λ-*connected.* The family $\Gamma_\lambda(A)$ of all λ-components of $\tilde{A}$ is a partition of $\tilde{A}$. For brevity, we shall write $\Gamma_\lambda(A)$ as Γ_λ or simply as Γ if there is no danger of misunderstanding.

Recall the notation π: $A \times \mathsf{R}_+ \to A$ and π_2: $A \times \mathsf{R}_+ \to \mathsf{R}_+$ for the projections. We define a partial ordering in $A \times \mathsf{R}_+$ by setting $z \leq z'$ if $\pi(z) = \pi(z')$ and $\pi_2(z) \leq \pi_2(z')$.

We also define a relation $\leq$ in Γ by setting $\gamma \leq \gamma'$ if $\pi(\gamma) \subset \pi(\gamma')$.

In Theorem 3.4 we collect together several basic properties of Γ. In particular, we show that the relation $\leq$ is a partial ordering of Γ and give two alternative characterizations for this relation. Geometrically, $\gamma \leq \gamma'$ means that γ lies below γ' in $A \times \mathsf{R}_+$.

3.3. *Notation.* If $x, y \in A$ and $x \neq y$, we set $\langle x, y\rangle = (x, |x-y|)$. Then $\langle x, y\rangle$ and $\langle y, x\rangle$ are in $\tilde{A}$, and $\varrho(\langle x, y\rangle, \langle y, x\rangle) = 1$ by 2.4. Hence the points $\langle x, y\rangle$ and $\langle y, x\rangle$ lie in the same λ-component. This simple observation is needed in several arguments, and it is the reason for our convention $\lambda \geq 1$.

3.4. **Theorem.** *Let $\gamma, \gamma' \in \Gamma = \Gamma_\lambda(A)$.*

(1) *If $\gamma \neq \gamma'$, $z \in \gamma$, $z' = \langle x, y\rangle \in \gamma'$, and $z < z'$, then $y \notin \pi\gamma$. Moreover, for each $w \in \gamma$ we have $\varrho(z', \langle \pi w, y\rangle) < 1 \leq \lambda$ and $w < \langle \pi w, y\rangle \in \gamma'$.*

(2) *$\gamma \leq \gamma'$ if and only if $z \leq z'$ for some $z \in \gamma$, $z' \in \gamma'$.*

(3) *$\gamma \leq \gamma'$ if and only if for each $z \in \gamma$ there is $z' \in \gamma'$ with $z \leq z'$.*

(4) *If $\gamma \neq \gamma'$, then either $\pi\gamma$ and $\pi\gamma'$ are disjoint or one of them is a proper subset of the other.*

(5) *$\leq$ is a partial ordering of Γ.*

(6) *If $z_1 \leq z_2 \leq z_3$ are points of $\tilde{A}$ with $z_1, z_3 \in \gamma$, then $z_2 \in \gamma$.*

(7) *If $\langle x, y\rangle \in \gamma$, then $x, y \in \pi\gamma$.*

(8) *$\widetilde{\pi\gamma} = \cup\{\beta \in \Gamma : \beta \leq \gamma\}$.*

(9) *If γ contains (x, r) and (x', r'), and if $r \wedge r' \leq |x - x'|$, then γ contains $\langle x, x'\rangle$ and $\langle x', x\rangle$.*

(10) *If $x_0, y \in \pi\gamma$ and $x \in A$ with $|x - x_0| \leq e^\lambda |y - x_0|$, then $x \in \pi\gamma$.*

(11) *If $d(\pi\gamma) = \infty$, then $\pi\gamma = A$, and γ is the greatest element of Γ.*

(12) *If $d(\pi\gamma) < \infty$, then $B\big(\pi\gamma, (e^\lambda - 1)d(\pi\gamma)\big) = \pi\gamma$.*

(13) *If $\gamma < \gamma'$, then $d(\pi\gamma') \geq (e^\lambda - 1)d(\pi\gamma)$.*

(14) *$\pi\gamma$ is closed and open in A.*

(15) *If A is separable, then Γ is countable.*

Proof. (1) Write $z = (x, r)$, $z' = (x, r')$, where $r' = |x - y|$. Suppose that $w = (u, t) \in \gamma$. We must show that $u \neq y$ and that $w < \langle u, y\rangle \in \gamma'$. We may assume that $\varrho(z, w) \leq \lambda$, since the general case follows from this by induction.

By 2.9 we have $|x-u| \le re^{\lambda-1}$, and hence

$$r' - re^{\lambda-1} \le |u-y| \le r' + re^{\lambda-1}.$$

Since $\lambda < \varrho(z,z') = \log(r'/r)$, we have $r' > re^{\lambda}$, and hence $|u-y| \ge r(e^{\lambda} - e^{\lambda-1}) > 0$. Thus the point $w' = \langle u,y\rangle$ is in $\tilde{A}$. Moreover,

$$\varrho(z',w') \le \frac{|u-x|}{r'} + \log\frac{r'}{r'-re^{\lambda-1}} \le e^{-1} + \log\frac{e}{e-1} < 1 \le \lambda,$$

which yields $w' \in \gamma'$.

It remains to show that $w < w'$, that is, $t < |u-y|$. Assume that $t \ge |u-y|$. Since $\varrho(w,w') > \lambda$, we then have $t > e^{\lambda}|u-y| > e^{\lambda}(e^{\lambda} - e^{\lambda-1})r$. On the other hand, $\varrho(z,w) \le \lambda$ implies that $t \le re^{\lambda}$, and we get the contradiction $1 > e^{\lambda} - e^{\lambda-1} \ge e-1$. Part (1) is proved.

Observe that in the situation of (1) we have $\langle y,x\rangle \in \gamma'$ by 3.3. Hence $\pi\gamma \subsetneq \pi\gamma'$, and parts (2), (3), (4) follow easily. Furthermore, (4) implies (5), and (6) follows from (2) and (5). Part (7) is clear in view of 3.3.

(8) If $\beta,\gamma \in \Gamma$ and $z = \langle x,y\rangle \in \beta \le \gamma$, then $x,y \in \pi\beta \subset \pi\gamma$ by (7), and hence $x \in \widetilde{\pi\gamma}$. Conversely, assume that $x,y \in \pi\gamma$, $x \ne y$. Choose $w \in \gamma$ with $x = \pi(w)$. Let $\beta \in \Gamma$ be the λ-component of $\tilde{A}$ containing $z = \langle x,y\rangle$. Since $x \in \pi\beta \cap \pi\gamma$, we have either $\beta \le \gamma$ or $\beta > \gamma$ by (4). The latter case is impossible, since it implies that $w < z$ by (2) and then $y \notin \pi\gamma$ by (1). Hence $\beta \le \gamma$.

(9) We may assume that $r \le |x-x'|$. Choose $\beta \in \Gamma$ containing $\langle x,x'\rangle$ and $\langle x',x\rangle$. Then $\beta \le \gamma$ by (8). Since $z \le \langle x,x'\rangle$, we have $\gamma \le \beta$ by (2).

(10) By (8) there is $\beta \in \Gamma$ with $\langle x_0,y\rangle \in \beta \le \gamma$. By (3) there is $z = (x_0,r) \in \gamma$ with $\langle x_0,y\rangle \le z$. If $\langle x_0,x\rangle \ge z$, then

$$\varrho(\langle x_0,x\rangle,z) = \log\frac{|x-x_0|}{r} \le \lambda,$$

and hence $\langle x_0,x\rangle \in \gamma$, which gives $x \in \pi\gamma$ by (7). If $\langle x_0,x\rangle \le z$, then $\langle x_0,x\rangle \in \beta \le \gamma$ for some $\beta \in \Gamma$ by (2), and hence $x \in \pi\beta \subset \pi\gamma$.

(11) follows directly from (10).

(12) Observe that $d(\pi\gamma) > 0$ by (7). Suppose that $x \in B\big(\pi\gamma,(e^{\lambda}-1)d(\pi\gamma)\big)$. We can choose $\varepsilon > 0$ and points $x_1,x_0,y \in \pi\gamma$ such that

$$|x-x_1| \le (e^{\lambda}-1)d(\pi\gamma) - \varepsilon,$$
$$|x_0-y| \ge d(\pi\gamma) - \varepsilon e^{-\lambda}.$$

Then

$$|x-x_0| \le |x-x_1| + |x_1-x_0| \le (e^{\lambda}-1)d(\pi\gamma) - \varepsilon + d(\pi\gamma) \le e^{\lambda}|y-x_0|,$$

which yields $x \in \pi\gamma$ by (10).

(13) and (14) follow from (12).

(15) If A is separable, then so is $A \times \mathsf{R}_+$. Since the neighborhoods $B(\gamma,\lambda/2)$ of $\gamma \in \Gamma$ are disjoint, Γ is countable. □

3.5. *Remark.* If $A \subset \mathsf{R}^n$ is infinite, then $(\tilde{A}, \varrho)$ is unbounded. Indeed, A contains a sequence (x_j) of distinct points converging to a point in R^n or to ∞. Then $\varrho(z_1, z_j) \to \infty$, where $z_j = \langle x_j, x_{j+1} \rangle$ in the first case, and $z_j = \langle x_1, x_{j+1} \rangle$ in the second case.

This result is not true for arbitrary metric spaces. For example, if A is a space with the discrete metric $|x - y| = 1$ for $x \neq y$, then $\tilde{A} = A \times \{1\}$, and $\varrho(z, z') = 1$ for all $z \neq z'$.

4. Simple λ-sequences and relative connectedness

4.1. *Summary.* We show that each λ-sequence $(z_0, \ldots, z_N)$ in the upper set $\tilde{A}$ can be embedded into a λ-sequence of a special kind, called a *simple* λ-sequence. It turns out that the simple λ-sequences of $\tilde{A}$ are in one-to-one correspondence with certain sequences $(x_0, \ldots, x_N)$ of A, called M*-relative* sequences and characterized by the property

$$|x_{j-1} - x_j|/M \leq |x_j - x_{j+1}| \leq M|x_{j-1} - x_j|,$$

$M = e^\lambda$. These lead to *relatively connected* metric spaces, which are closely connected with uniformly perfect spaces.

4.2. *Simple sequences.* We say that a pair (z, z') of points in the upper set $\tilde{A}$ of a metric space A is *simple* if $\pi_2(z) = \pi_2(z') = |\pi(z) - \pi(z')|$. In other words, there are $x, y \in A$ such that $z = \langle x, y \rangle$, $z' = \langle y, x \rangle$. Recall that (z, z') is vertical if $\pi(z) = \pi(z')$. In particular, a pair (z, z) is always vertical (but never simple). Observe that $\varrho(z, z') = 1$ for a simple pair (z, z') and that

$$\varrho(z, z') = \left| \log \frac{\pi_2(z)}{\pi_2(z')} \right|$$

for a vertical pair.

A finite sequence $\bar{z} = (z_0, \ldots, z_N)$ in $\tilde{A}$ is called *simple* if N is odd and if the step (z_{j-1}, z_j) is simple for all odd j and (z_{j-1}, z_j) is vertical for all even j.

We first show that every λ-sequence in $\tilde{A}$ can be *simplified* to a simple λ-sequence by adding new points.

4.3. **Theorem.** *Every λ-sequence $\bar{z} = (z_0, \ldots, z_N)$ of $\tilde{A}$ can be embedded into a simple λ-sequence $\bar{u} = (u_0, \ldots, u_{N'})$. This means that $u_0 = z_0$, $u_{N'} = z_N$, and $\bar{z}$ is a subsequence of $\bar{u}$.*

Proof. We first remark that if $\langle x, y \rangle \in \tilde{A}$, then the sequence $(\langle x, y \rangle, \langle y, x \rangle, \langle y, x \rangle, \langle x, y \rangle)$ is a simple 1-sequence, called a *trick sequence.* It has three steps, and it joins $\langle x, y \rangle$ to itself.

The theorem is proved by induction on N. Assume first that $N = 1$. Then $\bar{z}$ is a pair (z, z') with $\varrho(z, z') \leq \lambda$. Write $z = \langle x, y \rangle = (x, r)$ and $z' = \langle x', y' \rangle = (x', r')$. If $x = x'$, we can use two trick sequences to embed $\bar{z}$ into a simple λ-sequence with 7 steps. We may thus assume that $|x - x'| = t > 0$. By symmetry, we may assume that $r \geq r'$. We consider two cases.

Case 1. $r \le e^\lambda t$. Consider the sequence $\bar v = (z, v_1, v_2, z')$, where $v_1 = \langle x, x'\rangle$, $v_2 = \langle x', x\rangle$. If $r \ge t$, then $\varrho(z, v_1) = \log(r/t) \le \lambda$. If $r \le t$, then 2.4 and 2.7 imply that

$$\varrho(z, v_1) < \varrho(z, v_1) + \varrho(v_1, v_2) = \varrho(z, v_2) = \varrho\big(z, L(x')\big) \le \varrho(z, z') \le \lambda.$$

A similar argument shows that $\varrho(v_2, z') \le \lambda$. Hence $\bar v$ is a λ-sequence. Extending $\bar v$ by two trick sequences we obtain the desired simple λ-sequence $\bar u$ from z to z'.

Case 2. $r \ge e^\lambda t$. We show that the sequence $\bar v = (z, v_1, v_2, v_3, z')$ with $v_1 = \langle y, x\rangle$, $v_2 = \langle y, x'\rangle$, $v_3 = \langle x', y\rangle$ is a λ-sequence.

Since

$$r - t \le |y - x'| \le r + t, \tag{4.4}$$

we obtain

$$\varrho(v_1, v_2) = \left|\log \frac{r}{|y - x'|}\right| \le \log \frac{r}{r - t} \le \log \frac{e}{e - 1} < 1.$$

To prove that $\varrho(v_3, z') \le \lambda$ we consider two subcases.

Subcase 2a. $r' \le r - t$. By (4.4) and by 2.4 we obtain

$$\varrho(v_3, z') = \log \frac{|y - x'|}{r'} \le \log \frac{r}{r'} + \log\Big(1 + \frac{t}{r}\Big) \le \log \frac{r}{r'} + \frac{t}{r} = \varrho(z, z') \le \lambda.$$

Subcase 2b. $r' \ge r - t$. Since $r \ge e^\lambda t$, it follows from (4.4) that the numbers r' and $|y - x'|$ are between $r(1 - e^{-\lambda})$ and $r(1 + e^{-\lambda})$. Hence

$$\varrho(v_3, z') \le \log \frac{1 + e^{-\lambda}}{1 - e^{-\lambda}} \le \log \frac{e + 1}{e - 1} < 1.$$

We have proved that $\bar v$ is a λ-sequence. Extending $\bar v$ by a trick sequence from z' to z' we obtain a simple λ-sequence from z to z'.

Next assume that p is a positive integer and that the theorem is true whenever $N \le p$. Suppose that $N = p + 1$. Applying the induction hypothesis to the λ-sequences $(z_0, \ldots, z_{N-1})$ and (z_{N-1}, z_N) we embed them into simple λ-sequences. Linking these sequences with the trivial vertical step (z_{N-1}, z_{N-1}) we obtain the desired simple λ-sequence $\bar u$. □

4.5. *Remark.* The construction of the proof of 4.3 is not the most economical. It gives the bound $N' \le 10N$.

4.6. *Relative connectedness.* We say that a finite sequence $\bar x = (x_0, \ldots, x_N)$ in a metric space A is *proper* if $N \ge 1$ and if $x_{j-1} \ne x_j$ for all $1 \le j \le N$. In particular, a pair (x, y) is proper if $x \ne y$. The sequence $\bar x$ is called *M-relative*, $M \ge 1$, if $\bar x$ is proper and if

$$|x_{j-1} - x_j|/M \le |x_j - x_{j+1}| \le M|x_{j-1} - x_j|$$

or equivalently,

$$\left|\log \frac{|x_j - x_{j+1}|}{|x_{j-1} - x_j|}\right| \le \log M$$

for all $1 \le j \le N-1$. In the trivial case $N = 1$, $\bar{x}$ is a pair (x_0, x_1), and the condition for M-relativity is vacuously true for all $M \ge 1$.

We say that the sequence $\bar{x} = (x_0, \dots, x_N)$ *joins* the pairs (x_0, x_1) and (x_{N-1}, x_N), and that A is M*-relatively connected* if each pair of proper pairs (x, y) and (x', y') in A can be joined by an M-relative sequence in A. The space A is *relatively connected* if it is M-relatively connected for some $M \ge 1$.

A connected space is M-relatively connected for all $M > 1$, but so are many other spaces as well, for example, the Cantor middle-third set.

4.7. *Associated sequences.* We shall show that the e^λ-relative sequences of A are in one-to-one correspondence with the simple λ-sequences of the upper set $\tilde{A}$. Suppose that $\bar{x} = (x_0, \dots, x_N)$ is a proper sequence in A. Define a sequence $\bar{z} = (z_0, \dots, z_{2N-1})$ in $\tilde{A}$ by

$$z_{2j} = \langle x_j, x_{j+1} \rangle, \quad z_{2j+1} = \langle x_{j+1}, x_j \rangle$$

for $0 \le j \le N-1$. Thus $\bar{z}$ is the sequence

$$(\langle x_0, x_1 \rangle, \langle x_1, x_0 \rangle, \langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \dots, \langle x_{N-1}, x_N \rangle, \langle x_N, x_{N-1} \rangle).$$

Clearly $\bar{z}$ is a simple sequence. Moreover, $\bar{z}$ is a λ-sequence if and only if $\bar{x}$ is e^λ-relative; remember that we always assume that $\lambda \ge 1$. We write $\bar{z} = \mathrm{As}\,\bar{x}$ and say that $\bar{z}$ is the simple sequence *associated* with $\bar{x}$.

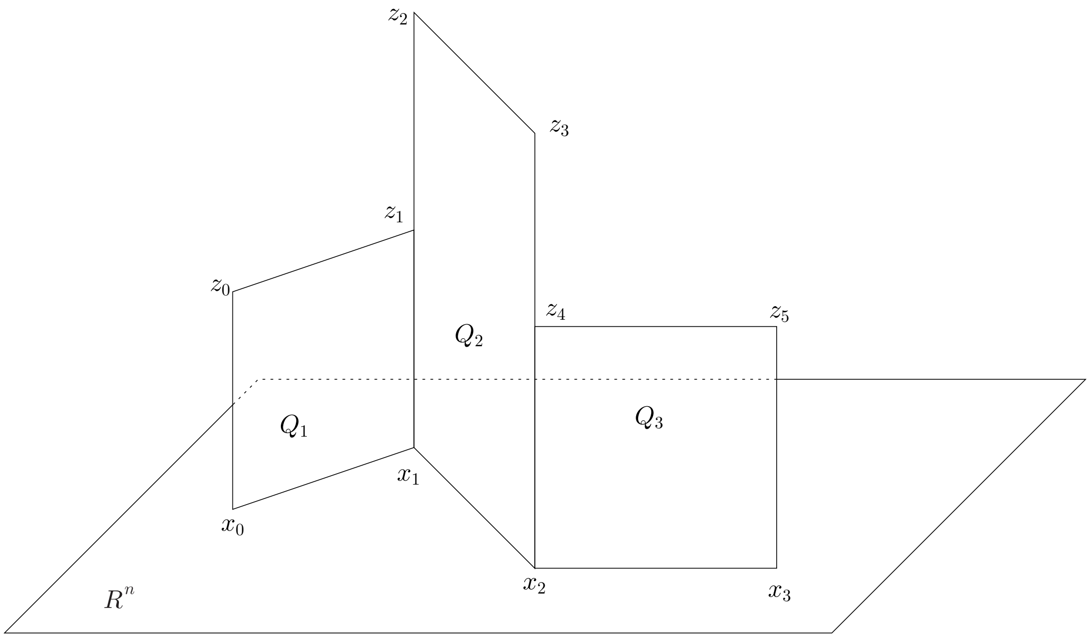


Figure 2. $\bar{z} = \mathrm{As}\,\bar{x}$

Conversely, if $\bar{z} = (z_0, \dots, z_{2N-1})$ is a simple sequence in $\tilde{A}$, we associate with $\bar{z}$ the proper sequence $\bar{x} = \mathrm{As}^{-1}\bar{z} = (x_0, \dots, x_N)$, where $x_0 = \pi(z_0)$, $x_N = \pi(z_{2N-1})$, and $x_j = \pi(z_{2j-1}) = \pi(z_{2j})$ for $1 \le j \le N-1$. Then $\mathrm{As}\,\mathrm{As}^{-1}\bar{z} = \bar{z}$ and $\mathrm{As}^{-1}\mathrm{As}\,\bar{x} = \bar{x}$ for all $\bar{z}$ and $\bar{x}$. We have proved:

4.8. **Theorem.** *The function $\bar{x} \mapsto \mathrm{As}\,\bar{x}$ gives a bijective correspondence between the proper sequences of A and the simple sequences of $\tilde{A}$. Moreover, $\bar{x}$ is e^λ-relative if and only if $\mathrm{As}\,\bar{x}$ is a λ-sequence.* □

4.9. **Theorem.** *For $\lambda \ge 1$, a metric space A is e^λ-relatively connected if and only if $\tilde{A}$ is λ-connected.*

Proof. This follows from 4.3 and 4.8. □

4.10. *Remark.* In the case $A \subset \mathsf{R}^n$, we can find $\mathrm{As}\,\bar{x}$ by erecting on each line segment $[x_{j-1}, x_j]$ a square Q_j, orthogonal to R^n (identified with $\mathsf{R}^n \times \{0\} \subset \mathsf{R}^{n+1}$). The sequence $\bar{z} = \mathrm{As}\,\bar{x}$ consists of vertices of these squares as shown in Figure 2.

We next give an alternative characterization for relative connectedness.

4.11. **Theorem.** *For a metric space A, the following conditions are quantitatively equivalent*:

(1) *A is M-relatively connected.*

(2) *There is $c \ge 1$ such that if $x \in A$ and $\overline{B}(x,r) \ne A$, then either $\overline{B}(x,r) = \{x\}$ or $\overline{B}(x,r) \setminus B(x,r/c) \ne \varnothing$.*

More precisely, (1) *implies* (2) *with $c = 2M+1$, and* (2) *implies* (1) *with all $M > c$.*

Proof. Assume that A is M-relatively connected and that (2) is not true for $c = 2M+1 \ge 3$. Then there are x and r such that $\{x\} \ne \overline{B}(x,r) \ne A$ and such that $\overline{B}(x,r) \setminus B(x,r/c) = \varnothing$. Pick points y and y' in A such that

$$0 < |x-y| < r/c, \quad |x-y'| > r.$$

Since A is M-relatively connected, there is an M-relative sequence $(x_0, \dots, x_N)$ joining (x,y) to (x,y'). Let k be the smallest number with $|x_k - x| \ge r$. Then $k \ge 2$, and

$$M \ge \frac{|x_k - x_{k-1}|}{|x_{k-1} - x_{k-2}|} > \frac{r - r/c}{2r/c} = \frac{c-1}{2},$$

which gives $c < 2M+1$, a contradiction.

Conversely, assume that (2) is true, that $M > c$, and that (x,y) and (x',y') are proper pairs in A. We show that they can be joined by an M-relative sequence.

We first assume that $x = x'$. By symmetry, we may assume that $|x-y'| \le |x-y|$. Applying condition (2) inductively, we find a sequence of points $y = y_0, \dots, y_N$ such that

$$|x - y_j|/M \le |x - y_{j+1}| \le c|x - y_j|/M$$

for $0 \le j \le N-1$ and $|x-y_N|/M \le |x-y'| \le |x-y_N|$. Now $(x, y_0, x, y_1, \dots, x, y_N, x, y')$ is an M-relative sequence from (x, y) to (x', y').

If $x \ne x'$, we can join the pairs (x, y) and (x', y') to (x, x') and (x', x), respectively, by c-relative sequences. These can be joined in a natural way to a c-relative sequence from (x, y) to (x', y'). □

4.12. *Uniformly perfect spaces.* A metric space A is *c-uniformly perfect* if $\overline{B}(x, r) \ne A$ implies that $\overline{B}(x, r) \setminus B(x, r/c) \ne \varnothing$. The concept was introduced by C. Pommerenke [Po] for closed unbounded sets in R^2, and it has turned out to be useful in various questions in analysis.

A uniformly perfect set containing more than one point has no isolated points. For example, a finite metric space containing more than one point is not uniformly perfect although it is relatively connected. The following corollary of 4.11 gives relations between uniform perfectness and relative connectedness.

4.13. **Theorem.** *A c-uniformly perfect space is M-relatively connected for all $M > c$. If a space A is M-relatively connected and has no isolated points, then A is c-uniformly perfect with $c = 2M + 1$.* □

We apply the results of this section to show that if points $z < z'$ can be joined by a λ-sequence in $\tilde{A}$, they can be joined by a vertical $(\lambda + 1)$-sequence. Recall the notation $L(x) = \{x\} \times (0, \infty)$ from 2.6.

4.14. **Lemma.** *Suppose that $z, z' \in \gamma \in \Gamma_\lambda(A)$ and that $\pi(z) = \pi(z') = x$. Then z and z' can be joined by a $(\lambda + 1)$-sequence in $\gamma \cap L(x)$.*

Proof. Write $z = \langle x, y \rangle$, $z' = \langle x, y' \rangle$, and assume, for example, that $z < z'$. By 4.3 and 4.8, there is an e^λ-relative sequence $(x_0, \dots, x_N)$ in A joining the pairs (x, y) and (x, y'). For $1 \le i \le N$ we set $p_i = \max\{|x_k - x| : 1 \le k \le i\}$. Then

$$|x_i - x_{i-1}| \le |x_i - x| + |x - x_{i-1}| \le 2p_i,$$

and hence

$$|x_{i+1} - x| \le |x_{i+1} - x_i| + |x_i - x| \le 2p_i e^\lambda + p_i,$$

which yields $p_{i+1} \le (2e^\lambda + 1)p_i$. Consequently,

$$\log \frac{p_{i+1}}{p_i} \le \log(2e^\lambda + 1) = \lambda + \log(2 + e^{-\lambda}) < \lambda + 1.$$

For each $i \in [1, N]$ choose $j(i) \le i$ with $|x_{j(i)} - x| = p_i$, and set $z_i = \langle x, x_{j(i)} \rangle$. Then $z_i \in \tilde{A}$, $z = z_1 \le z_2 \le \dots \le z_N$, and $z' \le z_N$. Choose k with $z_k \le z' \le z_{k+1}$. Since

$$\varrho(z_i, z_{i+1}) = \log \frac{p_{i+1}}{p_i} < \lambda + 1,$$

$(z_1, \dots, z_k, z')$ is a $(\lambda + 1)$-sequence joining z and z' in $\tilde{A} \cap L(x)$. Since $z, z' \in \gamma$, all z_i lie in γ by 3.4(6). □

We apply 4.14 to prove the following more general result.

4.15. **Theorem.** *If $z = (x, r)$ and $z' = (x', r')$ are in a λ-component γ, then z and z' can be joined by a $(\lambda+1)$-sequence in $\gamma \cap \big(L(x) \cup L(x')\big)$.*

Proof. By 4.14 we may assume that $x \neq x'$. Choose $\beta \in \Gamma$ containing $\langle x, x'\rangle$ and $\langle x', x\rangle$. If $\beta \neq \gamma$, then $\beta < \gamma$ by 3.4(9). Choose $y \in A$ with $z = \langle x, y\rangle$. Writing $z_1 = \langle x', y\rangle$ we have $\varrho(z, z_1) \le \lambda$ by 3.4(1). Then $z_1 \in \gamma$, and z' can be joined to z_1 by a $(\lambda+1)$-sequence in $\gamma \cap L(x')$ by 4.14. □

5. Order properties of $\Gamma_\lambda(A)$

5.1. *Summary.* We continue the study of the set $\Gamma = \Gamma_\lambda(A)$, especially from the order-theoretic point of view. To this end, we introduce the order-theoretic concept of a *family tree* and show that Γ has this property.

5.2. *Family trees.* We consider an arbitrary partially ordered set $(P, \le)$. We recall that P is (upwards) *directed* if for each pair $x, y \in P$ there is $z \in P$ with $x \le z \ge y$. If P is directed, then a maximal element in P is also the greatest element of P, written as $\max P$ if it exists.

We say that P is a *family tree* if

(1) P is directed,

(2) for each pair $a, b \in P$, the set $[a, b] = \{x \in P : a \le x \le b\}$ is finite and linearly ordered.

Suppose that P is a family tree and that $a \in P$, $a \neq \max P$. We show that the set $P^+(A) = \{x \in P : x > a\}$ has the least element, written as $p(a)$. Choosing $b \in P$ with $b > a$ we can write $[a, b] = \{x_0, \dots, x_N\}$, where $a = x_0 < x_1 < \dots < x_N = b$. If $y > a$, there is $z \in P$ with $y \le z \ge x_1$. Since $[a, z]$ is linearly ordered, we have either $x_1 \le y$ or $x_1 > y$. The second case is impossible, since $[a, x_1] = \{a, x_1\}$. Hence x_1 is the least element of $P^+(a)$.

If $b = p(a)$, we say that b is the *parent* of a and a is a *child* of b. An element $a \neq \max P$ has precisely one parent, but each element of P may have several children. Writing $p^2(a) = p\big(p(a)\big)$ etc., we have

$$P^+(a) = \{p(a), p^2(a), p^3(a), \dots\}. \tag{5.3}$$

The sequence may be infinite or finite. The latter case occurs if and only if $\max P$ exists; then the sequence ends with $\max P$. We see that $P^+(a)$ is always linearly ordered.

A family tree is obviously a *semilattice*, that is, each pair $a, b \in P$ has the least upper bound $a \vee b$.

We remark that the Hasse diagram of a family tree is a tree in the graph-theoretic sense: it is connected and contains no cycles.

We now return to the family $\Gamma = \Gamma_\lambda(A)$ of all λ-components of the upper set $\tilde{A}$ of a metric space A. We recall from 3.2 and from 3.4 the following three equivalent characterizations for the ordering $\gamma \le \gamma'$ of elements of Γ:

(1) $\pi\gamma \subset \pi\gamma'$.
(2) $z \le z'$ for some $z \in \gamma$ and $z' \in \gamma'$.
(3) For each $z \in \gamma$ there is $z' \in \gamma'$ with $z \le z'$.

5.4. **Theorem.** *$\Gamma_\lambda(A)$ is a family tree.*

Proof. To show that Γ is directed, assume that γ and γ' are incomparable elements of Γ. Choose $\langle x, y\rangle \in \gamma$ and $\langle x', y'\rangle \in \gamma'$. Then $x, y \in \pi\gamma$. By 3.4(10), we have $|x - x'| \ge e^\lambda |x - y| > |x - y|$, and hence $\langle x, y\rangle \le \langle x, x'\rangle$. Similarly $\langle x', y'\rangle \le \langle x', x\rangle$. Since there is $\gamma'' \in \Gamma$ containing $\langle x, x'\rangle$ and $\langle x', x\rangle$, Γ is directed.

Next assume that $\gamma, \gamma' \in \Gamma$ with $\gamma < \gamma'$. We can choose points $(x, r) \in \gamma$ and $(x, r') \in \gamma'$ with $r < r'$. If $\beta \in \Gamma$ with $\gamma < \beta < \gamma'$, then β contains a point (x, s) with $r < s < r'$. Hence $[\gamma, \gamma']$ is linearly ordered. Moreover, $[\gamma, \gamma']$ is finite, since $|\log(s/s')| > \lambda$ whenever (x, s) and (x, s') belong to different λ-components of $\tilde{A}$. □

5.5. *Remark.* It follows from 5.4 that each $\gamma \in \Gamma$ with $\gamma \ne \max \Gamma$ has a parent $p(\gamma)$. To find $p(\gamma)$, choose an arbitrary point $(x, r) \in \gamma$. Then $(x, r') \in p(\gamma)$ whenever $(x, r') \in \tilde{A}$ and $t \le r' < te^\lambda$, where $t = \inf\{s : r < s,\ (x, s) \in \tilde{A} \setminus \gamma\}$.

In the next result we give an alternative way to find $p(\gamma)$. Observe that $\gamma = \max \Gamma$ if and only if $\pi\gamma = A$.

5.6. **Lemma.** *Let $\gamma \in \Gamma$, and let $x \in \pi\gamma$, $y' \in A \setminus \pi\gamma$ be such that $|x - y'| \le ed(\pi\gamma, A \setminus \pi\gamma)$. Then $\langle x, y'\rangle \in p(\gamma)$.*

Proof. Choose $y \in \pi\gamma$ and $\gamma' \in \Gamma$ such that $\langle x, y\rangle \in \gamma$ and $\langle x, y'\rangle \in \gamma'$. By 3.4(10) we have $|x - y'| > e^\lambda |x - y|$, and hence $\langle x, y\rangle < \langle x, y'\rangle$. This implies $\gamma < \gamma'$, and thus $p(\gamma) \le \gamma'$. If $p(\gamma) \ne \gamma'$, then $p(\gamma)$ contains a point $\langle x, y''\rangle$ such that

$$|x - y''| < e^{-\lambda} |x - y'| \le e^{1-\lambda} d(\pi\gamma, A \setminus \pi\gamma) \le d(\pi\gamma, A \setminus \pi\gamma).$$

By 3.4(1) we have $y'' \in A \setminus \pi\gamma$. Since $x \in \pi\gamma$, this gives a contradiction. □

5.6. **Lemma.** *Let $\gamma \in \Gamma$, and let $x, y \in \pi\gamma$, $z \in A \setminus \pi\gamma$. Then*

$$|x - y| \le e^{-\varrho(\gamma, p(\gamma))} |x - z|.$$

Proof. Let β and γ' be the members of Γ containing $\langle x, y\rangle$ and $\langle x, z\rangle$, respectively. By 3.4(8), we have $\beta \le \gamma$. Since $x \in \pi\gamma \cap \pi\gamma'$ and since $z \notin \pi\gamma$, we have $\gamma < \gamma'$ by 3.4(4). Hence $p(\gamma) \le \gamma'$. By 3.4(3), there is $w \in p(\gamma)$ with $\langle x, y\rangle < w \le \langle x, z\rangle$. Consequently,

$$\varrho\big(\gamma, p(\gamma)\big) \le \varrho(\langle x, y\rangle, w) \le \log \frac{|x - z|}{|x - y|},$$

and the lemma follows. □

The following order-theoretic result is needed in Section 6.

5.8. **Lemma.** *Let $(P, \leq)$ be a family tree, and let $g\colon P \to [0, \infty)$ be an unbounded function. Then there is a sequence (x_j) in P such that $g(x_j) \to \infty$ and such that one of the following three conditions is satisfied*:

(1) $x_1 > x_2 > \cdots$;
(2) $x_1 < x_2 < \cdots$;
(3) x_i *and* x_j *are incomparable for all* $i \neq j$.

Proof. For $x \in P$ we set

$$h(x) = \sup\{g(y) : y < x\} \in [0, \infty]$$

with the agreement that $h(x) = 0$ for all minimal elements x of P. We consider three cases.

Case 1. $h(x) < \infty$ for all $x \in P$. Now $\max P$ does not exist. Fix an arbitrary $y_1 \in P$ and set $y_{n+1} = p^n(y_1)$ for $n \in \mathsf{N}$. Then $y_n < y_{n+1}$ and $h(y_n) \leq h(y_{n+1})$ for all n. We first show that $h(y_n) \to \infty$. Let $M > 0$, and choose $x \in P$ with $g(x) > M$. Setting $y = p(x \vee y_1)$ we have $y \in P^+(y_1)$, and hence $y = y_n$ for some n by (5.3). Thus $h(y_n) \geq g(x) > M$.

If $\sup\{g(y_n) : n \in \mathsf{N}\} = \infty$, there is a subsequence (x_n) of (y_n) such that $g(x_n) \to \infty$ and (2) is true. Suppose that $\sup\{g(y_n) : n \in \mathsf{N}\} = c < \infty$. Since $h(y_n) \to \infty$, there is a sequence $n_0 < n_1 < n_2 < \cdots$ in N such that $h(y_{n_0}) > c$ and $h(y_{n_{i+1}}) > h(y_{n_i})$ for all $i \geq 0$. Choose elements $x_1, x_2, \ldots$ in P such that $x_{i+1} < y_{n_{i+1}}$ and $g(x_{i+1}) > h(y_{n_i})$ for all $i \geq 0$. Then $g(x_i) \to \infty$. It suffices to show that x_j and x_{i+1} are incomparable whenever $1 \leq j \leq i$.

If $x_{i+1} \leq x_j$, then $x_{i+1} < y_{n_j}$, and we obtain the contradiction

$$h(y_{n_i}) < g(x_{i+1}) \leq h(y_{n_j}) \leq h(y_{n_i}).$$

Assume that $x_j < x_{i+1}$. Now x_{i+1} and y_{n_j} are in $P^+(x_j)$. Since this is linearly ordered by (5.3), either $x_{i+1} < y_{n_j}$ or $y_{n_j} \leq x_{i+1}$. As above, the first case is impossible. In the second case, $x_{i+1} = y_k$ for some k, and we get the contradiction

$$c < h(y_{n_0}) \leq h(y_{n_i}) < g(x_{i+1}) = g(y_k) \leq c.$$

Case 2. There is $y \in P$ such that $h(y) = \infty$ and such that $h(z) < \infty$ for every child z of y. Let $C = p^{-1}\{y\}$ be the set of all children of y. If g is unbounded in C, we choose a sequence (x_n) in C with $g(x_n) \to \infty$; then (x_n) satisfies (3). If g is bounded in C, then h is unbounded in C, and there is a sequence $z_0, z_1, \ldots$ in C such that $h(z_{i+1}) > h(z_i)$ and $h(z_i) \to \infty$. For $i \geq 0$ we choose elements $x_{i+1} < z_{i+1}$ with $g(x_{i+1}) > h(z_i)$. Now $g(x_i) \to \infty$. For $i \neq j$ we have $x_i \vee x_j = y$, and hence x_i and x_j are incomparable.

Case 3. Cases 1 and 2 do not occur. Then there is a sequence $y_1, y_2, \dots$ in P such that $h(y_n) = \infty$ and $y_n = p(y_{n+1})$ for all $n \in \mathsf{N}$. If $\sup\{g(y_n) : n \in \mathsf{N}\} = \infty$, then (y_n) has a subsequence (x_n) such that $g(x_n) \to \infty$ and (1) holds. Suppose that

$$\sup\{g(y_n) : n \in \mathsf{N}\} = c < \infty.$$

We define a sequence $n_1 < n_2 < \cdots$ in N and elements $x_i < y_{n_i}$ inductively as follows. Set $n_1 = 1$ and choose $x_1 < y_1$ with $g(x_1) > c$. Assume that n_i and $x_i < y_{n_i}$ have been defined. Since $[x_i, y_{n_i}]$ is finite, there is $n_{i+1} > n_i$ such that x_i is not smaller than $y_{n_{i+1}}$. Choose $x_{i+1} < y_{n_{i+1}}$ such that $g(x_{i+1}) > g(x_i) + 1$. Now $g(x_i) \to \infty$, and it suffices to show that x_j and x_{i+1} are incomparable whenever $1 \le j \le i$.

If $x_j \le x_{i+1}$, then $x_j < y_{n_{i+1}} \le y_{n_{j+1}}$, a contradiction. If $x_j > x_{i+1}$, then x_j and $y_{n_{i+1}}$ are in the linearly ordered set $P^+(x_{i+1})$, and hence either $x_j < y_{n_{i+1}}$ or $x_j \ge y_{n_{i+1}}$. In the first case we again get the contradiction $x_j < y_{n_{j+1}}$. In the second case, $x_j \in [y_{n_{i+1}}, y_1]$, and hence $x_j = y_k$ for some k. This is impossible, since $g(y_k) \le c < g(x_j)$. □

6. Quasisymmetric maps

6.1. *Summary*. We characterize the spaces A such that every quasisymmetric map of A is power quasisymmetric.

6.2. *Quasisymmetric maps*. By a *triplet* in a metric space A we mean a triple $T = (x, a, b)$ of *distinct* points in A. For a triplet T we write

$$|T| = \frac{|a - x|}{|b - x|}.$$

If A' is another metric space and if $f\colon A \to A'$ is injective, we let fT denote the triplet $\big(f(x), f(a), f(b)\big)$.

Let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. An injective map $f\colon A \to A'$ is η-*quasisymmetric* if

$$|fT| \le \eta(|T|) \tag{6.3}$$

for all triplets T in A, and f is quasisymmetric if it is η-quasisymmetric for some η. If f is η-quasisymmetric with $\eta(t) = C(t^\alpha \vee t^{1/\alpha})$ where $C > 0$ and $0 < \alpha \le 1$, we say that f is (C, α)-*quasisymmetric*. If f is (C, α)-quasisymmetric for some (C, α), then f is *power quasisymmetric*.

A quasisymmetric map is always an embedding. The basic theory of quasisymmetric maps is given in [TV]. A (C, α)-quasisymmetric map $f\colon A \to A'$ of a bounded space A satisfies a Hölder condition

$$|fx - fy| \le c|x - y|^\alpha,$$

where $c = 2Cd(fA)/d(A)^\alpha$; see the proof of [TV, 3.14], where α denotes our $1/\alpha$.

6.4. *Remark.* An η-quasisymmetric map $f: A \to A'$ satisfies the double inequality

$$\eta_L(|T|) \le |fT| \le \eta(|T|)$$

for all triplets T in A, where the lower bound is $\eta_L(t) = \eta(t^{-1})^{-1}$. On the other hand, it suffices that this condition holds for all triplets T with $|T| \le 1$. More precisely, assume that $f: A \to A'$ is injective and that

$$\varphi(|T|) \le |fT| \le \psi(|T|)$$

for all triplets T with $|T| \le 1$, where $\varphi, \psi: [0,1] \to \mathsf{R}$ are continuous strictly increasing functions with $\phi(0) = \psi(0) = 0$. Then (6.3) holds with $\eta(t)$ defined as $\psi(t)$ for $t \le 1$ and as $\varphi(t^{-1})^{-1}$ for $t \ge 1$. To get a continuous η, set $K = \varphi(1)\psi(1)$. If $K \le 1$, we replace $\eta(t)$ by $\eta(t)/K$ for all $t \le 1$. If $K \ge 1$, we replace $\eta(t)$ by $K\eta(t)$ for all $t \ge 1$.

Recall from 4.9 that A is e^λ-relatively connected if and only if $\tilde{A}$ is λ-connected, that is, $\Gamma = \Gamma_\lambda(A)$ consists of one element. Relative connectedness was defined in 4.6, and an alternative characterization was given in 4.11.

We show that a metric space A is relatively connected if and only if every quasisymmetric map of A is power quasisymmetric. The 'only if' part is not essentially new:

6.5. **Theorem.** *Suppose that A is M-relatively connected. Then every η-quasisymmetric map $f: A \to A'$ is (C,α)-quasisymmetric with (C,α) depending only on η and M.*

Proof. Suppose that A is M-relatively connected and that $f: A \to A'$ is η-quasisymmetric. Then condition 4.11(2) holds for $c = 2M + 2$. Set $q = 1/c$. It follows that if $a \ne b$ are points in A and if $B(a, q|a-b|) \ne \{a\}$, then there is $x \in A$ with $q^2|a-b| \le |a-x| \le q|a-b|$. This is all that is needed for the proof of [TV, 3.10], and we obtain the theorem. □

6.6. **Theorem.** *Suppose that E is a Banach space and that $A \subset E$ is not relatively connected. Then there is an η-quasisymmetric map $f: A \to E$ with a universal η such that f is not power quasisymmetric.*

Proof. Since the closure of A is not relatively connected, we may assume that A is closed and hence complete. We consider the family $\Gamma = \Gamma_1(A)$ of all 1-components of the upper set $\tilde{A}$. As in 5.5, we let $p(\gamma)$ denote the parent of a nonmaximal member $\gamma \in \Gamma$. Define $g: \Gamma \to \mathsf{R}$ by $g(\gamma) = \varrho\big(\gamma, p(\gamma)\big)$. If $\max\Gamma$ exists, we set $g(\max\Gamma) = 0$.

If $g(\gamma) \le M$ for all $\gamma \in \Gamma$, then $\tilde{A}$ is $2M$-connected, and hence A is e^{2M}-relatively connected, a contradiction. Hence g is unbounded. By Theorem 5.8 we can find a sequence $\gamma_1, \gamma_2, \dots$ in Γ such that $g(\gamma_j) \to \infty$ and such that either

(γ_j) is strictly monotone or the elements γ_j are pairwise incomparable. Passing to a subsequence we may assume that

$$4 \le g(\gamma_1) < g(\gamma_2) < \cdots .$$

Writing $t_i = e^{-g(\gamma_i)}$ we have $e^{-4} \ge t_1 > t_2 > \cdots$. We consider three cases.

Case 1. $\gamma_1 > \gamma_2 > \cdots$. We have $d(\pi\gamma_j) \le d(\pi\gamma_{j-1})/(e-1) < \infty$ by (11) and (13) of 3.4, and hence $d(\pi\gamma_j) \to 0$. Since A is complete, the intersection of all $\pi\gamma_j$ contains precisely one point. We may assume that this point is the origin of E.

We express A as the disjoint union of $\{0\}$ and the nonempty sets

$$A_0 = A \setminus \pi\gamma_1, \quad A_i = \pi\gamma_i \setminus \pi\gamma_{i+1},$$

$i \in \mathsf{N}$. We first observe that if $i < j$ and if $x, y \in \pi\gamma_j$, $z \in A \setminus \pi\gamma_{i+1}$, then we can apply Lemma 5.7 $j-i$ times to obtain

$$|x - y| \le t_{i+1} \cdots t_j |x - z|. \tag{6.7}$$

Define a homeomorphism φ: $(0,1] \to (0,1]$ by $\varphi(t) = 1/(1-\log t)$. It is easy to see that φ has the following properties:

(1) $\varphi(st) \ge \varphi(s)\varphi(t)$ for all $s, t \in (0,1]$.

(2) $\varphi(t)/t$ in decreasing in t.

(3) $\lim_{t\to 0} t^{-\alpha}\varphi(t) = \infty$ for each $\alpha > 0$.

Moreover, $1/5 \ge \varphi(t_1) > \varphi(t_2) > \cdots$. We set $s_0 = 1$ and

$$s_i = \frac{\varphi(t_1)\cdots\varphi(t_i)}{t_1 \cdots t_i}$$

for $i \in \mathsf{N}$. Then $1 < s_1 < s_2 < \cdots$.

Define f: $A \to E$ by $f(0) = 0$ and by $f(x) = s_i x$ for $x \in A_i$, $i \in \mathsf{N}_0$. We show that f has the desired properties.

Let $x \in A_i$ and $y \in A_j$. We first show that

$$s_{i\wedge j}|x - y|/2 \le |f(x) - f(y)| \le 2 s_{i\wedge j}|x - y|. \tag{6.8}$$

If $i = j$, this is clear. Suppose that $i < j$. Since $\varphi(t_k) \le 1/5$ for all $k \in \mathsf{N}$, we can apply (6.7) to the triple $(0, y, x)$ to get

$$s_j|y|/s_i \le \varphi(t_{i+1})\cdots\varphi(t_j)|x| \le |x|/5.$$

Consequently,

$$|f(x) - f(y)| = |s_i x - s_j y| \le s_i(|x| + s_j|y|/s_i) \le 6 s_i(|x - y| + |y|)/5.$$

Since

$$|y| \le t_j|x| \le e^{-4}|x| < |x|/5 \le |x-y|/5 + |y|/5,$$

we have $|y| \le |x-y|/4$, and we get the second inequality of (6.8). Similar estimates yield the first inequality:

$$|f(x)-f(y)| \ge s_i(|x| - s_j|y|/s_i) \ge 4s_i(|x-y| - |y|)/5 \ge 3s_i|x-y|/5.$$

To prove that f is quasisymmetric we assume that $T = (x,y,z)$ is a triplet in A with $|T| = |x-y|/|x-z| \le 1$. By 6.4 it suffices to show that

$$|T|/4 \le |fT| \le 4\varphi(|T|). \tag{6.9}$$

We assume that $x \in A_i$, $y \in A_j$, $z \in A_k$. The case where one of the points is the origin follows then by continuity. Setting $h = s_{i\wedge j}/s_{i\wedge k}$ we obtain by (6.8)

$$h|T|/4 \le |fT| \le 4h|T|. \tag{6.10}$$

It is not possible that $j < i\wedge k$, since then $x, z \in \pi\gamma_{j+1}$, $y \in A \setminus \pi\gamma_{j+1}$, and (6.7) gives the contradiction $|x-z| \le t_j|x-y| < |x-y|$. Hence $i\wedge j \ge i\wedge k$, which implies that $h \ge 1$, and the first inequality of (6.9) follows from (6.10).

If $i \le k$, then $i \le j$ and $h = 1$. Since $\varphi(t) \ge t$ for all t, (6.9) follows from (6.10). Assume that $i > k$. If $j = k$, we again have $h = 1$. It remains to consider the case $i > k$, $j > k$. Setting $n = i\wedge j$ we have $n > k$, $x, y \in \pi\gamma_n$ and $z \in A \setminus \pi\gamma_{k+1}$. Writing $t = t_{k+1}\cdots t_n$ we have $|T| \le t$ by (6.7). By (1) and (2) this implies

$$h = \frac{s_n}{s_k} = \frac{\varphi(t_{k+1})\cdots\varphi(t_n)}{t_{k+1}\cdots t_n} \le \frac{\varphi(t)}{t} \le \frac{\varphi(|T|)}{|T|},$$

which yields the second inequality of (6.9) by (6.10). Thus f is η-quasisymmetric with a universal η.

We assume that f is (C,α)-quasisymmetric for some (C,α) and show that this leads to a contradiction. Fix $j \in \mathsf{N}$ and set

$$r = \sup\{|x| : x \in \pi\gamma_j\}, \quad R = \inf\,\{|x| : x \in A \setminus \pi\gamma_j\}.$$

We show that

$$t_j/3 \le r/R \le t_j. \tag{6.11}$$

The second inequality follows from (6.7). It implies that $r \le e^{-4}R < R/3$. For the first inequality, choose $u \in \gamma_j$ and $z \in p(\gamma_j)$. We must show that

$$\varrho(z,u) \ge \log(R/3r). \tag{6.12}$$

First observe that $\pi_2(u) \le d(\pi\gamma_j) \le 2r$. If $|\pi(z)| > r$, then $|\pi(z)| \ge R$, and hence $|\pi(z) - \pi(u)| \ge R - r \ge 2R/3$. By 2.9 we obtain

$$2R/3 \le \pi_2(u)e^{\varrho(z,u)-1} < 2re^{\varrho(z,u)},$$

and (6.12) follows.

If $|\pi(z)| \le r$, then $z = \langle x, y\rangle$ with $|x| \le r$, $y \in A \setminus \pi\gamma_j$. Now $|y| \ge R$, and hence

$$\pi_2(z) = |x - y| \ge |y| - |x| \ge R - r \ge 2R/3,$$

which yields

$$\varrho(z,u) \ge \log\frac{\pi_2(z)}{\pi_2(u)} \ge \log\frac{2R/3}{2r} = \log\frac{R}{3r},$$

and (6.12) is proved.

Pick $a \in \pi\gamma_j$ and $b \in A \setminus \pi\gamma_j$ with $|a| \ge r/2$ and $|b| \le 2R$. Then $a \in A_j$ and $b \in A_{j-1}$ by (6.7). Setting $|T| = (0, a, b)$ we obtain in view of (6.11)

$$|T| = \frac{|a|}{|b|} \le \frac{r}{R} \le t_j < 1, \quad |T| \ge \frac{r}{4R} \ge \frac{t_j}{12}.$$

Hence

$$|fT| = \frac{|f(a)|}{|f(b)|} = \frac{s_j|a|}{s_{j-1}|b|} = \frac{\varphi(t_j)|T|}{t_j} \ge \frac{\varphi(t_j)}{12}.$$

Since $|fT| \le C|T|^\alpha$, these inequalities imply that $t_j^{-\alpha}\varphi(t_j) \le 12C$. As $j \to \infty$, the left-hand side tends to ∞, and we obtain a contradiction.

Case 2. $\gamma_1 < \gamma_2 < \cdots$. Corresponding to (6.7), we have now

$$|x - y| \le t_i \cdots t_j |x - z| \tag{6.13}$$

whenever $i \le j$ and $x, y \in \pi\gamma_i$, $z \in A \setminus \pi\gamma_j$.

Let $\psi = \varphi^{-1}$: $(0,1] \to (0,1]$, $\psi(t) = e^{1-1/t}$, be the inverse of the function φ of Case 1. The following properties of ψ are easily verified:

(i) $\psi(st) \le \psi(s)\psi(t)$ for all $s, t \in (0,1]$.
(ii) $\psi(t)/t$ is increasing in t.
(iii) $\lim_{t\to 0} t^{-\alpha}\psi(t) = 0$ for all $\alpha > 0$.

We may assume that $0 \in \pi\gamma_1$. Writing

$$A_0 = \pi\gamma_1, \quad A_i = \pi\gamma_{i+1} \setminus \pi\gamma_i$$

we express A as the disjoint union of the sets A_i, $i \in \mathsf{N}$. Set $r_0 = 1$ and

$$r_i = \frac{t_1 \cdots t_i}{\psi(t_1)\cdots\psi(t_i)}$$

for $i \in \mathsf{N}$, where $t_i = e^{-g(\gamma_i)}$ as before. Observe that $1 < r_1 < r_2 < \cdots$. We show that the desired map $f\colon A \to E$ is obtained by setting $f(x) = r_i x$ for $x \in A_i$.

We first show that

$$r_{i\vee j}|x-y|/2 \le |f(x)-f(y)| \le 2r_{i\vee j}|x-y| \tag{6.14}$$

whenever $x \in A_i$, $y \in A_j$.

If $i = j$, this is clear. Assume that $i < j$. Applying (6.13) to the triple $(0, x, y)$ we get

$$|x| \le t_j|y| \le t_1|y| < |y|/5 \le |x-y|/5 + |x|/5,$$

and hence $|x| \le |x-y|/4$. Consequently,

$$|f(x)-f(y)| = |r_i x - r_j y| \le r_i|x| + r_j|y| \le r_j(|x-y| + 2|x|) < 2r_j|x-y|.$$

Similarly

$$|f(x)-f(y)| \ge r_j(|y|-|x|) \ge r_j(|x-y| - 2|x|) \ge r_j|x-y|/2,$$

and (6.14) is proved.

Let $T = (x, y, z)$ be a triplet in A with $|T| \le 1$, $x \in A_i$, $y \in A_j$, $z \in A_k$. To prove that f is η-quasisymmetric it suffices to show that

$$\psi(|T|)/4 \le |fT| \le 4|T|. \tag{6.15}$$

Setting $h = r_{i\vee j}/r_{i\vee k}$ we obtain from (6.14)

$$h|T|/4 \le |fT| \le 4h|T|. \tag{6.16}$$

The case $i \vee k < j$ is impossible, since then $x, z \in \pi\gamma_j$, $y \in A \setminus \pi\gamma_j$, and (6.13) would give $|x-z| \le t_j|x-y| < |x-y|$. Hence $i \vee j \le i \vee k$, which gives $h \le 1$, and the second inequality of (6.15) follows from (6.16).

If $i \ge k$ or if $i < k = j$, then $h = 1$, and (6.15) follows from (6.16). It remains to consider the case $i < k$, $j < k$. Setting $n = i \vee j$ we have $n < k$. Now $x, y \in \pi\gamma_{n+1}$ and $z \in A \setminus \pi\gamma_k$. Writing $t = t_{n+1}\cdots t_k$ we have $|T| \le t$ by (6.13). By (i) and (ii) this implies

$$h = \frac{r_n}{r_k} = \frac{\psi(t_{n+1})\cdots\psi(t_k)}{t_{n+1}\cdots t_k} \ge \frac{\psi(t)}{t} \ge \frac{\psi(|T|)}{|T|},$$

which yields the first part of (6.15) by (6.16).

We assume that f is (C,α)-quasisymmetric and show that this leads to a contradiction. Fix $j \in \mathsf{N}$ and define r and R as in Case 1. The formula (6.11) is again valid. Pick $a \in A_j$, $b \in A_{j-1}$ with $|a| \leq 2R$, $|b| \geq r/2$. Setting $T = (0, a, b)$ we obtain by (6.11)

$$|T| = \frac{|a|}{|b|} \geq \frac{R}{r} \geq \frac{1}{t_j} > 1, \quad |T| \leq \frac{4R}{r} \leq \frac{12}{t_j},$$
$$|fT| = \frac{r_j|a|}{r_{j-1}|b|} = \frac{t_j|T|}{\psi(t_j)} \geq \frac{1}{\psi(t_j)}.$$

Since $|fT| \leq C|T|^{1/\alpha}$, these estimates yield

$$t_j^{-1/\alpha}\psi(t_j) \geq 12^{-1/\alpha}C^{-1}.$$

As $j \to \infty$, the left-hand side tends to 0 by (iii), and we reach a contradiction.

Case 3. γ_i and γ_j are incomparable for $i \neq j$. We express A as a disjoint union $A = \cup\{A_i : i \in \mathsf{N}_0\}$, where $A_i = \pi\gamma_i$ for $i \in \mathsf{N}$ and $A_0 = A \setminus \cup\{A_i : i \in \mathsf{N}\}$. Let φ be as in Case 1, and set $s_0 = 1$, $s_i = \varphi(t_i)/t_i$ for $i \in \mathsf{N}$. Fix arbitrary points $x_i \in A_i$, $i \in \mathsf{N}$, and define f: $A \to E$ by $f(x) = x$ for $x \in A_0$ and by

$$f(x) = x_i + s_i(x - x_i)$$

for $x \in A_i$, $i \geq 1$. If $x, y \in A_i$, $i \geq 1$, and $z \in A \setminus A_i$, then 5.7 gives

$$|x - y| \leq t_i|x - z|. \tag{6.17}$$

If $x, y \in A_i$ for some i, then $|f(x) - f(y)| = s_i|x - y|$. We show that

$$|x - y|/2 \leq |f(x) - f(y)| \leq 2|x - y| \tag{6.18}$$

whenever $x \in A_i$ and $y \in A_j$, $i \neq j$. We assume that $1 \leq i < j$; the case $i = 0$ is similar but easier. By (6.17) we have $|x - x_i| \leq t_i|x - y|$, and hence

$$|f(x) - x| \leq |f(x) - x_i| = s_i|x - x_i| \leq \varphi(t_i)|x - y| \leq |x - y|/5,$$

and similarly $|f(y) - y| \leq |x - y|/5$. These estimates imply (6.18).

Let $T = (x, y, z)$ be a triplet in A with $|T| \leq 1$, $x \in A_i$, $y \in A_j$, $z \in A_k$. To prove that f is η-quasisymmetric it suffices to show that

$$|T|/4 \leq |fT| \leq 4\varphi(|T|). \tag{6.19}$$

If $j \neq i \neq k$, then (6.18) implies that $|T|/4 \leq |fT| \leq 4|T|$, and (6.19) holds. If $i = j = k$, then $|fT| = |T|$. The case $1 \leq i = k \neq j$ is impossible, since then $|T| > 1$ (6.17). If $0 = i = k \neq j$ or $0 = i = j \neq k$, then (6.18) gives $|T|/2 = |fT| \leq 2|T|$. It remains to consider the case $1 \leq i = j \neq k$. Now $|fT| = s_i|x-y|/|f(x)-f(z)|$. Since $s_i \geq 1$, this and (6.18) give $|fT| \geq |T|/2$. Since $|T| \leq t_i$ by (6.17) and since $\varphi(t)/t$ is decreasing in t, we obtain $s_i = \varphi(t_i)/t_i \leq \varphi(|T|)/|T|$, and hence $|fT| \leq 2\varphi|T|$.

Finally, assume that f is (C, α)-quasisymmetric. Arguing as in Case 1 we can find points $a \in A_j$ and $b \in A \setminus A_j$ such that the triplet $T = (x_j, a, b)$ satisfies the inequalities $t_j/12 \leq |T| \leq t_j$ and $|fT| \geq \varphi(t_j)/24$, which give the contradicition as in Case 1. □

6.20. **Theorem.** *A metric space A is relatively connected if and only if every quasisymmetric map of A is power quasisymmetric.*

Proof. Since A can be isometrically embedded into a Banach space [Du, XIII.5.2], the theorem follows from 6.5 and from 6.6. □

We next give a quantitative version of 6.20. The proof is easier than in 6.6.

6.21. **Theorem.** *The following conditions are quantitatively equivalent for a metric space A:*

(1) *A is M-relatively connected,*

(2) *Every η-quasisymmetric map of A is (C,α)-quasisymmetric with (C,α) depending only on η.*

Proof. Observe that the data for (1) is M, and the data for (2) is the function $\eta \mapsto (C,\alpha)$.

By Theorem 6.5, (1) implies (2). Conversely, assume that (2) is true. We may again assume that A lies in a Banach space E.

Let $\varphi\colon (0,1] \to (0,1]$ be the function $\varphi(t) = 1/(1-\log t)$, considered in the proof of 6.6. By 6.4, there is η such that an injective map $f\colon A \to E$ is η-quasisymmetric whenever it satisfies the inequalities

$$|T|/4 \le |fT| \le 4\varphi(|T|) \tag{6.22}$$

for each triplet T in A with $|T| \le 1$. Let (C,α) be the pair given by (2) for this η. Choose q such that $0 < q \le 1/4$ and such that $t^{-\alpha}\varphi(t) \ge 13C$ for $0 < t \le q$. Set $c = 1/q$ and $M = c+1$. Then M depends only on the function $\eta \mapsto (C,\alpha)$.

We show that A is M-relatively connected. Assume that this is not true. Since $M > c$, condition 4.11(2) is not true with this c. Hence there are $x_0 \in A$ and $r > 0$ such that

$$A \not\subset \overline{B}(x_0,r), \quad A \cap \overline{B}(x_0,r) \ne \{x_0\}, \quad A \cap \big(\overline{B}(x_0,r) \setminus B(x_0,qr)\big) = \varnothing.$$

Replacing r by a larger number we may assume that A meets $B(x_0,2r)\setminus B(x_0,r)$. With an auxiliary similarity map of E we can further assume that $x_0 = 0$ and $r = 1$.

Setting $h = \sup\,\{|x| : x \in A \cap B(0,1)\}$ we have $0 < h \le q \le 1/4$. Write $s = \varphi(3h)/3h$ and define $f\colon A \to E$ by $f(x) = x$ for $|x| \ge 1$ and by $f(x) = sx$ for $|x| \le h$. We show that f is η-quasisymmetric but not (C,α)-quasisymmetric, which will give a contradiction.

If $|x| \le h$, then $|f(x)| \le sh = \varphi(3h)/3 < 1/3$. Hence

$$|x-y|/2 \le |f(x) - f(y)| \le 2|x-y|$$

whenever $x, y \in A$ and $|x| \le h$, $|y| \ge 1$. Let $T = (x, y, z)$ be a triplet in A with $|T| \le 1$. We have $|T|/4 \le |fT| \le 4|T|$ except for the second inequality in the case $x, y \in \overline{B}(0, h)$, $|z| \ge 1$. In this case we have

$$|T| = \frac{|x-y|}{|x-z|} \le \frac{2h}{1-h} < 3h,$$

since $h \le 1/4$. Since $\varphi(t)/t$ is decreasing in t, this implies that $s \le \varphi(|T|)/|T|$, and hence

$$|fT| \le \frac{s|x-y|}{|x-z|/2} \le 2\varphi(|T|).$$

We have proved that (6.22) holds, and hence f is η-quasisymmetric.

Choose points $a, b \in A$ with $h/2 \le |a| \le h$ and $1 \le |b| \le 2$. For the triplet $T = (0, a, b)$ we have $|T| = |a|/|b| \le h$ and

$$|fT| = \frac{s|a|}{|b|} \ge \frac{sh}{4} = \frac{\varphi(3h)}{12} > \frac{\varphi(h)}{12}.$$

Since $\varphi(h) \ge 13Ch^\alpha$, we obtain $|fT| > 13C|T|^\alpha/12$, and hence f is not (C, α)-quasisymmetric. □

References

[AT] Aseyev, V.V., and D.A. Trotsenko: Quasisymmetric imbeddings, quadruples of points and distortions of moduli. - Sibirsk. Mat. Zh. 28, 1987, 32–38 (Russian).

[BS] Bonk, M., and O. Schramm: Embeddings of Gromov hyperbolic spaces. - Preprint, 1997.

[Du] Dugundji, J.: Topology. - Allyn and Bacon, 1966.

[Po] Pommerenke, C.: Uniformly perfect sets and the Poincaré metric. - Arch. Math. 32, 1979, 192–199.

[Tr] Trotsenko, D.A.: Conditions for extensions of analogs of quasiconformal maps. - Proceedings of the second conference for young scientists of Siberia and Far East, Novosibirsk, 1988, 157–160 (Russian).

[TV] Tukia, P., and J. Väisälä: Quasisymmetric embeddings of metric spaces. - Ann. Acad. Sci. Fenn. Math. 5, 1984, 97–114.

Received 6 October 1997