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CRITERIA ON BOUNDEDNESS OF MATRIX OPERATORS IN WEIGHTED SPACES OF SEQUENCES AND THEIR APPLICATIONS

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ABSTRACT. In this paper we prove a new discrete Hardy type inequality involving a kernel which has a more general form than those known in the literature. We obtain necessary and sufficient conditions for the boundedness of a matrix operator from the weighted $l_{p,v}$ space into the weighted $l_{q,u}$ space defined by $(Af)_j := \sum_{i=j}^{\infty} a_{i,j} f_i$, for all $f = \{f_i\}_{i=1}^{\infty} \in l_{p,v}$ in case $1 < q < p < \infty$ and $a_{i,j} \geq 0$. Then we deduce a corresponding dual statement.

1. Introduction and preliminaries

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $u = \{u_i\}_{i=1}^{\infty}$, $v = \{v_i\}_{i=1}^{\infty}$ be positive sequences of real numbers, which will be referred to as weight sequences. Let $1 < p < \infty$. We denote by $l_{p,v}$ the space of sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,v} := \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{\frac{1}{p}} < \infty.$$

Moreover, let $(a_{i,j})$ be a non-negative triangular matrix with entries $a_{i,j} \geq 0$, if $i \geq j \geq 1$ and $a_{i,j} = 0$, if $i < j$.

We consider an estimate of the following form

$$\|Af\|_{q,u} \leq C \|f\|_{p,v}, \ \forall f \in l_{p,v}, \tag{1.1}$$

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where the matrix operator A is defined by

$$(Af)_i := \sum_{j=1}^{i} a_{i,j} f_j, \;\; i \geq 1, \tag{1.2}$$

or

$$(Af)_j := \sum_{i=j}^{\infty} a_{i,j} f_i, \;\; j \geq 1, \tag{1.3}$$

and C is a positive finite constant independent of f.

When one of parameters p or q is equal to 1 or ∞, necessary and sufficient conditions of the validity of (1.1) with the exact value of the best constant $C > 0$ have been obtained in [8]. In case $1 < p, q < \infty$ inequalities as (1.1) have not been established yet for arbitrary matrices $(a_{i,j})$. Instead inequality (1.1) has been established with certain restrictions on the matrix $(a_{i,j})$.

When $a_{i,j} = 1$, $i \geq j \geq 1$, the operators (1.2), (1.3) coincide with the discrete Hardy operators of the forms $(A_0 f)_i := \sum_{j=1}^{i} f_j$, $(A_0 f)_j := \sum_{i=j}^{\infty} f_i$, respectively. References about generalizations of the original forms of the discrete and continuous Hardy inequalities can be found in different books, see e.g. [1].

In [4], [5] necessary and sufficient conditions for the validity of (1.1) have been obtained for $1 < p, q < \infty$ under the assumption that there exists $d \geq 1$ such that the inequalities

$$\frac{1}{d}(a_{i,k} + a_{k,j}) \leq a_{i,j} \leq d(a_{i,k} + a_{k,j}), \;\; i \geq k \geq j \geq 1. \tag{1.4}$$

hold.

A sequence $\{a_i\}_{i=1}^{\infty}$ is called almost non-decreasing (non-increasing), if there exists $c > 0$ such that $ca_i \geq a_k$ ($a_k \leq ca_j$) for all $i \geq k \geq j \geq 1$.

In [6] estimate (1.1) has been studied under the assumption that there exist $d \geq 1$ and a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$, and a non-negative matrix $(b_{i,j})$, where $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, such that the inequalities

$$\frac{1}{d}(b_{i,k}\omega_j + a_{k,j}) \leq a_{i,j} \leq d(b_{i,k}\omega_j + a_{k,j}), \tag{1.5}$$

hold for all $i \geq k \geq j \geq 1$.

In this paper we consider inequality (1.1) under the following assumption.

Assumption A: There exist $d \geq 1$, a sequence of positive numbers $\{\omega_k\}_{k=1}^{\infty}$, and a non-negative matrix $(b_{i,j})$, whose entries $b_{i,j}$ are almost non-decreasing in i and almost non-increasing in j such that the inequalities

$$\frac{1}{d}(a_{i,k} + b_{k,j}\omega_i) \leq a_{i,j} \leq d(a_{i,k} + b_{k,j}\omega_i), \tag{1.6}$$

hold for all $i \geq k \geq j \geq 1$.

Let $\alpha > 0$. Let $a_{i,j} = (b_i - d_j)^\alpha$, if $i \geq j \geq 1$, where the sequences $\{b_i\}_{i=1}^\infty$ and $\{d_i\}_{i=1}^\infty$ are such that $b_i \geq d_j$, $i \geq j \geq 1$. If $\{b_i\}_{i=1}^\infty$ is a non-decreasing sequence and $\{d_i\}_{i=1}^\infty$ is an arbitrary sequence, then the entries of the matrix $(a_{i,j})$ satisfy condition (1.5), i.e. $a_{i,j} \approx (b_i - b_k)^\alpha + a_{k,j}$, $i \geq k \geq j \geq 1$. In general, the entries $a_{i,j}$ do not satisfy condition (1.6). If $\{d_i\}_{i=1}^\infty$ is a non-decreasing sequence and $\{b_i\}_{i=1}^\infty$ is an arbitrary sequence, then the entries $a_{i,j}$ satisfy condition (1.6), but in general, condition (1.5) does not hold for the entries of the matrix $(a_{i,j})$.

Thus, conditions (1.5), (1.6) include condition (1.4) and complement each other.

We also note that from (1.6) it easily follows that

$$da_{i,j} \geq a_{i,k}, \tag{1.7}$$

$$da_{i,j} \geq b_{k,j}\omega_i, \tag{1.8}$$

for $i \geq k \geq j \geq 1$.

A continuous analogue of (1.5)-(1.6) has been considered by R. Oinarov in [3].

Notation: If M and K are real valued functionals of sequences, then we understand that the symbol $M \ll K$ means that there exists $c > 0$ such that $M \leq cK$, where c is a constant which may depend only on parameters such as p, q and d. If $M \ll K \ll M$, then we write $M \approx K$.

For the proof of our main theorem we need the following well-known result for the discrete weighted Hardy inequality (see [1], [7]). For the sake of completeness, we include a statement of such result.

Theorem A. *Let $1 < q < p < \infty$. Then the inequality*

$$\left(\sum_{i=1}^\infty \left|\sum_{j=i}^\infty \mu_j f_j\right|^q u_i^q\right)^{\frac{1}{q}} \leq C\left(\sum_{i=1}^\infty |v_i f_i|^p\right)^{\frac{1}{p}} \quad \forall f \in l_{p,v}, \tag{1.9}$$

holds if and only if

$$H = \left(\sum_{k=1}^\infty \left(\sum_{i=1}^k u_i^q\right)^{\frac{p}{p-q}} \left(\sum_{j=k}^\infty \mu_j^{p'} v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} \mu_k^{p'} v_k^{-p'}\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $H \approx C$, where C is the best constant in (1.9).

We also need the following well-known result (see [4]).

Lemma A. *Let $\gamma > 0$. Then there exists $c > 0$ such that*

$$\frac{1}{c}\left(\sum_{k=1}^j \beta_k\right)^\gamma \leq \sum_{k=1}^j \beta_k \left(\sum_{i=1}^k \beta_i\right)^{\gamma-1} \leq c\left(\sum_{k=1}^j \beta_k\right)^\gamma \quad \forall j \in \mathbb{N}, \tag{1.10}$$

for all sequences $\{\beta_k\}_{k=1}^{\infty}$ *of positive real numbers.*

Moreover, there exists $c_1 \geq 1$ *such that*

$$\frac{1}{c_1}\left(\sum_{k=j}^{N}\beta_k\right)^{\gamma} \leq \sum_{k=j}^{N}\beta_k\left(\sum_{i=k}^{N}\beta_i\right)^{\gamma-1} \leq c_1\left(\sum_{k=j}^{N}\beta_k\right)^{\gamma} \tag{1.11}$$

for all $j,k \in \{1,2,...,N\}$, $N \in \mathbb{N}\cup\{\infty\}$ *and for all sequences* $\{\beta_k\}_{k=1}^{\infty}$ *of positive real numbers such that* $\sum\limits_{k=1}^{\infty}\beta_k < \infty$.

2. Main results

Theorem 2.1. *Let* $1 < q < p < \infty$. *Let the entries of the matrix* $(a_{i,j})$ *satisfy Assumption A. Then estimate (1.1) for the operator defined by (1.3) holds if and only if* $F = \max\{F_1,\ F_2\} < \infty$, *where*

$$F_1 = \left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}b_{i,j}^q u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{k=i}^{\infty}\omega_k^{p'}v_k^{-p'}\right)^{\frac{p(q-1)}{p-q}}\omega_i^{p'}v_i^{-p'}\right)^{\frac{p-q}{pq}}$$

and

$$F_2 = \left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i}u_j^q\right)^{\frac{q}{p-q}}\left(\sum_{k=i}^{\infty}a_{k,i}^{p'}v_k^{-p'}\right)^{\frac{q(p-1)}{p-q}}u_i^q\right)^{\frac{p-q}{pq}}.$$

Moreover $F \approx C$, *where* C *is the best constant in (1.1).*

Proof. Necessity. Let us assume that (1.1) holds for a finite constant C. Let $m \geq 1$. Then we take a test sequence $\widetilde{f_m} = \{\widetilde{f}_{m,k}\}_{k=1}^{\infty}$ such that

$$\widetilde{f}_{m,k} = \left(\sum_{j=1}^{k}b_{k,j}^q u_j^q\right)^{\frac{1}{p-q}}\left(\sum_{i=k}^{m}\omega_i^{p'}v_i^{-p'}\right)^{\frac{q-1}{p-q}}\omega_k^{p'-1}v_k^{-p'} \quad \text{if} \quad 1 \leq k \leq m,$$

$$\widetilde{f}_{m,k} = 0 \quad \text{if } k > m.$$

Then

$$\|\widetilde{f}_m\|_{p,v} = \left(\sum_{k=1}^{\infty}\widetilde{f}_{m,k}^p v_k^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{m}\left(\sum_{j=1}^{k}b_{k,j}^q u_j^q\right)^{\frac{p}{p-q}}\left(\sum_{i=k}^{m}\omega_i^{p'}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\omega_k^{p'}v_k^{-p'}\right)^{\frac{1}{p}} \tag{2.1}$$

Substituting $\widetilde{f}_m$ in the left hand side of inequality (1.1) and using (1.11) and (1.8), we deduce that

$$\|A\widetilde{f}_m\|_{q,u}^q \gg \sum_{k=1}^{m}\sum_{j=k}^{m}a_{j,k}\widetilde{f}_{m,j}\left(\sum_{i=j}^{m}a_{i,k}\widetilde{f}_{m,i}\right)^{q-1}u_k^q = \sum_{j=1}^{m}\widetilde{f}_{m,j}\sum_{k=1}^{j}u_k^q a_{j,k}\left(\sum_{i=j}^{m}a_{i,k}\widetilde{f}_{m,i}\right)^{q-1}$$

$$\gg \sum_{j=1}^{m}\widetilde{f}_{m,j}\omega_j\sum_{k=1}^{j}u_k^q b_{j,k}^q\left(\sum_{i=j}^{m}\omega_i\widetilde{f}_{m,i}\right)^{q-1} = \sum_{j=1}^{m}\widetilde{f}_{m,j}\omega_j\sum_{k=1}^{j}u_k^q b_{j,k}^q\left(\sum_{i=j}^{m}\omega_i\left(\sum_{s=1}^{i}b_{i,s}^q u_s^q\right)^{\frac{1}{p-q}}\right.$$

$$\times\left(\sum_{k=i}^{m}\omega_k^{p'}v_k^{-p'}\right)^{\frac{q-1}{p-q}}\omega_i^{p'-1}v_i^{-p'}\Bigg)^{q-1}\gg\sum_{j=1}^{m}\widetilde{f}_{m,j}\omega_j\sum_{k=1}^{j}u_k^qb_{j,k}^q\left(\sum_{s=1}^{j}b_{j,s}^qu_s^q\right)^{\frac{q-1}{p-q}}$$

$$\times\left(\sum_{i=j}^{m}\omega_i^{p'}v_i^{-p'}\left(\sum_{k=i}^{m}\omega_k^{p'}v_k^{-p'}\right)^{\frac{q-1}{p-q}}\right)^{q-1}$$

$$\gg\sum_{j=1}^{m}\widetilde{f}_{m,j}\omega_j\left(\sum_{s=1}^{j}b_{j,s}^qu_s^q\right)^{\frac{p-1}{p-q}}\left(\sum_{i=j}^{m}\omega_i^{p'}v_i^{-p'}\right)^{\frac{(p-1)(q-1)}{p-q}}$$

$$=\sum_{j=1}^{m}\left(\sum_{s=1}^{j}b_{j,s}^qu_s^q\right)^{\frac{p}{p-q}}\left(\sum_{i=j}^{m}\omega_i^{p'}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\omega_j^{p'}v_j^{-p'},$$

i.e.

$$\|A\widetilde{f}_m\|_{q,u}\gg\left(\sum_{j=1}^{m}\left(\sum_{s=1}^{j}b_{j,s}^qu_s^q\right)^{\frac{p}{p-q}}\left(\sum_{i=j}^{m}\omega_i^{p'}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\omega_j^{p'}v_j^{-p'}\right)^{\frac{1}{q}}. \tag{2.2}$$

From (1.1), (2.1) and (2.2) it follows that

$$\left(\sum_{j=1}^{m}\left(\sum_{s=1}^{j}b_{j,s}^qu_s^q\right)^{\frac{p}{p-q}}\left(\sum_{i=j}^{m}\omega_i^{p'}v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}}\omega_j^{p'}v_j^{-p'}\right)^{\frac{p-q}{pq}}\ll C$$

for all $m\geq 1$. Since $m\geq 1$ is arbitrary we have that

$$F_1\ll C. \tag{2.3}$$

Inequality (1.1) holds if and only if the following dual inequality

$$\|A^*g\|_{p',v^{-1}}\leq C\|g\|_{q',u^{-1}},\ \ g\in l_{q',u^{-1}} \tag{2.4}$$

holds for the conjugate operator A^*, which is defined by (1.2).
Moreover, the best constants in (1.1) and (2.4) coincide.

Now let $m\geq 1$. By taking a test sequence $\widetilde{g}_m=\{\widetilde{g}_{m,k}\}_{k=1}^{\infty}$ such that

$$\widetilde{g}_{m,k}=\left(\sum_{j=1}^{k}u_j^q\right)^{\frac{q-1}{p-q}}\left(\sum_{i=k}^{m}a_{i,k}^{p'}v_i^{-p'}\right)^{\frac{(q-1)(p-1)}{p-q}}u_k^q\quad\text{for } 1\leq k\leq m,$$

$\widetilde{g}_{m,k}=0\quad$ for $k>m$,
we have that

$$\|\widetilde{g}_m\|_{q',u^{-1}}=\left(\sum_{k=1}^{m}\left(\sum_{j=1}^{k}u_j^q\right)^{\frac{q}{p-q}}\left(\sum_{i=k}^{m}a_{i,k}^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}u_k^q\right)^{\frac{1}{q'}}. \tag{2.5}$$

By using (1.10) and (1.7) we have that

$$\|A^*\widetilde{g}_m\|_{p',v^{-1}}^{p'} \geq \sum_{i=1}^{m}\left(\sum_{j=1}^{i} a_{i,j}\widetilde{g}_{m,j}\right)^{p'} v_i^{-p'} \gg \sum_{i=1}^{m}\sum_{j=1}^{i} a_{i,j}\widetilde{g}_{m,j}\left(\sum_{k=1}^{j} a_{i,k}\widetilde{g}_{m,k}\right)^{p'-1} v_i^{-p'}$$

$$\geq \sum_{j=1}^{m}\widetilde{g}_{m,j}\sum_{i=j}^{m} a_{i,j}\left(\sum_{k=1}^{j} a_{i,k}\widetilde{g}_{m,k}\right)^{p'-1} v_i^{-p'} \gg \sum_{j=1}^{m}\widetilde{g}_{m,j}\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\left(\sum_{k=1}^{j}\widetilde{g}_{m,k}\right)^{p'-1}$$

$$= \sum_{j=1}^{m}\widetilde{g}_{m,j}\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\left(\sum_{k=1}^{j}\left(\sum_{s=1}^{k}u_s^q\right)^{\frac{q-1}{p-q}}\left(\sum_{i=k}^{m} a_{i,k}^{p'}v_i^{-p'}\right)^{\frac{(q-1)(p-1)}{p-q}} u_k^q\right)^{p'-1}$$

$$\gg \sum_{j=1}^{m}\widetilde{g}_{m,j}\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\left(\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\right)^{\frac{q-1}{p-q}}\left(\sum_{k=1}^{j}u_k^q\left(\sum_{s=1}^{k}u_s^q\right)^{\frac{q-1}{p-q}}\right)^{p'-1}$$

$$\gg \sum_{j=1}^{m}\widetilde{g}_{m,j}\left(\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\right)^{\frac{p-1}{p-q}}\left(\sum_{k=1}^{j}u_k^q\right)^{\frac{1}{p-q}}$$

$$= \sum_{j=1}^{m}\left(\sum_{k=1}^{j}u_k^q\right)^{\frac{q}{p-q}}\left(\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q,$$

i.e.,

$$\|A^*\widetilde{g}_m\|_{p',v^{-1}} \gg \left(\sum_{j=1}^{m}\left(\sum_{k=1}^{j}u_k^q\right)^{\frac{q}{p-q}}\left(\sum_{i=j}^{m} a_{i,j}^{p'}v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q\right)^{\frac{1}{p'}}. \tag{2.6}$$

Since $m \geq 1$ is arbitrary, then (2.4), (2.5), (2.6) imply that $F_2 \ll C$. Hence, (2.3) implies that

$$F \ll C. \tag{2.7}$$

The proof of the necessity is thus complete.

Sufficiency. Let $F < \infty$ and $0 \leq f \in l_{p,v}$.

For all $j \geq 1$ we define the following set:

$$T_j = \{k \in \mathbb{Z} : (d+1)^{-k} \leq (Af)_j\},$$

where d is the constant from (1.6) and $\mathbb{Z}$ is the set of integers. We assume that $\inf T_j = \infty$, if $T_j = \emptyset$ and $k_j = \inf T_j$, if $T_j \neq \emptyset$. We can clearly assume that $(Af)_1 \neq 0$. Without loss of generality, we may assume that $a_{i,j}$ is non-increasing in j, otherwise we take $a_{i,j} \approx \widetilde{a}_{i,j} = \sup\limits_{j\leq k\leq i} a_{i,k}$. Therefore $k_j < k_{j+1}$. If $k_j < \infty$, then

$$(d+1)^{-k_j} \leq (Af)_j < (d+1)^{-(k_j-1)}, \; j \geq 1. \tag{2.8}$$

Let $m_1 = 0$, $k_1 = k_{m_1+1}$ and $M_1 = \{j \in \mathbb{N} : k_j = k_1 = k_{m_1+1}\}$, where $\mathbb{N}$ is the set of natural numbers. Suppose that m_2 is such that $\sup M_1 = m_2$. Obviously

$m_2 > m_1$ and if the set M_1 is upper bounded, then $m_2 < \infty$ and $m_2 = \max M_1$. We now define inductively the numbers $0 = m_1 < m_2 < \ldots < m_s < \infty$, $s \geq 1$. We set $m_{s+1} = \sup M_s$, where $M_s = \{j \in N : k_j = k_{m_s+1}\}$.

Let $N_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Further, we assume that $k_{m_{s+1}} = n_{s+1},\ s \in N_0$. From the definition of m_s and from (2.8) it follows that

$$(d+1)^{-n_{s+1}} \leq (Af)_j < (d+1)^{-n_{s+1}+1},\ m_s+1 \leq j \leq m_{s+1} \tag{2.9}$$

for all $s \in N_0$. Then

$$\mathbb{N} = \bigcup_{s\in N_0} [m_s+1, m_{s+1}),\ \text{where } [m_s+1, m_{s+1}) \cap [m_l+1, m_{l+1}) = \emptyset,\ s \neq l.$$

Therefore

$$\|Af\|_{q,u}^q = \sum_{s\in N_0} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q\, u_j^q. \tag{2.10}$$

We assume that $\sum\limits_{j=m_s+1}^{m_{s+1}} = 0$, if $m_s = \infty$.

There are two possible cases: $N_0 = \mathbb{N}$ and $N_0 \neq \mathbb{N}$.

1. If $N_0 = \mathbb{N}$, then we estimate the left hand side of (1.1) in the following way.

Clearly inequalities $n_{s+1} < n_{s+2} < n_{s+3}$ imply that $-n_{s+3}+1 \leq -n_{s+1}-1$ for all $s \in \mathbb{N}$. Hence, (2.9), (1.6) imply that

$$\begin{aligned}
(d+1)^{-n_{s+1}-1} &= (d+1)^{-n_{s+1}} - d(d+1)^{-n_{s+1}-1} \\
\leq (d+1)^{-n_{s+1}} &- d(d+1)^{-n_{s+3}+1} < (Af)_{m_{s+1}} - d\,(Af)_{m_{s+3}} \\
&= \sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}} f_i - d \sum_{i=m_{s+3}}^{\infty} a_{i,m_{s+3}} f_i \\
&\leq \sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + \sum_{i=m_{s+3}}^{\infty} [a_{i,m_{s+1}} - d a_{i,m_{s+3}}] f_i \\
&\leq \sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + \sum_{i=m_{s+3}}^{\infty} [d(a_{i,m_{s+3}} + b_{m_{s+3},m_{s+1}}\omega_i) - d a_{i,m_{s+3}}] f_i \\
&= \sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + d b_{m_{s+3},m_{s+1}} \sum_{i=m_{s+3}}^{\infty} \omega_i f_i.
\end{aligned} \tag{2.11}$$

Now, by using (2.9) and (2.11), we can estimate the summand on the left hand side in (1.1) in the following way:

$$\begin{aligned}
\sum_{s\in\mathbb{N}} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q\, u_j^q &< \sum_{s\in\mathbb{N}} \sum_{j=m_s+1}^{m_{s+1}} (d+1)^{(-n_{s+1}+1)q} u_j^q \\
&= (d+1)^{2q} \sum_{s\in\mathbb{N}} (d+1)^{(-n_{s+1}-1)q} \sum_{j=m_s+1}^{m_{s+1}} u_j^q
\end{aligned}$$

$$\ll \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i + d b_{m_{s+3},m_{s+1}} \sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q$$

$$\times \sum_{j=m_s+1}^{m_{s+1}} u_j^q \ll \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$+ \sum_{s\in\mathbb{N}} b^q_{m_{s+3},m_{s+1}} \left(\sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q := S_1 + S_2, \tag{2.12}$$

where

$$S_1 = \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a_{i,m_{s+1}} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q,$$

and

$$S_2 = \sum_{s\in\mathbb{N}} b^q_{m_{s+3},m_{s+1}} \left(\sum_{i=m_{s+3}}^{\infty} \omega_i f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q.$$

To estimate S_1, we apply the Hölder Inequality in the inner summand with the powers p, p' and in the outer summand with the powers $\frac{p}{p-q}$, $\frac{p}{q}$, and we obtain that

$$S_1 \le \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a^{p'}_{i,m_{s+1}} v_i^{-p'} \right)^{\frac{q}{p'}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} |v_i f_i|^p \right)^{\frac{q}{p}} \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\le \left(\sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a^{p'}_{i,m_{s+1}} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\times \left(\sum_{s\in\mathbb{N}} \sum_{i=m_{s+1}}^{m_{s+3}} |v_i f_i|^p \right)^{\frac{q}{p}} \ll (\widetilde{F_2})^{\frac{p-q}{p}} \|f\|^q_{p,v}. \tag{2.13}$$

By (1.11) and (1.7) we can estimate $\widetilde{F_2}$ as follows:

$$\widetilde{F_2} = \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a^{p'}_{i,m_{s+1}} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q \right)^{\frac{p}{p-q}}$$

$$\ll \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+1}}^{m_{s+3}} a^{p'}_{i,m_{s+1}} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=j}^{m_{s+1}} u_k^q \right)^{\frac{q}{p-q}} u_j^q$$

$$\ll \sum_{s\in\mathbb{N}} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=1}^{m_{s+1}} u_k^q\right)^{\frac{q}{p-q}} \left(\sum_{i=m_s+1}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q$$

$$\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} u_k^q\right)^{\frac{q}{p-q}} \left(\sum_{i=j}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q = F_2^{\frac{pq}{p-q}}. \tag{2.14}$$

By (2.13) and (2.14) we deduce that

$$S_1 \ll F_2^q \|f\|_{p,v}^q. \tag{2.15}$$

Next we introduce the sequence $\{\Delta_j\}_{j=1}^{\infty}$ such that $\Delta_j = b^q_{m_{s+3},m_{s+1}} \sum\limits_{i=m_s+1}^{m_{s+1}} u_i^q$, $j = m_{s+3}$ and $\Delta_j = 0$, $j \neq m_{s+3}$, $s \in \mathbb{N}$. Hence, we can rewrite S_2 in the following form:

$$S_2 = \sum_{s\in\mathbb{N}} \left(\sum_{i=m_{s+3}}^{\infty} \omega_i f_i\right)^q b^q_{m_{s+3},m_{s+1}} \sum_{i=m_s+1}^{m_{s+1}} u_i^q = \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \omega_i f_i\right)^q \Delta_j.$$

Thus, by Theorem A, we have that

$$S_2 \ll \widetilde{H^q} \|f\|_{p,v}^q, \tag{2.16}$$

where

$$\widetilde{H} = \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} \Delta_i\right)^{\frac{p}{p-q}} \left(\sum_{j=k}^{\infty} \omega_j^{p'} v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} \omega_k^{p'} v_k^{-p'}\right)^{\frac{p-q}{pq}}. \tag{2.17}$$

By Assumption A, $b_{i,j}$ is almost non-decreasing in i and almost non-increasing in j, and accordingly,

$$\begin{aligned} \sum_{i=1}^{k} \Delta_i &= \sum_{m_{s+3}\leq k} b^q_{m_{s+3},m_{s+1}} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \ll \\ &\ll \sum_{m_{s+3}\leq k} \sum_{j=m_s+1}^{m_{s+1}} b^q_{k,j} u_j^q \leq \sum_{j=1}^{k} b^q_{k,j} u_j^q. \end{aligned} \tag{2.18}$$

By combining (2.16), (2.17) and (2.18), we obtain

$$S_2 \ll F_1^q \|f\|_{p,v}^q. \tag{2.19}$$

Thus, from (2.10), (2.12), (2.15) and (2.19) it follows that

$$\|Af\|_{q,u} \ll F\|f\|_{p,v}, \ \ f \geq 0, \tag{2.20}$$

i.e inequality (1.1) is valid and we see that the best constant in (1.1) $C \ll F$.

2. If $N_0 \neq \mathbb{N}$, i.e. $\max N_0 < \infty$ and $N_0 = \{1, 2, ..., s_0\}$, $s_0 \geq 1$. Therefore $m_{s_0} < \infty$ and $m_{s_0+1} = \infty$. We assume that $\sum\limits_{s=k}^{n} = 0$, if $k > n$ and $\sum\limits_{s=k}^{n} = \sum\limits_{s=1}^{n}$, if

$k \le 0$. We have two possible cases: $n_{s_0+1} < \infty$ and $n_{s_0+1} = \infty$. We consider such cases separately:

1) If $n_{s_0+1} < \infty$, then from (2.10) it follows that

$$\|Af\|_{q,u}^q = \sum_{s\in N_0} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q u_j^q = \sum_{s=1}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q u_j^q \tag{2.21}$$

$$= \sum_{s=1}^{s_0-3} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q u_j^q + \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q u_j^q = I_1 + I_2.$$

If $I_1 \ne 0$ then we estimate I_1 using (2.11) and the previous proof for the case $N_0 = N$. Hence, we obtain

$$I_1 \ll F^q \|f\|_{p,v}^q. \tag{2.22}$$

By using (2.9) and applying the Hölder Inequality with the powers p, p' and with the powers $\frac{p}{p-q}$, $\frac{p}{q}$, we obtain the following inequality

$$I_2 = \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q u_j^q < \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} (d+1)^{(-n_{s+1}+1)q} u_j^q \tag{2.23}$$

$$= (d+1)^q \sum_{s=s_0-2}^{s_0} (d+1)^{-n_{s+1}q} \sum_{j=m_s+1}^{m_{s+1}} u_j^q \ll \sum_{s=s_0-2}^{s_0} (Af)_{m_{s+1}}^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$= \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}} f_i \right)^q \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\le \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q}{p'}} \left(\sum_{i=m_{s+1}}^{\infty} |v_i f_i|^p \right)^{\frac{q}{p}} \sum_{j=m_s+1}^{m_{s+1}} u_j^q$$

$$\le \left(\sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}}$$

$$\times \left(\sum_{s=s_0-2}^{s_0} \sum_{i=m_{s+1}}^{\infty} |v_i f_i|^p \right)^{\frac{q}{p}} \ll (\widehat{F_2})^{\frac{p-q}{p}} \|f\|_{p,v}^q.$$

Using (1.11) and (1.7) we can estimate $\widehat{F_2}$ as follows:

$$\widehat{F_2} = \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=m_s+1}^{m_{s+1}} u_j^q \right)^{\frac{p}{p-q}}$$

$$\ll \sum_{s=s_0-2}^{s_0} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,m_{s+1}}^{p'} v_i^{-p'} \right)^{\frac{q(p-1)}{p-q}} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=j}^{m_{s+1}} u_k^q \right)^{\frac{q}{p-q}} u_j^q$$

$$\ll \sum_{s=s_0-2}^{s_0} \sum_{j=m_s+1}^{m_{s+1}} \left(\sum_{k=1}^{m_{s+1}} u_k^q\right)^{\frac{q}{p-q}} \left(\sum_{i=m_{s+1}}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q$$

$$\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} u_k^q\right)^{\frac{q}{p-q}} \left(\sum_{i=j}^{\infty} a_{i,j}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} u_j^q = F_2^{\frac{pq}{p-q}}. \tag{2.24}$$

From (2.23) and (2.24) we obtain

$$I_2 \ll F_2^q \|f\|_{p,v}^q. \tag{2.25}$$

From (2.21), (2.22) and (2.25) we have (2.20).

2) If $n_{s_0+1} = \infty$, which means that $k_{m_{s_0}+1} = \infty$, then we have $k_j = \infty$ and $T_j = \emptyset$, if $j \geq m_{s_0} + 1$, i.e. $(Af)_j = 0$, if $j \geq m_{s_0+1}$ and $(Af)_j = \sum_{i=j}^{m_{s_0}} a_{i,j} f_i$, $1 \leq j \leq m_{s_0}$. Therefore $m_2 < \infty$ and $s_0 \geq 2$. Then from (2.10) we have

$$\|Af\|_{q,u}^q = \sum_{s\in N_0} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q \, u_j^q = \sum_{s=1}^{s_0-1} \sum_{j=m_s+1}^{m_{s+1}} (Af)_j^q \, u_j^q \tag{2.26}$$

Similarly, we can exploit (2.26) to prove (2.20). Then (2.20) together with (2.7) implies that $C \approx F$ and thus the proof is complete. □

It is known that inequality (1.1) holds if and only if the dual inequality defined by (2.4) holds for the conjugate operator A^*, which coincides with operator defined by (1.2). Moreover, the best constants in (1.1) and (2.4) coincide.

Therefore by using Theorem 2.1 with p', q', v^{-1} and u^{-1} replaced by q, p, u and v, respectively, we obtain the following dual version of Theorem 2.1:

Theorem 2.2. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. Then estimate (1.1) for the operator defined by (1.2) holds if and only if $F^* = \max\{F_1^*,\ F_2^*\} < \infty$, where*

$$F_1^* = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} b_{i,j}^{p'} v_j^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{k=i}^{\infty} \omega_k^q u_k^q\right)^{\frac{q}{p-q}} \omega_i^q u_i^q\right)^{\frac{p-q}{pq}},$$

$$F_2^* = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} \left(\sum_{k=i}^{\infty} a_{k,i}^q u_k^q\right)^{\frac{p}{p-q}} v_i^{-p'}\right)^{\frac{p-q}{pq}}.$$

Moreover $F^* \approx C$, where C is the best constant in (1.1).

3. Application of the main results

Our main results can be used to derive other inequalities. We consider an additive estimate of the form

$$\|Af\|_{q,u} \leq C\left(\|f\|_{p,v} + \|A_0 f\|_{p,\rho}\right), \ \forall f \geq 0, \tag{3.1}$$

where the matrix operator A is defined by (1.2) and the operator A_0 is defined by $(A_0 f)_i := \sum\limits_{j=1}^{i} f_j, \ i \geq 1$.

We assume that the weighted sequences v and ρ satisfy the following conditions

$$v_k > 0, k \geq 1, \sum_{k=1}^{\infty} \rho_k < \infty.$$

We denote by $\Delta\varphi_i$ the difference $\varphi_i - \varphi_{i-1}$ and we set

$$\varphi_n = \left\{ \min_{1\leq k\leq n} \left[\left(\sum_{i=k}^{n} v_i^{-p'} \right)^{-\frac{1}{p'}} + \left(\sum_{i=k}^{\infty} \rho_i^{p} \right)^{\frac{1}{p}} \right] \right\}^{-1}, \ \varphi_0 = 0,$$

for all $n \geq 1$.

Next we introduce the following result of R. Oinarov [2] on the equivalence of inequalities (2.17) and (1.1) which we exploit below.

Theorem C *Let $1 < p, q < \infty$. Let the entries of the matrix $(a_{k,i})$ of the operator A be non-negative and non-increasing in i, i.e. $a_{k,i+1} \leq a_{k,i}$, if $k \geq 1$, $i \geq 1$. Then inequality (3.1) holds if and only if the inequality*

$$\left(\sum_{k=1}^{\infty} \omega_k^q \left(\sum_{i=1}^{k} a_{k,i} f_i \right)^q \right)^{\frac{1}{q}} \leq \widetilde{C} \left(\sum_{k=1}^{\infty} f_k^p \left(\varphi_k^{p'} - \varphi_{k-1}^{p'} \right)^{1-p} \right)^{\frac{1}{p}}, \ f \geq 0 \tag{3.2}$$

holds. Moreover, $C \approx \widetilde{C}$, where C and $\widetilde{C}$ are the best constants in (3.1) and (3.2), respectively.

By exploiting Theorem C, we obtain the following statement:

Theorem 3.1. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{i,j})$ satisfy Assumption A. Then inequality (3.1) holds if and only if $E = \max\{E_1, \ E_2\} < \infty$, where*

$$E_1 = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} b_{i,j}^{p'} \Delta\varphi_j^{p'} \right)^{\frac{q(p-1)}{p-q}} \left(\sum_{k=i}^{\infty} \omega_k^q u_k^q \right)^{\frac{q}{p-q}} \omega_i^q u_i^q \right)^{\frac{p-q}{pq}},$$

$$E_2 = \left(\sum_{i=1}^{\infty} \Delta\varphi_i^{\frac{pq}{p-q}} \left(\sum_{k=i}^{\infty} a_{k,i}^q u_k^q \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{pq}}.$$

Moreover $E \approx C$, where C is the best constant in (3.1).

Proof. We set $\sup\limits_{j\le k\le i} a_{i,k} = \widetilde{a}_{i,j}$. Obviously,

$$a_{i,j} \le \widetilde{a}_{i,j}. \tag{3.3}$$

According to (1.7) we have

$$da_{i,j} \ge \sup_{j\le k\le i} a_{i,k} = \widetilde{a}_{i,j}. \tag{3.4}$$

From (3.3) and (3.4) it follows that $a_{i,j} \approx \widetilde{a}_{i,j}$. The matrix operator $\left(\widetilde{A}f\right)_i = \sum\limits_{j=1}^{i} \widetilde{a}_{i,j} f_j$, $i \ge 1$, is equivalent to the operator A, i.e. $(Af)_i \le \left(\widetilde{A}f\right)_i \le d\,(Af)_i$ or $(Af)_i \approx \left(\widetilde{A}f\right)_i$ for all $f \ge 0$, $i \ge 1$. Then inequality (3.1) is equivalent to

$$\|\widetilde{A}f\|_{q,u} \le C_1\left(\|f\|_{p,v} + \|A_0 f\|_{p,\rho}\right)\ \forall f \ge 0, \tag{3.5}$$

Moreover, $C \approx C_1$, where C and C_1 are the best constants in (3.1) and (3.5), respectively. It is easy to see that the entries of the matrix $(\widetilde{a}_{i,j})$ satisfy the following condition $\widetilde{a}_{i,j} \ge \widetilde{a}_{i,k}$, $i \ge k \ge j \ge 1$. Then according to Theorem C inequality (3.5) holds if and only if the inequality

$$\left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{i=1}^{k} \widetilde{a}_{k,i} f_i\right)^q\right)^{\frac{1}{q}} \le C_2 \left(\sum_{k=1}^{\infty} f_k^p \left(\Delta\varphi_k^{p'}\right)^{1-p}\right)^{\frac{1}{p}}\ \forall f \ge 0, \tag{3.6}$$

holds. Moreover, $C_1 \approx C_2$, where C_2 is the best constant in (3.6).

Since (3.5) is equivalent to inequality (3.1), inequality (3.6) is equivalent to inequality (3.1). By Theorem 2.2 inequality (3.6) (and, thus, (3.5) and (3.1)) holds if and only if $E = \max\{E_1,\ E_2\} < \infty$.

Hence, the proof is complete. □

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