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STABILITY ON A CONE IN TERMS OF TWO MEASURES FOR DIFFERENTIAL EQUATIONS WITH "MAXIMA"

ANGEL DISHLIEV[1] AND SNEZHANA HRISTOVA[2*]

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ABSTRACT. Stability in terms of two measures for nonlinear differential equations with "maxima" is studied. A special type of stability in terms of two measures is defined. The new type of stability generalizes some of the known in the literature. Sufficient conditions for the defined stability are obtained. Cone-valued continuous Lyapunov functions are applied. Method of Razumikhin as well as comparison method for scalar ordinary differential equations have been employed. The usefulness of the introduced definition and the obtained sufficient conditions is illustrated through an example.

1. Introduction and preliminaries

One type of functional differential equations is the case when the evolutionary equations using the maximum of the studied function on a past time interval. Since the maximum function has very specific properties, it makes the equations strongly nonlinear. As a result these equations gain an important place in the theory of differential equations and are called differential equations with "maxima"([2, 3, 7]). Differential equations with "maxima" first appeared as an object of investigation about thirty years ago in connection with the solution of some applied problems. For example, in the theory of automatic control of various technical systems it often occurs that the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals. Popov ([13]) in 1966 considered the system for regulating the voltage of a generator of constant current. The object of the experiment was a generator of constant

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* Corresponding author.

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current with parallel simulation and the regulated quantity was the voltage at the source electric current. The equation describing the work of the regulator involves the maximum of the unknown function and it has the form ([13])

$$T_0 u'(t) + u(t) + q \max_{s\in[t-h,t]} u(s) = f(t),$$

where T_0 and q are constants characterizing the object, $u(t)$ is the voltage regulated and $f(t)$ is the perturbed effect.

It is relevant to mention here the opinion of Myshkis that "the specific character of these equations is not yet sufficiently clear". In his survey in 1975 ([12]) he also distinguishes the equations with maxima as differential equations with deviating argument of complex structure.

The problems of stability of solutions of differential equations via Lyapunov functions have been successfully investigated in the past. One type of stability, very useful in real world problems, deals with two different measures. Stability in terms of two measures for differential equations has been studied by means of various types of Lyapunov functions ([4, 5, 8, 9, 10, 11]).

One of the main problems of Lyapunov's second method is related to the construction of an appropriate Lyapunov function. Often, it is easier to construct a vector Lyapunov function rather than a scalar one. However, vector functions require comparison systems of differential equations. In order to involve scalar differential equations in the comparison method instead of comparison systems, we use a special type of stability that combines the ideas of two different measures and a dot product. In the present paper d-stability in terms of two different measures is defined for differential equations with "maxima". Cone-valued Lyapunov functions and comparison results for scalar ordinary differential equations are employed to study the introduced stability.

2. Preliminary notes and definitions

Consider the system of nonlinear differential equations with "maxima"

$$x' = F(t, x(t), max_{s\in[t-r,t]} x(s)) \quad \text{for} \quad t \geq 0\,, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $F = (F_1, F_2, \dots, F_n)$, $r > 0$ is a constant.

Let $\phi \in C([-r,0], \mathbb{R}^n)$ and $t_0 \in \mathbb{R}_+$ be a fixed point. We denote by $x(t; t_0, \phi)$ the solution of system (2.1) with initial conditions

$$x(t) = \phi(t - t_0), \quad t \in [t_0 - r, t_0]. \tag{2.2}$$

Let $x, y \in \mathbb{R}^n$. Denote by $(x \bullet y)$ the dot product of both vectors x and y. Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone. Consider the set

$$\mathcal{K}^* = \{\varphi \in \mathbb{R}^n : \ (\varphi \bullet x) \geq 0 \text{ for any } \ x \in \mathcal{K}\}.$$

We assume that $\mathcal{K}^*$ is a cone.

Consider following sets

$$K = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) \ : \ a(s) \text{ is strictly increasing and } a(0) = 0\};$$

$$\mathcal{G} = \{h \in C([-r,\infty) \times \mathbb{R}^n, \mathcal{K}) \ : \ \inf_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for each } t \geq -r\}.$$

Note, for example, $\mathcal{K} \equiv \mathbb{R}^n_+$ is a cone and the function

$$h(t,x) = (e^{-t}|x_1|, e^{-t}|x_2|, \dots, e^{-t}|x_n|), \quad x \in \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n)$$

is from the set $\mathcal{G}$.

Let ρ be positive constant, $\varphi_0 \in \mathcal{K}^*$, $h \in \mathcal{G}$. Define

$$\mathcal{S}(h, \rho, \varphi_0) = \{(t,x) \in [0,\infty) \times \mathbb{R}^n : \quad (\varphi_0 \bullet h(t,x)) < \rho\}.$$

We will introduce the definition of a new type of stability for differential equations with "maxima", that combines the ideas of stability in terms of two measures ([10, 11]) and a dot product.

Definition 2.1. System of differential equations with "maxima" (2.1) is said to be

(S1) *d-stable in terms of two measures* if there exist a vector $\varphi_0 \in \mathcal{K}^*$ and two functions $h_0, h \in \mathcal{G}$ such that for every $\epsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for any $\phi \in C([-r,0], \mathbb{R}^n)$ inequality $(\varphi_0 \bullet h_0(t_0 + s, \phi(s))) < \delta$ for $s \in [-r, 0]$ implies $(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < \epsilon$ for $t \geq t_0$, where $x(t; t_0, \phi)$ is a solution of differential equations with "maxima"(2.1) with initial condition (2.2);

(S2) *uniformly d-stable in terms of two measures* if (S1) is satisfied, where δ is independent on t_0;

Remark 2.1. The vector φ_0 , that is introduced into Definition 1, is a proxy for the weights of the solution's components.

Remark 2.2. In the partial case of one dimensional cone $\mathcal{K} = \mathbb{R}_+$ the measures h and h_0 are scalar valued nonnegative functions. The above-defined d-stability in terms of two measures reduces to stability in terms of two measures for differential equations with "maxima".

In our further investigations we will use following comparison scalar ordinary differential equations

$$u' = g(t,u)\,, \tag{2.3}$$

where $u \in \mathbb{R}$, $g(t,0) \equiv 0$.

We will use some properties of the functions from class $\mathcal{G}$.

Definition 2.2. Let functions $h, h_0 \in \mathcal{G}$ and vector $\varphi_0 \in \mathcal{K}^*$. The function h_0 is *uniformly* φ_0*-finer* than the function h if there exist a constant $\sigma > 0$ and a function $p \in K$ such that for any point $(t,x) \in [-r,\infty) \times \mathbb{R}^n :\quad (\varphi_0 \bullet h_0(t,x)) < \sigma$ the inequality $(\varphi_0 \bullet h(t,x)) \leq p((\varphi_0 \bullet h_0(t,x))$ holds.

We will introduce the following class of functions:

Definition 2.3. We will say that function $V(t,x) : \Omega \times \mathbb{R}^n \to \mathcal{K}, \quad \Omega \subset \mathbb{R}_+$, $V = (V_1, V_2, \dots, V_n)$, belongs to the class $\mathcal{L}$ if:

1. $V(t,x) \in C^1(\Omega \times \mathbb{R}^n, \mathcal{K})$;
2. There exist constants $M_i > 0,\ i = 1, 2, \dots, n$, such that $|V_i(t,x) - V_i(t,y)| \leq M_i||x - y||$ for any $t \in \Omega,\ x, y \in \mathbb{R}^n$.

Let function $V \in \mathcal{L}$, $V = (V_1, V_2, \dots, V_n)$ and $\phi \in C([-r,0],\mathbb{R}^n)$. We define a derivative $\mathcal{D}_{(2.1)}V(t,x)$ of the function V among the system (2.1) by the equalities

$$\mathcal{D}_{(2.1)}V_i(t,\phi(0)) = \frac{\partial V_i(t,\phi(0))}{\partial t} + \sum_{j=1}^{n} \frac{\partial V_i(t,\phi(0))}{\partial x_j} F_j(t,\phi(0), sup_{s\in[-r,0]}\phi(s))$$

$1 \le i \le n$, where $\mathcal{D}_{(2.1)}V(t,x) = (\mathcal{D}_{(2.1)}V_1(t,x), \mathcal{D}_{(2.1)}V_2(t,x), \dots, \mathcal{D}_{(2.1)}V_n(t,x))$.

In the further investigations we will use following comparison result:

Lemma 2.1. *Let following conditions be fulfilled:*

1. Vector $\varphi_0 \in \mathcal{K}^*$ and function $V(t,x) : [t_0,T] \times \mathbb{R}^n \to \mathcal{K}$, $V \in \mathcal{L}$ are such that for any function $\psi \in C([-r,0],\mathbb{R}^n)$ and any number $t \in [t_0,T]$ such that $(\varphi_0 \bullet V(t,\psi(0))) > (\varphi_0 \bullet V(t+s,\psi(s)))$ for $s \in [-r,0)$ the inequality

$$\Big(\varphi_0 \bullet \mathcal{D}_{(2.1)}V(t,\psi(0))\Big) \le g(t, (\varphi_0 \bullet V(t,\psi(0))))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), g(t,0) \equiv 0$.

2. Function $x(t) = x(t;t_0,\varphi)$ is a solution of (2.1) with initial condition $x(t_0 + s) = \varphi(s)$, $s \in [-r,0]$, that is defined for $t \in [t_0 - r, T]$ where $\varphi \in C([-r,0],\mathbb{R}^n)$.

3. Function $u^*(t) = u^*(t;t_0,u_0)$ is the maximal solution of (2.3) with initial condition $u^*(t_0) = u_0$, that is defined for $t \in [t_0,T]$.

Then the inequality $\max_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\varphi(s))) \le u_0$ implies the validity of the inequality $(\varphi_0 \bullet V(t,x(t))) \le u^*(t)$ for $t \in [t_0,T]$.

Proof. Let $u_n(t)$ be the maximal solution of the initial value problem

$$u' = g(t,u) + \frac{1}{n},$$

$$u(t_0) = u_0 + \frac{1}{n},$$

where $\max_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s,\varphi(s))) \le u_0$ and n is a natural number. Assume that $u_n(t)$ is defined for $t \in [t_0,T]$.

Define a function $m(t) \in C([t_0 - r, T], \mathbb{R}_+)$ by $m(t) = (\varphi_0 \bullet V(t,x(t)))$. Because $u^*(t;t_0,u_0) = lim_{n\to\infty} u_n(t)$ it is enough to prove that for any natural number n the inequality

$$m(t) \le u_n(t) \quad \text{for} \ \ t \in [t_0,T] \tag{2.4}$$

holds.

Note that for any natural number n the inequality $m(t_0+s) < u_n(t_0)$, $s \in [-r,0]$ holds.

Assume that inequality (2.4) is not true. Let n be a natural number such that there exists a point $\eta \in (t_0,T) : m(\eta) > u_n(\eta)$. Let $t_n^* = max\{t \in [t_0,T] : m(s) < u_n(s)$ for $s \in [t_0,t)\}$, $t_n^* < T$.

Therefore

$$m(t_n^*) = u_n(t_n^*), \quad m(t) < u_n(t) \ \text{for}\ t \in [t_0,t_n^*), \quad m(t) \ge u_n(t) \ \text{for}\ t \in (t_n^*, t_n^*+\delta), \tag{2.5}$$

where $\delta > 0$ is enough small number.

From inequalities (2.5) it follows that

$$D_{-}m(t_n^*) \geq u_n'(t_n^*) = g(t, u_n(t_n^*)) + \frac{1}{n} = g(t, m(t_n^*)) + \frac{1}{n}. \tag{2.6}$$

From $g(t,u)+\frac{1}{n} > 0$ on $[t_n^*-r, t_n^*]$ it follows that function $u_n(t)$ is nondecreasing on $[t_n^* - r, t_n^*] \cap [t_0, T]$.

Let $t_n^* - r \geq t_0$. Then $m(t_n^*) = u_n(t_n^*) \geq u_n(s) \geq m(s)$ for $s \in [t_n^* - r, t_n^*)$.

Let $t_n^* - r < t_0$. Then as above $m(t_n^*) \geq m(s)$ for $s \in [t_0, t_n^*)$. For $s \in [t_n^* - r, t_0)$ from the monotonicity of the function $u_n(t)$ we get $m(t_n^*) = u_n(t_n^*) \geq u_n(t_0+0) = u_0 + \frac{1}{n} > u_0 \geq \sup_{s\in[-r,0]}(\varphi_0 \bullet V(t_0+s, \phi(s))) \geq m(s)$.

Therefore, $m(t_n^*) = u_n(t_n^*) \geq u_n(s) > m(s)$ for $s \in [t_n^* - r, t_n^*)$. According to condition (i) using standard argument we get $D_{-}m(t_n^*) \leq g(t_n^*, m(t_n^*)) < g(t_n^*, m(t_n^*)) + \frac{1}{n}$ that contradicts (2.6). Therefore the inequality (2.4) holds and hence the conclusion of Lemma 1 follows. □

Remark 2.3. Note the condition $(\varphi_0 \bullet V(t_0, \varphi(0))) \leq u_0$ is not enough for validity of the conclusion of Lemma 1. It is necessary the inequality to be fulfilled on the whole initial interval.

3. Main results

We will use Razhumikhin method and Lyapunov functions to obtain sufficient conditions for stability of solutions of differential equations with "maxima". We will consider both cases of multidimensional and one dimensional cone.

3.1. Multidimensional case.

We will consider the cone $\mathcal{K} \subset \mathbb{R}^n$, $n > 1$ and we will obtain sufficient conditions for d-stability in terms of two measures of systems of differential equations with "maxima". We will employ cone valued Lyapunov functions from class $\mathcal{L}$. The proof is based on Razumikhin method combined by comparison method with scalar ordinary differential equations.

Theorem 3.1. *Let the following conditions be fulfilled:*

1. The function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

2. The vector $\varphi_0 \in \mathcal{K}^*$ and the functions $h_0, h \in \mathcal{G}$, h_0 is uniformly φ_0-finer than h.

3. There exists a function $V(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathcal{K}$, $V \in \mathcal{L}$ such that

(i) $b((\varphi_0 \bullet h(t,x))) \leq (\varphi_0 \bullet V(t,x)) \leq a((\varphi_0 \bullet h_0(t,x)))$, $\quad (t,x) \in \mathcal{S}(h, \rho, \varphi_0)$ where $a, b \in K$;

(ii) for any function $\psi \in C([-r,0], \mathbb{R}^n)$ and any number $t \geq 0$ such that $(\varphi_0 \bullet V(t, \psi(0))) > (\varphi_0 \bullet V(t+s, \psi(s)))$ for $s \in [-r, 0)$ and $(t, \psi(0))) \in \mathcal{S}(h, \rho, \varphi_0)$ the inequality

$$\Big(\varphi_0 \bullet \mathcal{D}_{(2.1)} V(t, \psi(0))\Big) \leq g(t, (\varphi_0 \bullet V(t, \psi(0))))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t,0) \equiv 0$, $\rho > 0$ is a constant.

4. For any initial function $\phi \in C([-r,0], \mathbb{R}^n)$ the solution of the initial value problem for systems of differential equations with "maxima" (2.1),(2.2) exists on $[t_0 - r, \infty)$, $t_0 \geq 0$.

5. For any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar equation (2.3) exists on $[t_0, \infty)$, $t_0 \geq 0$.

6. Zero solution of scalar differential equation (2.3) is equi-stable.

Then system of differential equations with "maxima" (2.1) is d-stable in terms of two measures.

Proof. Let $\epsilon > 0$ be a fixed number, $\epsilon < \rho$, and $t_0 \geq 0$ be a fixed point.

From condition 2 of Theorem 3.1 follows there exist a constant $\sigma > 0$ and a function $p \in K$. Therefore we can find $\delta_1 = \delta_1(\epsilon) > 0$, $\delta_1 < \rho$ such that the inequality

$$p(\delta_1) < \epsilon \tag{3.1}$$

holds.

Since the zero solution of scalar impulsive differential equation (2.3) is equi-stable there exists $\delta_2 = \delta_2(t_0, \epsilon) > 0$ such that inequality $|u_0| < \delta_2$ implies

$$|u(t; t_0, u_0)| < b(\epsilon), \quad t \geq t_0,$$

where $u(t; t_0, u_0)$ is the maximal solution of (2.3) with initial condition $u(t_0) = u_0$.

Since the function $a \in K$, that is defined in condition (i), we can find $\delta_3 = \delta_3(t_0, \epsilon) > 0$, $\delta_3 < \rho$ such that the inequality

$$a(\delta_3) < \delta_2 \tag{3.2}$$

holds.

Now let function $\phi \in C([-r, 0], \mathbb{R}^n)$ be such that

$$(\varphi_0 \bullet h_0(t_0 + s, \phi(s))) < \delta_4 \quad \text{for} \quad s \in [-r, 0], \tag{3.3}$$

where $\delta_4 = min\{\delta_1, \delta_3, \sigma\}$, $\delta_4 = \delta_4(t_0, \epsilon) > 0$.

From condition 2, inequalities (3.1), (3.3), and the choice of δ_4 follows that for $s \in [-r, 0]$ the inequality

$$(\varphi_0 \bullet h(t_0 + s, x(t_0 + s; t_0, \phi))) < p((\varphi_0 \bullet h_0(t_0 + s, \phi(s)))) < p(\delta_4) \leq p(\delta_1) < \epsilon \tag{3.4}$$

holds, where $x(t; t_0, \phi)$ is a solution of initial value problem (2.1),(2.2).

We will prove that if inequality (3.3) is satisfied then

$$(\varphi_0 \bullet h(t, x(t; t_0, \phi))) < \epsilon \quad \text{for} \quad t \geq t_0. \tag{3.5}$$

Suppose inequality (3.5) is not true. From inequality (3.4) follows that there exists a point $t^* > t_0$ such that

$$(\varphi_0 \bullet h(t^*, x(t^*; t_0, \phi))) = \epsilon, \quad (\varphi_0 \bullet h(t, x(t; t_0, \phi))) < \epsilon, \quad t \in [t_0 - r, t^*). \tag{3.6}$$

Inequalities (3.4), (3.6), $\epsilon < \rho$ and the inclusion $\mathcal{S}(h, \epsilon, \varphi_0) \subset \mathcal{S}(h, \rho, \varphi_0)$ prove that

$$(t, x(t; t_0, \phi)) \in \mathcal{S}(h, \rho, \varphi_0) \quad \text{for} \quad t \in [t_0 - r, t^*]. \tag{3.7}$$

From inequality (3.2), condition (i) and the inclusion (3.7) follows that

$$max_{s \in [-r,0]}(\varphi_0 \bullet V(t_0 + s, \varphi(s))) \leq a((\varphi_0 \bullet h_0(t_0 + s, x(t_0 + s; t_0, \phi))) \leq a(\delta_3) < \delta_2. \tag{3.8}$$

Therefore according to inequalities (3.1), (3.8) we get

$$u^*(t; t_0, u_0^*) < b(\epsilon), \quad t \geq t_0, \tag{3.9}$$

where $u^*(t; t_0, u_0^*)$ is the maximal solution of the scalar equation (2.3) with initial condition $u(t_0) = u_0^*$, $u_0^* = max_{s\in[-r,0]}(\varphi_0 \bullet V(t_0 + s, \varphi(s)))$.

According to Lemma 2.1 the inequality

$$(\varphi_0 \bullet V(t, x(t; t_0, \phi))) \le u^*(t; t_0, u_0^*) \quad \text{for} \quad t \in [t_0, t^*] \tag{3.10}$$

holds.

From inequalities (3.9), (3.10), the choice of the point t^*, and condition (i) of Theorem 3.1 we obtain

$$\begin{aligned} b(\epsilon) &> u^*(t^*; t_0^*, u_0^*) \ge (\varphi_0 \bullet V(t^*, x(t^*; t_0, \phi))) \\ &\ge b((\varphi_0 \bullet h(t^*, x(t^*; t_0, \phi)))) = b(\epsilon). \end{aligned}$$

The obtained contradiction proves the validity of inequality (3.5) for $t \ge t_0$.

Inequality (3.5) proves d-stability in terms of two measures of the considered system of differential equations with “maxima”. □

Theorem 3.2. *Let the following conditions be fulfilled:*

1. Conditions 1, 2, 3, 4, 5 of Theorem 3.1 are satisfied.

2. Zero solution of scalar differential equation (2.3) is uniformly stable.

Then system of differential equations with “maxima” (2.1) is uniformly d-stable in terms of two measures.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1 and we omit it.

3.2. **One dimensional case.** Now consider the cone $\mathcal{K} = \mathbb{R}_+$.

In this case the measures are scalar functions from the class

$$\tilde{\mathcal{G}} = \{h \in C([-r, \infty) \times \mathbb{R}^n, \mathbb{R}_+) \ : \ \inf_{x\in\mathbb{R}^n} h(t, x) = 0 \text{ for each } t \ge -r\},$$

the set

$$\tilde{\mathcal{S}}(h, \rho) = \{(t, x) \in [0, \infty) \times \mathbb{R}^n : \quad h(t, x) < \rho\},$$

and the Lyapunov functions $V(t, x) : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ are such that

$$\mathcal{D}_{(2.1)}V(t, \phi(0)) = \frac{\partial V(t, \phi(0))}{\partial t} + \sum_{j=1}^{n} \frac{\partial V(t, \phi(0))}{\partial x_j} F_j(t, \phi(0), sup_{s\in[-r,0]}\phi(s)).$$

As partial cases of the proved above Theorem 3.1 and Theorem 3.2 we obtain sufficient conditions for stability in terms of two measures of systems of differential equations with “maxima”.

Theorem 3.3. *Let following conditions be fulfilled:*

1. The function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

2. The functions $h_0, h \in \tilde{\mathcal{G}}$ *and there exist a constant* $\sigma > 0$ *and a function* $p \in K$ *such that for any point* $(t, x) \in [-r, \infty) \times \mathbb{R}^n : \quad h_0(t, x) < \sigma$ *the inequality* $h(t, x) \le p(h_0(t, x))$ *holds.*

3. There exists a function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $V \in \mathcal{L}$, $(n = 1)$, *such that*

(i) $b(h(t, x)) \le V(t, x) \le a(h_0(t, x)), \quad (t, x) \in \tilde{\mathcal{S}}(h, \rho)$

where $a, b \in K$*;*

(ii) for any function $\psi \in C([-r,0],\mathbb{R}^n)$ and any number $t \geq 0$ such that $V(t,\psi(0)) > V(t+s,\psi(s))$ for $s \in [-r,0)$ and $(t,\psi(0))) \in \tilde{\mathcal{S}}(h,\rho)$ the inequality

$$\mathcal{D}_{(2.1)}V(t,\psi(0)) \leq g(t,V(t,\psi(0)))$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t,0) \equiv 0$, $\rho > 0$ is a constant.

4. For any initial function $\phi \in C([-r,0],\mathbb{R}^n)$ the solution of the initial value problem for systems of differential equations with "maxima" (2.1),(2.2) exists on $[t_0 - r, \infty)$, $t_0 \geq 0$.

5. For any initial point $(t_0,u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar equation (2.3) exists on $[t_0,\infty)$, $t_0 \geq 0$.

6. Zero solution of scalar differential equation (2.3) is equi-stable.

Then system of differential equations with "maxima" (2.1) is stable in terms of two measures.

Theorem 3.4. *Let the following conditions be fulfilled:*

1. Conditions 1, 2, 3, 4, 5 of Theorem 3 are satisfied.

2. Zero solution of scalar differential equation (2.3) is uniformly stable.

Then system of differential equations with "maxima" (2.1) is uniformly stable in terms of two measures.

Remark 3.5. As partial cases of the above results we obtain:

Case 1. Let the cone $\mathcal{K} \subset \mathbb{R}^n$, $n > 1$, $r = 0$ and $h(t,y) = h_0(t,y) = (|y_1|,|y_2|,\ldots,|y_n|)$. Then the above results reduce to results for φ_0-stability for ordinary differential equations, that is defined and studied in [1, 6] by employing comparison systems instead of comparison scalar equations.

Case 2. Let the cone $\mathcal{K} \subset \mathbb{R}^n$, $n > 1$, $h(t,y) = h_0(t,y) = (|y_1|,|y_2|,\ldots,|y_n|)$ and φ_0 is the unit vector. Then results of Theorem 1 and Theorem 2 reduce to results for stability of differential equations with "maxima"([2, 14]).

Case 3. Let the cone $\mathcal{K} \subset \mathbb{R}^n$, $n > 1$, the constant $r = 0$, the measures $h(t,y) = h_0(t,y) = (|y_1|,|y_2|,\ldots,|y_n|)$ and the vector φ_0 is the unit vector. Then results of Theorem 1 and Theorem 2 reduce to results for stability of ordinary differential equations, studied by many authors.

Case 4. Let the cone $\mathcal{K} \subset \mathbb{R}^n$, $n > 1$, the constant $r = 0$, the measures $h(t,y) = h_0(t,y) = (|y_1|,|y_2|,\ldots,|y_n|)$ and vector φ_0 consists only of 0s and 1s. Then the above results reduce to results about partial stability of differential equations with "maxima".

4. Applications.

Now we will illustrate the application of the defined above stability in terms of two measures and the obtained sufficient conditions on an example.

Example 4.1. Consider the system of differential equations with "maxima"

$$x'(t) = -x(t) + 4y(t) + \frac{1}{2} \max_{s\in[t-r,t]} x(s) \tag{4.1}$$

$$y'(t) = -x(t) - y(t) + \frac{1}{2} \max_{s\in[t-r,t]} y(s), \qquad t \geq t_0, \tag{4.2}$$

with initial conditions

$$x(t) = \phi_1(t-t_0), \quad y(t) = \phi_2(t-t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{4.3}$$

where $x, y \in \mathbb{R}$, $r > 0$ is small enough constant, $t_0 \geq 0$.

We will study the stability of the solution of (4.1), (4.2) by applying different Lyapunov functions and two measures for the initial function and the solution. Note the corresponding system of differential equations without perturbations of maximum functions is stable.

Case 1. (vector Lyapunov function and a dot product).

Consider the cone $\mathcal{K} = \{(x,y) : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$. Let functions $h_0(t,x,y) = (|x|,|y|)$, $h(t,x,y) = (x^2, y^2)$ and vector $\varphi_0 = (1,4)$. Note that the functions $h, h_0 \in \mathcal{G}$, vector $\varphi_0 \in \mathcal{K}^*$, and the function h_0 is uniformly φ_0-finer than the function h since there exist a constant $\delta = \frac{1}{4} > 0$ and a function $p \in K$, $p(u) \equiv u$ such that for any point $(t,x,y) \in [-r,\infty) \times \mathbb{R}^2 : \quad |x| + 4|y| < \delta$ the inequality $x^2 + 4y^2 \leq |x| + 4|y|$ holds.

Consider the set $\mathcal{S}(h,1,\varphi_0) = \{(x,y) \in \mathbb{R}^2 : \quad x^2 + 4y^2 < 1\}$. Define the Lyapunov function $V : \mathbb{R}^2 \to \mathcal{K}$, $\ V = (V_1, V_2)$, where $V_1(x,y) = \frac{1}{2}x^2$, $V_2(x,y) = \frac{1}{2}y^2$.

Then the condition (i) of Theorem 1 is fulfilled for functions $b(s) = \frac{1}{2}s \in K$ and $a(s) = \frac{1}{2}s^2 \in K$ since $\frac{1}{2}(x^2 + 4y^2) \leq \frac{1}{2}(|x| + 4|y|)^2$ for $(x,y) \in \mathcal{S}(h,1,\varphi_0)$.

Let function $\psi \in C([-r,0],\mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that the inequality

$$(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))) = \frac{1}{2}\psi_1^2(0) + 2\psi_2^2(0)$$

$$> \frac{1}{2}\psi_1^2(s) + 2\psi_2^2(s) = (\varphi_0 \bullet V(\psi_1(s), \psi_2(s)) \tag{4.4}$$

holds for $s \in [-r, 0)$. Then

$$\psi_1(0) \max_{s\in[t-r,t]} \psi_1(s) \leq |\psi_1(0)| \mid \max_{s\in[t-r,t]} \psi_1(s)| = \sqrt{(\psi_1(0))^2} \sqrt{(\max_{s\in[t-r,t]} \psi_1(s))^2}$$
$$\leq \sqrt{2(\varphi_0 \bullet V(\psi_1(0), \psi_2(0)))} \sqrt{2(\varphi_0 \bullet V(\psi_1(s), \psi_2(s))))}$$
$$\leq 2(\varphi_0 \bullet V(\psi_1(0), \psi_2(0)))$$

and

$$\psi_2(0) \max_{s\in[t-r,t]} \psi_2(s) \leq |\psi_2(0)| \mid \max_{s\in[t-r,t]} \psi_2(s)| = \sqrt{(\psi_2(0))^2} \sqrt{(\max_{s\in[t-r,t]} \psi_2(s))^2}$$
$$\leq \sqrt{\frac{1}{2}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))} \sqrt{\frac{1}{2}(\varphi_0 \bullet V(\psi_1(s), \psi_2(s)))}$$
$$\leq \frac{1}{2}(\varphi_0 \bullet V(\psi_1(0), \psi_2(0))).$$

Therefore if inequality (4.4) is fulfilled then

$$\begin{aligned}&\Big(\varphi_0 \bullet \mathcal{D}_{(4.1),(4.2)}V(\psi_1(0),\psi_2(0))\Big)\\&= -(\psi_1(0))^2 - 4(\psi_2(0))^2 + \frac{1}{2}\psi_1(0)\max_{s\in[t-r,t]}\psi_1(s) + 2\psi_2(0)\max_{s\in[t-r,t]}\psi_2(s)\\&\le -(\psi_1(0))^2 - 4(\psi_2(0))^2 + \frac{1}{2}(\varphi_0 \bullet V(\psi_1(0),\psi_2(0)))\\&\quad+\frac{1}{2}(\varphi_0 \bullet V(\psi_1(0),\psi_2(0)))\end{aligned}$$

or

$$\Big(\varphi_0 \bullet \mathcal{D}_{(4.1),(4.2)}V(\psi_1(0),\psi_2(0))\Big) \le -(\varphi_0 \bullet V(\psi_1(0),\psi_2(0)))) \le 0.$$

Consider the scalar comparison equation $u' = 0$ which zero solution is uniformly stable and according to Theorem 2 the system of differential equations with "maxima" (4.1), (4.2) is uniformly d- stable in terms of two measures, i.e. for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that inequality $\max_{s\in[-r,0]}\Big(|\phi_1(s)|+4|\phi_2(s)|\Big) < \delta$ implies

$$(x(t))^2 + 4(y(t))^2 < \epsilon \quad \text{for} \quad t \ge t_0, \tag{4.5}$$

where $x(t), y(t)$ is the solution of initial value problem (4.1)-(4.3).

Case 2. (scalar Lyapunov function).

Consider the scalar Lyapunov function $\tilde{V}(x,y) = \frac{1}{2}x^2 + 2y^2$ and both measures $h_0(t,x,y) = |x|+|y|$, $h(t,x,y) = x^2+y^2$. Note that construction of the Lyapunov function is easier in the case of cone valued functions, since it is easier to check conditions component wisely. In this case, as above, we could check that if (4.4) is satisfied then $\mathcal{D}_{(2.1)}\tilde{V}(\phi_1(0),\phi_2(0)) = \frac{\partial\tilde{V}(\phi_1(0),\phi_2(0))}{\partial x}\Big(-\phi_1(0) + 4\phi_2(0) + \frac{1}{2}\max_{s\in[t-r,t]}\phi_1(s)\Big) + \frac{\partial\tilde{V}(\phi_1(0),\phi_2(0))}{\partial y}\Big(-\phi_1(0) - \phi_2(0) + \frac{1}{2}\max_{s\in[t-r,t]}\phi_2(s)\Big) \le 0.$ According to Theorem 4 the solution of initial value problem (4.1)-(4.3) is stable since the zero solution of the scalar equation $u' = 0$ is stable, i.e.

$$h(t,x(t),y(t)) = (x(t))^2 + (y(t))^2 < \epsilon \tag{4.6}$$

provided that $|\phi_1(s)| + |\phi_2(s)| < \delta$. Note that the estimate (4.5) is better than (4.6). Therefore, the application of dot product gives us a better estimate of the solution, This is especially the case when one of the components of the solution plays a more important role on the stability and thus we can assign a higher weight to that component.

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[1] Department of Mathematics , University of Chemical Technology and Metallurgy, 1756 Sofia, Bulgaria.
E-mail address: dishliev@uctm.edu

[2] Department of Applied Mathematics and Modeling, Plovdiv University, 4000 Plovdiv, Bulgaria.
E-mail address: snehri@uni-plovdiv.bg