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ON POINTWISE INVERSION OF THE FOURIER TRANSFORM OF BV_0 FUNCTIONS

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Abstract. Using a Riemann-Lebesgue lemma for the Fourier transform over the class of bounded variation functions that vanish at infinity, we prove the Dirichlet–Jordan theorem for functions on this class. Our proof is in the Henstock–Kurzweil integral context and is different to that of Riesz-Livingston [Amer. Math. Monthly 62 (1955), 434–437]. As consequence, we obtain the Dirichlet–Jordan theorem for functions in the intersection of the spaces of bounded variation functions and of Henstock–Kurzweil integrable functions. In this intersection there exist functions in $L^2(\mathbb{R})\backslash L(\mathbb{R})$.

1. Introduction

It is known that if $f:\mathbb{R}\to\mathbb{R}$ is a function such that for $s\in\mathbb{R}$ the product $f(t)e^{-ist}$ is integrable, in some sense, then its Fourier transform in s is defined as

$$\widehat{f}(s)=\int_{-\infty}^{\infty} f(t)e^{-ist}dt. \tag{1.1}$$

In the space of Lebesgue integrable functions, denoted by $L(\mathbb{R})$, we have that f is integrable if and only if its Fourier tranform exists for each $s\in\mathbb{R}$. However, this situation is not true for functions that are not in $L(\mathbb{R})$. For example, the function $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\sin x/x$ for $x\neq 0$ and 1 for $x=0$, does not belong to $L(\mathbb{R})$, it is improper Lebesgue integrable and its Fourier transform, as (1.1), does not exist in $s=1$.

A fundamental problem for the Fourier transform is its pointwise inversion, which means recovering the function at given points from its Fourier transform.

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The Dirichlet-Jordan theorem in $L(\mathbb{R})$ solves the pointwise inversion for functions in $BV(\mathbb{R})$, the space of bounded variation functions on $\mathbb{R}$. This theorem tells us that if $f \in L(\mathbb{R}) \cap BV(\mathbb{R})$ then, for each $x \in \mathbb{R}$,

$$\lim_{M\to\infty} \frac{1}{2\pi} \int_{-M}^{M} e^{ixs} \widehat{f}(s) ds = \frac{1}{2}\{f(x+0) + f(x-0)\}.$$

In [4], M. Riesz and A. E. Livingston proved the following version of the previous theorem when f is of bounded variation but not necessarily Lebesgue integrable: *Suppose that* $f \in BV(\mathbb{R})$ and $\lim_{|t|\to\infty} f(t) = 0$ *then, for each* $x \in \mathbb{R}$,

$$\lim_{\substack{M\to\infty \\ \varepsilon\to 0}} \frac{1}{2\pi} \int_{\varepsilon<|s|<M} e^{ixs} \int_{-\infty}^{\infty} f(t) e^{-ist} dt\ ds = \frac{1}{2}\{f(x+0) + f(x-0)\}.$$

The class of functions in $BV(\mathbb{R})$ that vanish at infinity (its limit is zero at infinity) is represented as $BV_0(\mathbb{R})$.

In this paper we make a proof of the Dirichlet-Jordan theorem on $BV_0(\mathbb{R})$ in the Henstock-Kurzweil integral context, which is different to that of Riesz-Livingston. We first prove that if f is in $BV_0(\mathbb{R})$ then its Fourier transform exists for each $s \in \mathbb{R}\backslash\{0\}$. We prove a Riemann-Lebesgue lemma over $BV_0(\mathbb{R})$ and we employ it for prove our main result. The validity of this lemma is interesting because, in [5], E. Talvila proved that, in general, the Riemann-Lebesgue lemma is not valid in the space of Henstock-Kurzeil integrable functions on $\mathbb{R}$, denoted $HK(\mathbb{R})$. The Dirichlet-Jordan theorem on $L(\mathbb{R}) \cap BV(\mathbb{R})$ and $HK(\mathbb{R}) \cap BV(\mathbb{R})$ are consequences of our main theorem.

In the following section we will give the basic notions about the Henstock-Kurzweil integral and the bounded variation functions. We will explain our main results in section 3.

2. Preliminaries on the Henstock-Kurzweil integral

The Henstock-Kurzweil integral was defined in the mid-twentieth century by Jaroslav Kurzweil and Ralph Henstock. Both mathematicians developed their integrals independently and it was not until later that these were proven to be equal. This integral is also equivalent to Denjoy-Perron integral, but its construction follows the same pattern as the construction of the Riemann integral. Moreover, all Riemann and Lebesgue integrable functions and those that are improper Riemann or Lebesgue integrable, over $\mathbb{R}$, are Henstock-Kurzweil integrable. The basic notions of this integral are presented below.

Let $I \subset \mathbb{R}$ a closed interval, finite or infinite. A *partition* P of I is a increasing finite collection of points $\{t_1, t_2, ..., t_n\} \subset I$ such that if $I = [a, b]$, $a, b \in \mathbb{R}$, then $t_0 = a$ and $t_n = b$; if $I = [a, \infty)$, $t_0 = a$; and if $I = (-\infty, b]$ then $t_n = b$.

Suppose that I is a finite closed interval in $\mathbb{R}$. A *tagged partition* of I is a partition P of I such that for each subinterval $I_i = [t_{i-1}, t_i]$ there is assigned a point $s_i \in I_i$, which is called a *tag* of I_i. With this concept we define the Henstock-Kurzweil integral on finite intervals in $\overline{\mathbb{R}}$.

Definition 2.1. The function $f : [a, b] \to \mathbb{R}$ is *Henstock-Kurzweil integrable* if there exists $H \in \mathbb{R}$ which satisfies the following: for each $\varepsilon > 0$ exists a function $\gamma_\varepsilon : [a, b] \to (0, \infty)$ such that if $P = \{([t_{i-1}, t_i], s_i)\}_{i=1}^n$ is a tagged partition such that

$$[t_{i-1}, t_i] \subset [s_i - \gamma_\varepsilon(s_i), s_i + \gamma_\varepsilon(s_i)\,] \quad \text{for } i = 1, 2, ..., n., \tag{2.1}$$

then

$$|\Sigma_{i=1}^n f(s_i)(s_i - s_{i-1}) - H| < \varepsilon.$$

H should be the integral of f over $[a, b]$ and it should be denoted as

$$H = \int_a^b f \text{ or } \int_a^b f dt$$

A tagged partition that satisfies (2.1) is called $\gamma_\varepsilon-$fine.

The definition process for the Henstock-Kurzweil integral for functions over infinite intervals is the following.

Definition 2.2. Given a function $\gamma : [a, \infty] \to (0, \infty)$, we will say that the tagged partition $P = \{([t_{i-1}, t_i], s_i)\}_{i=1}^{n+1}$ *is $\gamma-$fine* if:

(a) $t_0 = a$, $t_{n+1} = \infty$.
(b) $[t_{i-1}, t_i] \subset [s_i - \gamma_\varepsilon(s_i), s_i + \gamma_\varepsilon(s_i)\,]$ for $i = 1, 2, ..., n$.
(c) $[t_n, \infty] \subset [1/\gamma(\infty), \infty]$.

If f is defined over an interval $[a, \infty)$, we may condition it to $f(\infty) = 0$, this allows us redefine to f over $[a, \infty]$. Thus we have the following definition for functions defined on $[a, \infty]$.

Definition 2.3. The function $f : [a, \infty] \to \mathbb{R}$ will be *Henstock-Kurzweil integrable* if it satisfies the Definition 2.1, but the partition P must be $\gamma_\varepsilon-$fine according to Definition 2.2.

For functions defined over intervals $[-\infty, a]$ and $[-\infty, +\infty]$ we make similar considerations. We denote as $HK(I)$ the vector space of Henstock-Kurzweil integrable functions on I.

Suppose that $I \subset \mathbb{R}$ is a closed interval, finite or infinite.

Definition 2.4. A function $f : I \to \mathbb{R}$ is of *bounded variation* over I if exists a $M > 0$ such that

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < M,$$

for all finite partition P of I.

Definition 2.5. If f is a function of bounded variation, its *total variation* over I is defined as

$$V_f(I) = \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : P \text{ is a partition of } I\right\}.$$

For infinite intervals, the above definition can be reformulated equivalently as follows: f is of bounded variation on $I = [a, \infty)$ if there exists $N > 0$ such that

$$V_f([a,t]) \leq N$$

for all $t \geq a$. The total variation of f on I should be equal to

$$V_f([a,\infty)) = \sup\{V_f([a,t]\,) : a \leq t\} \tag{2.2}$$

For $I = (-\infty, b]$ the equivalence is analogous. If $I = \mathbb{R}$, the total variation should be

$$V_f(\mathbb{R}) = \sup\{V_f([t,s]\,) : t, s \in \mathbb{R}, \quad s < t\} \tag{2.3}$$

Since $V_f([a,t])$ is increasing on $[a,\infty)$ and $V_f([t,b])$ is decreasing on $(-\infty, b]$, the expressions in (2.2) and (2.3) enables us to see that:

$$\begin{aligned} V_f([a,\infty)) &= \lim_{t\to\infty} V_f([a,t]\,), \\ V_f((-\infty,b] &= \lim_{t\to-\infty} V_f([t,b]\,) \end{aligned}$$

and

$$V_f(\mathbb{R}\,) = \lim_{\substack{t\to-\infty \\ s\to\infty}} V_f([t,s]\,).$$

The space of bounded variation functions over I will be denoted as $BV(I)$. Some properties of $BV(I)$ are the following:

- $BV(I)$ is a vector space.
- $f \in BV(I)$, if and only if, there exist f_1 and f_2 which are increasing bounded functions such that $f = f_1 - f_2$.
- If $I = [a,\infty)$, then $\lim_{t\to\infty} f(t)$ exists. For cases $(-\infty, b]$ and $\mathbb{R}$ we have similar results. A particular case is when f is in the class $BV_0(I)$, when $\lim_{|t|\to\infty} f(t) = 0$.

Remark 2.6. Since if $I = \mathbb{R}$, then $\lim_{|t|\to\infty} f(t)$ exists. Therefore, each $f \in BV(\mathbb{R})$ can be extended uniquely in a continuous way to a function $\widetilde{f} \in BV(\overline{\mathbb{R}})$. In this sense, we have that $BV(\mathbb{R}) \subset BV(\overline{\mathbb{R}})$. On the other hand, since if $f = g$ a.e., with respect to Lebesgue measure, their Henstock-Kurzweil integrals are the same, then there is not essential difference between $HK(\mathbb{R})$ and $HK(\overline{\mathbb{R}})$.

If I is a finite interval, we know that $BV(I) \subset L(I) \subset HK(I)$. However, if I is an infinite interval, we have the following inclusion relations, see [3].

(a) $L(I) \subsetneq HK(I)$
(b) $BV_0(I) \nsubseteq L(I)$ and $BV_0(I) \nsubseteq HK(I)$
(c) $BV(I) \cap HK(I) \nsubseteq L(I)$ and $L(I) \nsubseteq BV(I) \cap HK(I)$.
(d) $BV(I) \cap HK(I) \subset BV_0(I)$.
(e) If $f \in BV(I)$, then $f, f' \in HK_{loc}(I)$.

Now we state some of the fundamental theorems about the Henstock-Kurzweil integral that we will use frequently, [2, Theorems 10.12, 16.7 and 16.10].

Multiplier Theorem. *Let* $[a, b]$ *be a finite interval. If* $\varphi \in HK([a, b])$, $f \in BV([a, b])$ *and* $\Phi(x) = \int_a^x \varphi(t)$, *for* $x \in [a, b]$, *then* $\varphi f \in HK([a, b])$ *and*

$$\int_a^b \varphi f = \Phi(b)f(b) - \int_a^b \Phi df. \tag{2.4}$$

If $a \in R$ *and* $b = \infty$, *then* $\varphi f \in HK([a, \infty])$ *and* (2.4) *has the following form*

$$\int_a^\infty \varphi f = \lim_{b\to\infty} \left[\Phi(b)f(b) - \int_a^b \Phi df \right]. \tag{2.5}$$

The integrals on the right are Riemann-Stieljes integrals. For the case (2.5) $\lim_{b\to\infty} \int_a^b \Phi df = \int_a^\infty \Phi df$ *will be the improper Riemann-Stieljes integral. Analogously, if the integration is on the intervals* $[-\infty, b]$ *or* $[-\infty, \infty]$ *we take the respective limits in* (2.5). *This theorem is a version of the Integration by Parts theorem, which is also valid in* $HK(I)$.

Hake's Theorem. $\varphi \in HK([a, \infty])$ *if and only if for each* b, ε *such that* $b > a$, $b - a > \varepsilon > 0$, *it follows that* $\varphi \in HK([a + \varepsilon, b])$ *and* $\lim_{\varepsilon\to 0,\ b\to\infty} \int_{a+\varepsilon}^b \varphi(t)dt$ *exists. In this case, this limit shall be* $\int_a^\infty \varphi(t)dt$.

Similar results are valid for the cases $[-\infty, \infty]$ *and* $[-\infty, a]$.

Chartier-Dirichlet Test. *Let* φ, $f : [a, \infty] \to \mathbb{R}$ *and suppose that: (a)* $\varphi \in HK\left([a, c]\right)$ *for each* $c \geq a$, *and* $\Phi\left(t\right) = \int_a^t \varphi du$ *is bounded on* $[a, \infty)$; *and (b)* f *is monotonous with* $\lim_{t\to\infty} f\left(t\right) = 0$. *Then* $\varphi f \in HK\left([a, \infty]\right)$.

3. Main results

From the Multiplier theorem, Hake's theorem and the Chartier-Dirichlet test, the following lemma is deduced.

Lemma 3.1. *Suppose that* $f \in BV_0([a, \infty])$, $\varphi \in HK([a, b])$ *for every* $b > a$, *and* $\Phi(t) = \int_a^t \varphi du$ *is bounded on* $[a, \infty)$. *Then* $\varphi f \in HK([a, b])$,

$$\int_a^\infty \varphi f dt = -\int_a^\infty \Phi(t)df(t)$$

and

$$\left| \int_a^\infty \varphi f dt \right| \leq \sup_{a<t} |\Phi(t)|\, V_f([a, \infty]).$$

Similar results are valid for the cases $[-\infty, \infty]$ *and* $[-\infty, a]$.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 1 & \text{for} \quad t \in (-1,\, 1), \\ \frac{1}{\sqrt{|t|}} & \text{for} \quad |t| \in [1,\, \infty) \end{cases}$$

is in $BV_0(\mathbb{R})$ but $\widehat{f}(0)$ does not exist. Now, observe for Proposition 2,b) in [5] that if $f \in BV_0(\mathbb{R})$ then $\widehat{f}(s)$ exists for each $s \in \mathbb{R} \setminus \{0\}$.
We have the following lemma which is a generalization of some results obtained in [3].

Lemma 3.2 (Riemann-Lebesgue lemma)**.** *If $f \in BV_0(\mathbb{R})$, then the Fourier transform $\widehat{f}\,(s)$ exists for all $s \in \mathbb{R}\backslash\{0\}$, and has the following properties:*

(i) $\widehat{f} : \mathbb{R} \to \mathbb{C}$ *is continuous at* $\mathbb{R}\backslash\{0\}$.

(ii) $\lim\limits_{|s|\to\infty} \widehat{f}(s) = 0$.

Proof. Above, we have mentioned the existence of $\widehat{f}$ on $\mathbb{R} \setminus \{0\}$. Let $\widehat{f_+}(s) = \int_0^\infty f(t)e^{-ist}dt$. According to Lemma 3.1, we have

$$\widehat{f_+}(s) = -\int_0^\infty \left(\frac{e^{-ist}-1}{-is}\right) df(t). \tag{3.1}$$

Let $\beta > 0$ fixed. For each $t \in [0,\infty]$, the function $s \rightsquigarrow (e^{-ist}-1)/(-is)$ is continuous on $[\beta,\infty)$ and

$$\left|\frac{e^{-ist}-1}{-is}\right| \le \frac{2}{\beta}.$$

Applying the Lebesgue Dominate Convergence theorem on (3.1), we deduce that $\widehat{f_+}$ is continuous on $[\beta,\infty)$ for each $\beta > 0$. Therefore $\widehat{f_+}$ is continuous on $(0,\infty)$. Using a similar argument, we prove the case when $(-\infty, 0)$.

Also by (3.1), it holds

$$\left|\widehat{f_+}(s)\right| \le \frac{2}{|s|} V_f([a,\infty]),$$

therefore it follows that $\lim_{|s|\to\infty} \widehat{f_+}(s) = 0$.

Defining $\widehat{f_-}(s) = \int_{-\infty}^0 f(t)e^{-ist}dt$ and following a similar argument over $[-\infty, 0]$ we obtain (i) and (ii). □

If $g, h \in BV([a,\infty])$ and Q_1, Q_2 are upper bounds of g and h, respectively, then for every $b > a : V_{gh}([a,b]) \le Q_2 V_g([a,\infty]) + Q_1 V_h([a,\infty])$. It follows that $gh \in BV([a,\infty])$. Using this fact and Lemma 3.2 we formulated the following corollary.

Corollary 3.3. *Suppose that $\delta, \alpha > 0$ and $f \in BV(\mathbb{R})$, then*

$$\lim_{M\to\infty} \int_\delta^\infty \frac{f(t)}{t^\alpha} e^{-iMt} dt = 0.$$

The function defined in $\mathbb{R}$ by $\sin t/t$, for $t \ne 0$, and 1 for $t = 0$, belongs to $HK(\mathbb{R})$. It is known that from the previous function the Sine Integral is defined as

$$Si(x) = \frac{2}{\pi} \int_0^x \frac{\sin t}{t} dt,$$

which has the properties:

(1) $Si(0) = 0$, $\lim_{x\to\infty} Si(x) = 1$ and

(2) $Si(x) \le Si(\pi)$ for all $x \in [0,\infty]$.

We use the Sine Integral function in the proof of the following lemma.

Lemma 3.4. *Let $\delta > 0$. If $f \in BV_0(\mathbb{R})$, then*

$$\lim_{\varepsilon \to 0} \int_{\delta}^{\infty} f(t) \frac{\sin \varepsilon t}{t} dt = 0.$$

Proof. By Lemma 3.1 we have

$$\left| \int_{\delta}^{\infty} \frac{\sin \varepsilon t}{t} f(t) dt \right| \leq \left| \int_{\delta}^{\infty} \left(\int_{\delta\varepsilon}^{t\varepsilon} \frac{\sin u}{u} du \right) df(t) \right|. \tag{3.2}$$

Observe that for each $t \in [a, \infty)$: $\lim_{\varepsilon \to 0} \int_{\delta\varepsilon}^{t\varepsilon} \frac{\sin u}{u} du = 0$ and $\left| \int_{\delta\varepsilon}^{t\varepsilon} \frac{\sin u}{u} du \right| \leq \pi Si(\pi)$ for all $\varepsilon > 0$. Then, applying the Lebesgue Dominated Convergence theorem to the integral on the right in (3.2), we obtain the result. □

Lemma 3.5. *Suppose that $f \in BV_0(\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}$ are such that $0 < \alpha < \beta$ or $\alpha < \beta < 0$. For all $s \in [\alpha, \beta]$ we have*

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_{\alpha}^{\beta} e^{ixs} \int_{a}^{b} f(t) e^{-ist} dt ds = \int_{\alpha}^{\beta} e^{ixs} \int_{-\infty}^{\infty} f(t) e^{-ist} dt ds. \tag{3.3}$$

Proof. Suppose that $0 < \alpha < \beta$. Let $\widehat{f}_{0b}(s) = \int_0^b f(t) e^{-ist} dt$ and $\widehat{f}_0(s) = \int_0^{\infty} f(t) e^{-ist} dt$, which are continuous on $\mathbb{R} \setminus \{0\}$. Therefore the integrals in (3.3) exist. We know that there is $R > 0$ such that $|f(t)| \leq R$ for all $t \in \mathbb{R}$, and that for any $b > 0 : V_f([0, b]) \leq V_f([0, \infty))$. For each $s \in [\alpha, \beta]$ the Multiplier theorem implies

$$\left| \widehat{f}_{0b}(s) \right| \leq \frac{2}{\alpha} \{ R + V_f([0, \infty)) \} = N.$$

This inequality implies that for any $b > 0$ and all $s \in [\alpha, \beta]$: $\left| e^{ixs} \widehat{f}_{0b}(s) \right| \leq N$, for each $x \in \mathbb{R}$. Applying the theorem of Hake we have: $\lim_{b \to \infty} \widehat{f}_{0b}(s) = \widehat{f}_0(s)$. Then, by the Lebesgue Dominated Convergence theorem,

$$\lim_{b \to \infty} \int_{\alpha}^{\beta} e^{ixs} \widehat{f}_{0b}(s) ds = \int_{\alpha}^{\beta} e^{ixs} \widehat{f}_0(s) ds.$$

To obtain the result, we follow a similar process over the interval $[a, 0]$ leading a to minus infinity. □

To conclude this section, we state the following lemma [1, Theorem 11.8].

Lemma 3.6. *Let $\delta > 0$. If g is of bounded variation on $[0, \delta]$, then*

$$\lim_{M \to \infty} \frac{2}{\pi} \int_0^{\delta} g(t) \frac{\sin Mt}{t} dt = g(0+)$$

3.1. **Main theorem.** We do not know if $e^{ixs}\widehat{f}$ is Henstock-Kurzweil integrable around 0, so our main theorem is as follows:

Theorem 3.7. *If $f \in BV_0(\mathbb{R})$, then, for each $x \in \mathbb{R}$,*

$$\lim_{\substack{M\to\infty \\ \varepsilon\to 0}} \frac{1}{2\pi} \int\limits_{\varepsilon<|s|<M} e^{ixs}\widehat{f}(s)ds = \frac{1}{2}\{f(x+0)+f(x-0)\}. \tag{3.4}$$

Proof. Suppose that $\delta > 0$ and let $g(x,t) = f(x-t)+f(x+t)$. By Lemma 3.5 and by Fubini's theorem for the Lebesgue integral [1, Theorem 15.7], at $[-M,-\varepsilon] \times [a,b]$ and $[\varepsilon, M] \times [a,b]$, and we have

$$\begin{aligned}
\int\limits_{\varepsilon<|s|<M} e^{ixs} \int_{-\infty}^{\infty} f(t)e^{-ist}dtds &= \lim_{\substack{a\to-\infty \\ b\to\infty}} \left(\int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M}\right) e^{ixs} \int_a^b f(t)e^{-ist}dtds \\
&= \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(t) \left(\int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M}\right) e^{is(x-t)}dsdt \\
&= \int_{-\infty}^{\infty} f(t) \left(\int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M}\right) e^{is(x-t)}dsdt \\
&= 2\int_0^{\infty} \frac{g(x,t)}{t}(\sin M\ t - \sin\varepsilon t)dt \\
&= 2\int_{\delta}^{\infty} \frac{g(x,t)}{t}(\sin M\ t - \sin\varepsilon t)dt \\
&\quad +2\int_0^{\delta} \frac{g(x,t)}{t}(\sin M\ t - \sin\varepsilon t)dt.
\end{aligned}$$

In $[\delta, \infty]$, by Corollary 3.3 and Lemma 3.4, we obtain

$$\lim_{M\to\infty,\varepsilon\to 0} \int_{\delta}^{\infty} \frac{g(x,t)}{t}(\sin M\ t - \sin\varepsilon t)dt = 0. \tag{3.5}$$

In $[0, \delta]$, the Lebesgue Dominate Convergence theorem implies that

$$\lim_{\varepsilon\to 0} \int_0^{\delta} \frac{g(x,t)}{t} \sin\varepsilon t dt = 0. \tag{3.6}$$

Now, by Lemma 3.6,

$$\lim_{M\to\infty} \int_0^{\delta} g(x,t)\frac{\sin Mt}{t}dt = g(x,0+) = \frac{\pi}{2}\left[f(x-0)+f(x+0)\right].$$

Combining (3.5), (3.6) and the above expression we then conclude the proof. ☐

Remark 3.8. It is clear that the classic theorem of Dirichlet-Jordan on $L(\mathbb{R})$ is a particular case of Theorem 3.7. Since $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset BV_0(\mathbb{R})$, from Lemma 3.2 and Theorem 3.7 we get the following items. The first of them contains some results of [3].

- (Riemann-Lebesgue lemma) If $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then the Fourier transform $\widehat{f}(s)$ exists for each $s \in \mathbb{R}$ and $\widehat{f} \in C_0(\mathbb{R}\setminus\{0\})$.

- For every $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$ and each $x \in \mathbb{R}$, the expression (3.4) holds.

In ([3]) it is proved that if $h \in BV([-\pi^{1/\alpha}, \pi^{1/\alpha}])$, then, for $1 > \alpha > 0$, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} h(t) & \text{if } t \in (-\pi^{1/\alpha}, \pi^{1/\alpha}), \\ \dfrac{\sin |t|^\alpha}{|t|} & \text{if } t \in (-\infty, -\pi^{1/\alpha}] \cup [\pi^{1/\alpha}, \infty) \end{cases}$$

belongs to $\{HK(\mathbb{R}) \cap BV(\mathbb{R})\} \setminus L(\mathbb{R})$. By the Multiplier theorem it follows that $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset L^2(\mathbb{R})$, so the above function is in $\{BV_0(\mathbb{R}) \cap L^2(\mathbb{R})\} \setminus L(\mathbb{R})$. Therefore we have the following corollary.

Corollary 3.9. *There exist functions in $L^2(\mathbb{R}) \setminus L(\mathbb{R})$ such that their Fourier transforms exist as in (1.1) and, for each $x \in \mathbb{R}$, the expression (3.4) is true.*

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