Annals of Functional Analysis

Ann. Funct. Anal. 1 (2010), no. 1, 26–35
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

STABILITY OF A FUNCTIONAL EQUATION RELATED TO DISTANCE MEASURES - II

GWANG HUI KIM[1] AND PRASANNA K. SAHOO[2]*

Communicated by S.-M. Jung

ABSTRACT. The present work continues the study of the stability of the functional equations of the type $f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s)$ namely (i) $f(pr, qs)+f(ps, qr) = g(p, q)\, g(r, s)$, and (ii) $f(pr, qs)+f(ps, qr) = g(p, q)\, h(r, s)$ for all $p, q, r, s \in G$, where G is an abelian group. These functional equations arise in the characterization of symmetrically compositive sumform distance measures.

1. INTRODUCTION

Let G be an abelian group. Let I denote the open unit interval (0, 1). Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Further, let

$$\Gamma_n^o = \left\{ P = (p_1, p_2, ..., p_n) \,\middle|\, 0 < p_k < 1,\ \sum_{k=1}^{n} p_k = 1 \right\}$$

denote the set of all n-ary discrete complete probability distributions (without zero probabilities), that is Γ_n^o is the class of discrete distributions on a finite set Ω of cardinality n with $n \geq 2$. Over the years, many distance measures between discrete probability distributions have been proposed. Hellinger coeeficient, Jeffreys distance, Chernoff coefficient, directed divergence, and its symmetrization J-divergence are examples of such measures (see [1] and [8]).

Almost all similarity, affinity or distance measures $\mu_n : \Gamma_n^o \times \Gamma_n^o \to \mathbb{R}_+$ that have been proposed between two discrete probability distributions can be represented

Date: Received: August 11, 2010; Accepted: October 20, 2010.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 39B82; Secondary 39B72.

Key words and phrases. Distance measure, multiplicative function, sum form distance measure, stability of functional equation.

in the *sum form*

$$\mu_n(P,Q) = \sum_{k=1}^{n} \phi(p_k, q_k), \tag{1.1}$$

where $\phi : I \times I \longrightarrow \mathbb{R}$ is a real-valued function on unit square, or a monotonic transformation of the right side of (1.1), that is

$$\mu_n(P,Q) = \psi\left(\sum_{k=1}^{n} \phi(p_k, q_k)\right),$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is an increasing function on $\mathbb{R}$. The function ϕ is called a *generating function*. It is also referred to as the *kernel* of $\mu_n(P,Q)$.

In information theory, for P and Q in Γ_n^o, the symmetric divergence of degree α is defined as

$$J_{n,\alpha}(P,Q) = \frac{1}{2^{\alpha-1}-1}\left[\sum_{k=1}^{n} \left(p_k^{\alpha} q_k^{1-\alpha} + p_k^{1-\alpha} q_k^{\alpha}\right) - 2\right].$$

It is easy to see that $J_{n,\alpha}(P,Q)$ is symmetric. That is $J_{n,\alpha}(P,Q) = J_{n,\alpha}(Q,P)$ for all $P,Q \in \Gamma_n^o$. Moreover it satisfies the composition law

$$\begin{aligned} &J_{nm,\alpha}(P*R, Q*S) + J_{nm,\alpha}(P*S, Q*R) \\ &\qquad = 2J_{n,\alpha}(P,Q) + 2J_{m,\alpha}(R,S) + \lambda\, J_{n,\alpha}(P,Q)\, J_{m,\alpha}(R,S) \end{aligned}$$

for all $P,Q \in \Gamma_n^o$ and $R,S \in \Gamma_m^o$ where $\lambda = 2^{\alpha-1}-1$ and

$$P*R = (p_1r_1, p_1r_2, ..., p_1r_m, p_2r_1, ..., p_2r_m, ..., p_nr_m).$$

In view of this, symmetrically compositive statistical distance measures are defined as follows. A sequence of symmetric measures $\{\mu_n\}$ is said to be *symmetrically compositive* if for some $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\mu_{nm}(P\star R,\, Q\star S) + \mu_{nm}(P\star S,\, Q\star R) \\ &\qquad = 2\,\mu_n(P,\, Q) + 2\,\mu_m(R,\, S) + \lambda\,\mu_n(P,\, Q)\,\mu_m(R,\, S) \end{aligned}$$

for all $P,Q \in \Gamma_n^o$, $S,R \in \Gamma_m^o$, where

$$P*R = (p_1r_1, p_1r_2, ..., p_1r_m, p_2r_1, ..., p_2r_m, ..., p_nr_m).$$

Chung, Kannappan, Ng and Sahoo [1] characterized symmetrically compositive sumform distance measures with a measurable generating function. The following functional equation

$$f(pr, qs) + f(ps, qr) = f(p,q)\, f(r,s) \tag{FE}$$

holding for all $p,q,r,s \in I$ was instrumental in the characterization of symmetrically compositive sumform distance measures. They proved the following theorem giving the general solution of this functional equation (FE).

Theorem 1.1. *Suppose $f : I^2 \to \mathbb{R}$ satisfies the functional equation (FE), that is*

$$f(pr, qs) + f(ps, qr) = f(p,q)\, f(r,s)$$

for all $p, q, r, s \in I$. Then

$$f(p,q) = M_1(p)\, M_2(q) + M_1(q)\, M_2(p)$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.

The stability of the functional equation (FE) and two generalizations of (FE) namely,

$$f(pr, qs) + f(ps, qr) = f(p,q)g(r,s) \qquad (FE_{fg})$$

$$f(pr, qs) + f(ps, qr) = g(p,q)f(r,s) \qquad (FE_{gf})$$

for all $p, q, r, s \in G$, were studied in [5]. In this paper, we study the stability of two more generalizations of (FE), namely

$$f(pr, qs) + f(ps, qr) = g(p,q)g(r,s) \qquad (FE_{gg})$$

$$f(pr, qs) + f(ps, qr) = g(p,q)h(r,s) \qquad (FE_{gh})$$

for all $p, q, r, s \in G$. For other functional equations similar to (FE), the interested reader should refer to [3], [4], [6] and [7]. For an account on stability of functional equations, the book [2] is an excellent source for reference.

2. Stability of functional equation (FE_{gg})

The following theorem states that an approximate equation of (FE_{gg}) with the boundedness of $f(p,q) - g(p,q)$ and $f(p,q) - f(q,p)$ also implies the functional equation (FE_{gg}).

Theorem 2.1. *Let $f, g : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}$ be a nonzero function satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p,q)g(r,s)| \le \phi(p,q) \quad \forall\, p, q, r, s \in G \qquad (2.1)$$

and $|f(p,q) - g(p,q)| \le M$, and $|f(p,q) - f(q,p)| \le M'$ for all $p, q \in G$ and some constants M, M'. Then either g is bounded or g satisfy the equation (FE), that is

$$g(pr, qs) + g(ps, qr) = g(p,q)g(r,s).$$

Proof. Let g be an unbounded solution of the inequality (2.1). Then we can choose a sequence $\{(x_n, y_n) \,|\, n \in \mathbb{N}\}$ in G^2 such that $0 \ne |g(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $r = x_n$ and $s = y_n$ in (2.1), we have

$$|f(px_n, qy_n) + f(py_n, qx_n) - g(p,q)g(x_n, y_n)| \le \phi(p,q)$$

which is

$$\left| \frac{f(px_n, qy_n) + f(py_n, qx_n)}{g(x_n, y_n)} - g(p,q) \right| \le \frac{\phi(p,q)}{|g(x_n, y_n)|}. \qquad (2.2)$$

Taking the limit of the both sides of (2.2) as $n \to \infty$, we obtain

$$g(p,q) = \lim_{n\to\infty} \frac{f(px_n, qy_n) + f(py_n, qx_n)}{g(x_n, y_n)}. \qquad (2.3)$$

Next, letting $r = rx_n$ and $s = sy_n$ in (2.1), we have

$$|f(prx_n, qsy_n) + f(psy_n, qrx_n) - g(p,q)g(rx_n, sy_n)| \le \phi(p,q)$$

which is

$$\left| \frac{f(prx_n, qsy_n) + f(psy_n, qrx_n)}{g(x_n, y_n)} - g(p,q)\frac{g(rx_n, sy_n)}{g(x_n, y_n)} \right| \le \frac{\phi(p,q)}{|g(x_n, y_n)|}. \tag{2.4}$$

Further, letting $r = sx_n$ and $s = ry_n$ in (2.1), we have

$$|f(psx_n, qry_n) + f(pry_n, qsx_n) - g(p,q)g(sx_n, ry_n)| \le \phi(p,q)$$

which is

$$\left| \frac{f(psy_n, qry_n) + f(pry_n, qsx_n)}{g(x_n, y_n)} - g(p,q)\frac{g(sx_n, ry_n)}{g(x_n, y_n)} \right| \le \frac{\phi(p,q)}{|g(x_n, y_n)|}. \tag{2.5}$$

Thus from (2.3), (2.4), (2.5), boundedness by M, M', and $0 \neq |g(x_n, y_n)| \to \infty$, we obtain

$$\begin{aligned}
&g(pr, qs) + g(ps, qr)\\
&\quad= \lim_{n\to\infty} \frac{f(prx_n, qsy_n) + f(pry_n, qsx_n)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(psx_n, qry_n) + f(psy_n, qrx_n)}{g(x_n, y_n)}\\
&\quad= \lim_{n\to\infty} \left(\frac{f(prx_n, qsy_n) + f(psy_n, qrx_n)}{g(x_n, y_n)} + \frac{f(psx_n, qry_n) + f(pry_n, qsx_n)}{g(x_n, y_n)} \right)\\
&\quad= g(p,q) \lim_{n\to\infty} \frac{g(rx_n, sy_n) + g(sx_n, ry_n)}{g(x_n, y_n)}\\
&\quad= g(p,q) \Big[g(r,s) + \lim_{n\to\infty} \frac{g(rx_n, sy_n) - f(rx_n, sy_n)}{g(x_n, y_n)}\\
&\qquad\qquad + \lim_{n\to\infty} \frac{g(sx_n, ry_n) - f(sx_n, ry_n)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(sx_n, ry_n) - f(ry_n, sx_n)}{g(x_n, y_n)} \Big]\\
&\quad= g(p,q)g(r,s) + g(p,q) \Big[\lim_{n\to\infty} \frac{g(rx_n, sy_n) - f(rx_n, sy_n)}{g(x_n, y_n)}\\
&\qquad\qquad + \lim_{n\to\infty} \frac{g(sx_n, ry_n) - f(sx_n, ry_n)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(sx_n, ry_n) - f(ry_n, sx_n)}{g(x_n, y_n)} \Big]\\
&\quad= g(p,q)g(r,s).
\end{aligned}$$

The proof of the theorem is now complete. □

Theorem 2.2. *Let $f, g : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}$ be a nonzero function satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p,q)g(r,s)| \le \phi(r,s) \quad \forall\, p,q,r,s \in G. \tag{2.6}$$

Then g is bounded or g satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $|f(p,q) - g(p,q)| \le M$ for all $p, q \in G$ and some nonnegative constant M.

Proof. Suppose g is unbounded. We would like to show that g satisfies (FE) for all $p, q, r, s \in G$ if and only if $|f(p,q) - g(p,q)| \le M$ for all $p, q \in \mathbb{R}$ and for some $M \ge 0$.

Since g is unbounded, because of (2.6), f must be unbounded. Now we prove that the if part of the proof. Suppose g satisfies (FE) for all $p, q, r, s \in G$. Letting $p = q = r = s = 1$ in (FE), we see that that $g(1,1) = 0$ or $g(1,1) = 2$. We claim that $g(1,1) = 2$. Suppose not. Then $g(1,1) = 0$. Taking $r = s = 1$ in (2.6), we obtain

$$|2\,f(p,q)| = |2\,f(p,q) - g(p,q)\,g(1,1)| \leq \phi(1,1),$$

that is f is bounded contrary to the fact that f is unbounded. Hence $g(1,1) = 2$ and therefore

$$|2f(p,q) - g(p,q)\,g(1,1)| \leq \phi(1,1)$$

which implies

$$|f(p,q) - g(p,q)| \leq \frac{\phi(1,1)}{2} = M.$$

Next, let us prove the only if part. Since g is the unbounded solution of the inequality (2.6), therefore, there exists a sequence $\{(x_n, y_n)\,|\,n \in \mathbb{N}\}$ in $\mathbb{R}^2$ such that $0 \neq |g(x_n, y_n)\,| \to \infty$ as $n \to \infty$.

Letting $p = x_n$ and $q = y_n$ in (2.6), we have

$$|f(x_n r, y_n s) + f(x_n s, y_n r) - g(x_n, y_n)g(r,s)| \leq \phi(r,s).$$

Taking the limit as $n \to \infty$, we obtain that

$$g(r,s) = \lim_{n\to\infty} \frac{f(x_n r, y_n s) + f(x_n s, y_n r)}{g(x_n, y_n)} \tag{2.7}$$

Letting $p = x_n p$ and $q = y_n q$ in (2.6), we have

$$|f(x_n pr, y_n qs) + f(x_n ps, y_n qr) - g(x_n p, y_n q)g(r,s)| \leq \phi(r,s)$$

which is

$$\left| \frac{f(x_n pr, y_n qs) + f(x_n ps, y_n qr)}{g(x_n, y_n)} - \frac{g(x_n p, y_n q)}{g(x_n, y_n)} g(r,s) \right| \leq \frac{\phi(r,s)}{|g(x_n, y_n)|}. \tag{2.8}$$

Letting $p = x_n q$ and $q = y_n p$ in (2.6), dividing $g(x_n, y_n)$, passing to the limit as $n \to \infty$, we have

$$\left| \frac{f(x_n qr, y_n ps) + f(x_n qs, y_n pr)}{g(x_n, y_n)} - g(r,s) \frac{g(x_n q, y_n p)}{g(x_n, y_n)} \right| \leq \frac{\phi(r,s)}{|g(x_n, y_n)|}. \tag{2.9}$$

Thus from (2.7), (2.8), (2.9), boundedness by M, and $0 \neq |g(x_n, y_n)| \to \infty$, we obtain

$$\begin{aligned}
&g(pr, qs) + g(ps, qr) \\
&= \lim_{n\to\infty} \frac{f(x_npr, y_nqs) + f(x_nqs, y_npr)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(x_nps, y_nqr) + f(x_nqr, y_nps)}{g(x_n, y_n)} \\
&= \lim_{n\to\infty} \left(\frac{f(x_npr, y_nqs) + f(x_nps, y_nqr)}{g(x_n, y_n)} + \frac{f(x_nqs, y_npr) + f(x_nqr, y_nps)}{g(x_n, y_n)} \right) \\
&= \lim_{n\to\infty} \frac{g(x_np, y_nq) + g(x_nq, y_np)}{g(x_n, y_n)}\, g(r, s) \\
&= \left[g(p, q) + \lim_{n\to\infty} \frac{g(x_np, y_nq) - f(x_np, y_nq) + g(x_nq, y_np) - f(x_nq, y_np)}{g(x_n, y_n)} \right] g(r, s) \\
&= g(p, q)\, g(r, s) \\
&\qquad + \left[\lim_{n\to\infty} \frac{g(x_np, y_nq) - f(x_np, y_nq) + g(x_nq, y_np) - f(x_nq, y_np)}{g(x_n, y_n)} \right] g(r, s) \\
&= g(p, q)\, g(r, s).
\end{aligned}$$

This completes the proof of the theorem. □

The following corollary follows from the Theorem 2.2.

Corollary 2.3. *Let $f, g : G^2 \to \mathbb{R}$ be functions satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)g(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G$$

for some $\epsilon \geq 0$. Then the function g is bounded or it satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some nonnegative constant M.

3. Stability of functional equation (FE_{gh})

Theorem 3.1. *Let $f, g, h : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}$ be a nonzero function satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)| \leq \phi(r, s) \tag{3.1}$$

for all $p, q, r, s \in G$. If $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some constant M, then g is bounded or h satisfies (FE) for all $p, q, r, s \in G$.

Proof. Let g be unbounded. Then we can choose a sequence $\{(x_n, y_n) \,|\, n \in \mathbb{N}\}$ in $\mathbb{R}^2$ such that $0 \neq |g(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $p = x_n$ and $q = y_n$ in (3.1), we have

$$|f(x_nr, y_ns) + f(x_ns, y_nr) - g(x_n, y_n)h(r, s)| \leq \phi(r, s),$$

which is

$$\left| \frac{f(x_nr, y_ns) + f(x_ns, y_nr)}{g(x_n, y_n)} - h(r, s) \right| \leq \frac{\phi(r, s)}{|g(x_n, y_n)|}.$$

Taking the limit as $n \to \infty$, we obtain

$$h(r, s) = \lim_{n\to\infty} \frac{f(x_nr, y_ns) + f(x_ns, y_nr)}{g(x_n, y_n)} \tag{3.2}$$

Letting $p = x_n p$ and $q = y_n q$ in (3.1), we have

$$f(x_npr, y_nqs) + f(x_nps, y_nqr) - g(x_np, y_nq)h(r, s)| \leq \phi(r, s),$$

which is

$$\left| \frac{f(x_npr, y_nqs) + f(x_nps, y_nqr)}{g(x_n, y_n)} - \frac{g(x_np, y_nq)}{g(x_n, y_n)} h(r, s) \right| \leq \frac{\phi(r, s)}{|g(x_n, y_n)|}. \tag{3.3}$$

Letting $p = x_n q$ and $q = y_n p$ in (3.1) and proceeding as above, we have

$$|f(x_nqr, y_nps) + f(x_nqs, y_npr) - g(x_nq, y_np)h(r, s)| \leq \phi(r, s). \tag{3.4}$$

From the last inequality (3.4), we obtain

$$\left| \frac{f(x_nqr, y_nps) + f(x_nqs, y_npr)}{g(x_n, y_n)} - \frac{g(x_nq, y_np)}{g(x_n, y_n)} h(r, s) \right| \leq \frac{\phi(r, s)}{|g(x_n, y_n)|}. \tag{3.5}$$

Using (3.2), (3.3), and (3.5), we obtain

$$\begin{aligned}
&h(pr, qs) + h(ps, qr) \\
&= \lim_{n\to\infty} \frac{f(x_npr, y_nqs) + f(x_nqs, y_npr)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(x_nps, y_nqr) + f(x_nqr, y_nps)}{g(x_n, y_n)} \\
&= \lim_{n\to\infty} \frac{g(x_np, y_nq) + g(x_nq, y_np)}{g(x_n, y_n)} h(r, s) \\
&= \lim_{n\to\infty} \Big[\frac{g(x_np, y_nq) - f(x_np, y_nq) + g(x_nq, y_np) - f(x_nq, y_np)}{g(x_n, y_n)} + h(p, q)\Big] h(r, s) \\
&= h(p, q)h(r, s).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $f, g, h : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}_+$ be functions satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)| \leq \phi(p, q)$$

for all $p, q, r, s \in G$. If $|f(p, q) - h(p, q)| \leq M$ for all $p, q \in G$ and some constant M then h is bounded or g satisfies (FE) for all $p, q, r, s \in G$.

Proof. Let h be unbounded. Then we can choose a sequence $\{(x_n, y_n) \,|\, n \in \mathbb{N}\}$ in G^2 such that $0 \neq |h(x_n, y_n)| \to \infty$ as $n \to \infty$.

Proceeding as similar to the derivation of (3.2), we obtain

$$g(p, q) = \lim_{n\to\infty} \frac{f(pr_n, qs_n) + f(ps_n, qr_n)}{h(x_n, y_n)}.$$

The rest of the proof runs similar to the Theorem 3.1 and we see that g satisfies (FE) for all $p, q, r, s \in G$. □

Corollary 3.3. *Let $f, g, h : G^2 \to \mathbb{R}$ be functions satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)| \leq \varepsilon$$

for all $p, q, r, s \in G$ and for some $\epsilon \geq 0$.
(a) If $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some constant M then g is bounded or h satisfies (FE) for all $p, q, r, s \in G$.

(b) If $|f(p,q) - h(p,q)| \le M$ for all $p, q \in G$ and some constant M then h is bounded or g satisfies (FE) for all $p, q, r, s \in G$.

Remark 3.4. (i) Choosing g and h appropriately in Theorem 3.1 one can obtain the stability for the functional equations (FE_{fg}), (FE_{gf}) and (FE). For example, by letting first g to be f and then h to be g, the stability of (FE_{fg}) can be obtained which was studied in [5].

(ii) Theorems 2.1 and 3.1 hold if one replaces the domain of the functions f, g, h, ϕ by S^2, where S is an abelian semigroup.

4. Extension of the results to Banach spaces

In this section, let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. All results in the Section 2 and the Section 3 can be extended to the superstability on the Banach space. For simplicity, we will combine the two theorems of the same functional equation in Section 2 and Section 3 into the one theorem, respectively.

Theorem 4.1. *Let $f, g, h : G^2 \to E$ and $\phi : G^2 \to \mathbb{R}$ be functions satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p,q)h(r,s)\| \le \begin{cases} (i) & \phi(r,s) \\ (ii) & \phi(p,q) \end{cases} \tag{4.1}$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$:
(a) If $\|f(p,q) - g(p,q)\| \le M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ g$ is bounded or h satisfies (FE) for all $p, q, r, s \in G$ in the case (i) of (4.1).
(b) If $\|f(p,q) - h(p,q)\| \le M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ h$ is bounded or g satisfies (FE) for all $p, q, r, s \in G$ in the case (ii) of (4.1).

Proof. First we show (a). Assume that (i) of (4.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As well known we have $\|x^*\| = 1$ hence, for every $x, y \in G$, we have

$$\begin{aligned} \phi(r,s) &\ge \|f(pr,qs) + f(ps,qr) - g(p,q)h(r,s)\| \\ &= \sup_{\|y^*\|=1} \left| y^*\big(f(pr,qs) + f(ps,qr) - g(p,q)h(r,s)\big) \right| \\ &\ge \left| x^*\big(f(pr,qs)\big) - x^*\big(f(ps,qr)\big) - x^*\big(g(p,q)\big)x^*\big(h(r,s)\big) \right|, \end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield solutions of inequality (3.1). Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 3.1 shows that the function $x^* \circ h$ solves the equation (FE). In other words, bearing the linear multiplicativity of x^* in mind, for all $p, q, r, s \in G$, the difference

$$DFE := h(pr, qs) + h(ps, qr) - h(p,q)h(r,s)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$DFE(p,q,r,s) \in \bigcap \left\{ \ker x^* \mid x^* \text{ is a multiplicative member of } E^* \right\}$$

for all $p, q, r, s \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$h(pr, qs) + h(ps, qr) - h(p, q)h(r, s) = 0 \quad \text{for all} \quad p, q, r, s \in G,$$

as claimed. The other case (b) is similar, so its proof will be omitted. This completes the proof. □

Corollary 4.2. *Let $f, g, h : G^2 \to E$ and $\phi : G^2 \to \mathbb{R}$ be functions satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)\| \le \varepsilon$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$:
(a) If $\|f(p, q) - g(p, q)\| \le M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ g$ is bounded or h satisfies (FE) for all $p, q, r, s \in G$.
(b) If $\|f(p, q) - h(p, q)\| \le M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ h$ is bounded or g satisfies (FE) for all $p, q, r, s \in G$.

The proof of the following theorem follows similar to the proof of Theorem 4.1.

Theorem 4.3. *Let $f, g, h : G^2 \to E$ and $\phi : G^2 \to \mathbb{R}$ be functions satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p, q)g(r, s)\| \le \begin{cases} (i) & \phi(r, s) \\ (ii) & \phi(p, q) \end{cases}$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$:
(a) In case (i), the superposition $x^* \circ g$ is bounded or g satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $\|f(p, q) - g(p, q)\| \le M$ for all $p, q \in G$ and some nonnegative constant M.
(b) In case (ii), if $\|f(p, q) - g(p, q)\| \le M$ and $\|f(p, q) - f(q, p)\| \le M'$ for all $p, q \in G$ and for some nonnegative constants M, M', then either the superposition $x^* \circ g$ is bounded or g satisfy the equation (FE).

Corollary 4.4. *Let $f, g, h : G^2 \to E$ be a function satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p, q)g(r, s)\| \le \varepsilon$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$, the superposition $x^* \circ g$ is bounded or g satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $\|f(p, q) - g(p, q)\| \le M$ for all $p, q \in G$ and some nonnegative constant M.

Acknowledgement. The authors would like to thank the referee for valuable suggestions that improved the presentation of this paper. The first author was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2010-0010243). The second author was partially supported by an IRI grant from the Office of the Vice President for Research of the University of Louisville.

References

1. J.K. Chung, Pl. Kannappan, C.T. Ng and P.K. Sahoo, *Measures of distance between probability distributions*, J. Math. Anal. Appl. **138** (1989), 280–292.
2. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.
3. Pl. Kannappan and P.K. Sahoo, *Sum form distance measures between probability distributions and functional equations*, Int. J. of Math. & Stat. Sci. **6** (1997), 91–105.
4. Pl. Kannappan, P.K. Sahoo and J.K. Chung, *On a functional equation associated with the symmetric divergence measures*, Utilitas Math. **44** (1993), 75–83.
5. G.H. Kim and P.K. Sahoo, *Stability of a functional equation related to distance measures-I*, preprint.
6. T. Riedel and P.K. Sahoo, *On a generalization of a functional equation associated with the distance between the probability distributions*, Publ. Math. Debrecen **46** (1995), 125–135.
7. T. Riedel and P.K. Sahoo, *On two functional equations connected with the characterizations of the distance measures*, Aequationes Math. **54** (1998), 242–263.
8. P.K. Sahoo, *On a functional equation associated with stochastic distance measures*, Bull. Korean Math. Soc. **36** (1999), 287–303.

[1] Department of Mathematics, Kangnam University, Yongin, Gyeonggi, 446-702, Korea.

E-mail address: ghkim@kangnam.ac.kr

[2] Department of Mathematics, University of Louisville, Louisville, Kentucky, 40292 USA.

E-mail address: sahoo@louisville.edu