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RANK EQUALITIES FOR MOORE-PENROSE INVERSE AND DRAZIN INVERSE OVER QUATERNION

HUASHENG ZHANG

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ABSTRACT. In this paper, we consider the ranks of four real matrices $G_i(i = 0, 1, 2, 3)$ in $M^\dagger$, where $M = M_0 + M_1 i + M_2 j + M_3 k$ is an arbitrary quaternion matrix, and $M^\dagger = G_0 + G_1 i + G_2 j + G_3 k$ is the Moore-Penrose inverse of M. Similarly, the ranks of four real matrices in Drazin inverse of a quaternion matrix are also presented. As applications, the necessary and sufficient conditions for $M^\dagger$ is pure real or pure imaginary Moore-Penrose inverse and N^D is pure real or pure imaginary Drazin inverse are presented, respectively.

1. INTRODUCTION

Throughout this paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3\ \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate size by I, the conjugate transpose of a matrix A by A^*, the column right space, the row left space of a matrix A over $\mathbb{H}$ by $\mathcal{R}(A), \mathcal{N}(A)$, respectively. The Moore-penrose inverse of $A \in \mathbb{H}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X to the four matrix equations

$$(i)\, AXA = A, (ii)\, XAX = X, (iii)\, (AX)^* = AX, (iv)\, (XA)^* = XA.$$

Let $A \in \mathbb{H}^{m\times m}$ be given with $IndA = k$, the smallest positive integer such that $r\left(A^{k+1}\right) = r\left(A^k\right)$. The Drazin inverse of matrix A, denoted by A^D, is defined to be the unique solution X of the following three matrix equations

$$(i)\, A^k XA = A^k, (ii)\, XAX = X, (iii)\, XA = AX.$$

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Suppose

$$M = M_0 + M_1 i + M_2 j + M_3 k, N = N_0 + N_1 i + N_2 j + N_3 k \tag{1.1}$$

be a quaternion matrix, where $M_i \in \mathbb{R}^{m\times n}$, $N_i \in \mathbb{R}^{m\times m}$, $i = 0, 1, 2, 3$, and let

$$\overline{M} = \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix}, \overline{N} = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ N_2 & -N_3 & N_0 & N_1 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \tag{1.2}$$

and the Moore-Penrose inverse of M, the Drazin inverse of N are denoted by

$$M^\dagger = G_0 + G_1 i + G_2 j + G_3 k, N^D = D_0 + D_1 i + D_2 j + D_3 k, \tag{1.3}$$

respectively, where $G_i \in \mathbb{R}^{n\times m}$, $D_i \in \mathbb{R}^{m\times m}$, $i = 0, 1, 2, 3$.

Moore-Penrose inverse of matrix is an attractive topic in matrix theory and have extensively been investigated by many authors (see, e.g., [1]-[11]). Drazin inverse is also one of the important types of generalized inverses of matrices, and have well been examined in the literatures, (see, e.g., [1]-[2], [13]-[16]). For example, Campbell and Meyer gave a basic identity on Drazin inverse of a matrix in [1]

$$A^D = A^k \left(A^{2k+1}\right)^\dagger A^k. \tag{1.4}$$

L. Zhang presented a characterization of the Drazin inverse of any n×n singular matrix and proposed a method for solving the Drazin inverse and an algorithm with detailed steps to compute the Drazin inverse in [13].

As well known, the expressions of $G_i, D_i\,(i = 0, 1, 2, 3)$ in $M^\dagger, N^D$ are quite complicated if there are no restrictions (see, e.g., [3], [5]). In that case, it is difficult to find properties of $G_i, D_i\,(i = 0, 1, 2, 3)$ in $M^\dagger, N^D$. In this paper, we derived the ranks of $G_i, D_i\,(i = 0, 1, 2, 3)$ in $M^\dagger, N^D$ through a simpler method, and then give some interesting consequences.

As a continuation of the above works, we in this paper investigate the ranks of real matrices G_i, $D_i (i = 0, 1, 2, 3)$ in $M^\dagger$ and N^D. In Section 2, we derive the formulas of rank equalities of four real matrices G_0, G_1, G_2 and G_3 in $M^\dagger = G_0 + G_1 i + G_2 j + G_3 k$. Moreover, we established the necessary and sufficient conditions for $M^\dagger$ is pure real or pure imaginary Moore-Penrose inverse. In Section 3, the formulas of rank equalities of four real matrices D_0, D_1, D_2 and D_3 in $N^D = D_0 + D_1 i + D_2 j + D_3 k$ are established, and the necessary and sufficient conditions for N^D is pure real or pure imaginary Drazin inverse are presented. Some further research topics related to this paper are also given.

2. Rank equality for $G_i\,(i = 0, 1, 2, 3)$ in $M^\dagger$

We begin with the following lemmas which can be generalized to $\mathbb{H}$.

Lemma 2.1. *(see [6]) Let $A_1, A_2, \cdots, A_k \in \mathbb{H}^{m\times n}$. Then the Moore-Penrose inverse of their sum satisfies*

$$(A_1+A_2+\cdots+A_k)^{\dagger}=\frac{1}{k}\left[I_n,I_n,\cdots I_n\right]\begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{\dagger}\begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}.$$

Lemma 2.2. *(see [6]) Let $A_1, A_2, \cdots, A_k \in \mathbb{H}^{m\times n}$. Then the Drazin inverse of their sum satisfies*

$$(A_1+A_2+\cdots+A_k)^{D}=\frac{1}{k}\left[I_n,I_n,\cdots I_n\right]\begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{D}\begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}.$$

Lemma 2.3. *(see [7]) Let $A\in\mathbb{H}^{m\times n}$, $B\in\mathbb{H}^{m\times k}$, $C\in\mathbb{H}^{l\times n}$ and $D\in\mathbb{H}^{l\times k}$ be given. Then the rank of the Schur complement $S=D-CA^{\dagger}B$ satisfies the equality*

$$r\left(D-CA^{\dagger}B\right)=r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix}-r\left(A\right). \tag{2.1}$$

Lemma 2.4. *(see [8]) Let $A\in\mathbb{H}^{m\times n}, B\in\mathbb{H}^{m\times k}$ and $C\in\mathbb{H}^{l\times n}$ be given, and suppose that*

$$\mathcal{R}\left(AQ\right)=\mathcal{R}\left(A\right),\mathcal{R}\left[\left(PA\right)^*\right]=\mathcal{R}\left(A^*\right).$$

Then

$$r\left[AQ,B\right]=r\left[A,B\right],r\begin{bmatrix} PA \\ C \end{bmatrix}=r\begin{bmatrix} A \\ C \end{bmatrix}.$$

where P and Q are arbitrary matrices over $\mathbb{H}$.

Now we establish the main result about Moore-Penrose inverse.

Theorem 2.5. *Let $M,\overline{M}$ and M^{+}be given by (1.1), (1.2) and (1.3). Then the ranks of $G_i\,(i=0,1,2,3)$ in (1.3) can be determined by the following formulas*

$$r\left(G_0\right)=r\begin{bmatrix} \widehat{M_0} & \widetilde{M_0} \\ \widetilde{M} & 0 \end{bmatrix}-r\left(\overline{M}\right),\ r\left(G_1\right)=r\begin{bmatrix} \widehat{M_1} & \widetilde{M_1} \\ \widetilde{M} & 0 \end{bmatrix}-r\left(\overline{M}\right), \tag{2.2}$$

$$r\left(G_2\right)=r\begin{bmatrix} \widehat{M_2} & \widetilde{M_2} \\ \widetilde{M} & 0 \end{bmatrix}-r\left(\overline{M}\right),\ r\left(G_3\right)=r\begin{bmatrix} \widehat{M_3} & \widetilde{M_3} \\ \widetilde{M} & 0 \end{bmatrix}-r\left(\overline{M}\right), \tag{2.3}$$

where

$\widehat{M_0}$

$$=\begin{bmatrix} -M_1 & -M_2 & -M_3 \\ M_0 & M_3 & -M_2 \\ -M_3 & M_0 & M_1 \\ M_2 & -M_1 & M_0 \end{bmatrix}\begin{bmatrix} M_0^* & -M_3^* & M_2^* \\ M_3^* & M_0^* & -M_1^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^*\begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widehat{M_1} = \begin{bmatrix} M_0 & -M_2 & -M_3 \\ M_1 & M_3 & -M_2 \\ M_2 & M_0 & M_1 \\ M_3 & -M_1 & M_0 \end{bmatrix} \begin{bmatrix} M_1^* & M_2^* & M_3^* \\ M_3^* & M_0^* & -M_1^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widehat{M_2} = \begin{bmatrix} M_0 & -M_1 & -M_3 \\ M_1 & M_0 & -M_2 \\ M_2 & -M_3 & M_1 \\ M_3 & M_2 & M_0 \end{bmatrix} \begin{bmatrix} M_1^* & M_2^* & M_3^* \\ M_0^* & -M_3^* & M_2^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widehat{M_3} = \begin{bmatrix} M_0 & -M_1 & -M_2 \\ M_1 & M_0 & M_3 \\ M_2 & -M_3 & M_0 \\ M_3 & M_2 & -M_1 \end{bmatrix} \begin{bmatrix} M_1^* & M_2^* & M_3^* \\ M_0^* & -M_3^* & M_2^* \\ M_3^* & M_0^* & -M_1^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widetilde{M_0} = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \widetilde{M_1} = \begin{bmatrix} -M_1 \\ M_0 \\ -M_3 \\ M_2 \end{bmatrix}, \widetilde{M_2} = \begin{bmatrix} -M_2 \\ M_3 \\ M_0 \\ -M_1 \end{bmatrix}, \widetilde{M_3} = \begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \\ M_0 \end{bmatrix},$$

and

$$\widetilde{M} = [M_0, -M_1, -M_2, -M_3].$$

Proof. According to Lemma 1, we have

$$\begin{aligned}
&(M_0 + M_1 i + M_2 j + M_3 k)^\dagger \\
&= \frac{1}{4}[I_n, I_n, I_n, I_n] \begin{bmatrix} M_0 & M_1 i & M_2 j & M_3 k \\ M_1 i & M_0 & M_3 k & M_2 j \\ M_2 j & M_3 k & M_0 & M_1 i \\ M_3 k & M_2 j & M_1 i & M_0 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ I_m \\ I_m \\ I_m \end{bmatrix} \\
&= \frac{1}{4}[I_m, iI_m, jI_m, kI_m] \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix} \\
&= \frac{1}{4}[I_m, iI_m, jI_m, kI_m] \begin{bmatrix} G_0 & -G_1 & -G_2 & -G_3 \\ G_1 & G_0 & G_3 & -G_2 \\ G_2 & -G_3 & G_0 & G_1 \\ G_3 & G_2 & -G_1 & G_0 \end{bmatrix} \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix}.
\end{aligned}$$

Obviously, G_0 can be written as

$$G_0 = [I_n, 0, 0, 0]\, \overline{M}^\dagger \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix} = P\overline{M}^\dagger Q, \tag{2.4}$$

where

$$P = [I_m, 0, 0, 0], Q = \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then it follows by Lemma 2, Lemma 3, (1.4) and (2.4) we get

$$r(G_0) = \begin{bmatrix} \overline{M}^*\overline{M}\overline{M}^* & \overline{M}^*Q \\ P\overline{M}^* & 0 \end{bmatrix} - r(\overline{M})$$

$$= \begin{bmatrix} \overline{M}\overline{M}^*\overline{M} & \overline{M}P^* \\ Q^*\overline{M} & 0 \end{bmatrix} - r(\overline{M})$$

$$= \begin{bmatrix} \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} \overline{M}^* \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} & \begin{matrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{matrix} \\ \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \end{bmatrix} & 0 \end{bmatrix}$$

$$-r(\overline{M})$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & -M_1 & -M_2 & -M_3 \\ 0 & M_0 & M_3 & -M_2 \\ 0 & -M_3 & M_0 & M_1 \\ 0 & M_2 & -M_1 & M_0 \end{bmatrix} \overline{M}^* \begin{bmatrix} 0 & 0 & 0 & 0 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} & \begin{matrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{matrix} \\ \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \end{bmatrix} & 0 \end{bmatrix}$$
$$- r(\overline{M})$$

$$= \begin{bmatrix} \begin{bmatrix} -M_1 & -M_2 & -M_3 \\ M_0 & M_3 & -M_2 \\ -M_3 & M_0 & M_1 \\ M_2 & -M_1 & M_0 \end{bmatrix} \overline{M}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} & \begin{matrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{matrix} \\ \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \end{bmatrix} & 0 \end{bmatrix} - r(\overline{M})$$

which is the first equality in (2.2). The other equalities (2.2) and (2.3) can also be derived by the similar approach. □

If $M_0 = 0$, then the result in (2.2) and (2.3) can be simplified to the following.

Corollary 2.6. *Let $M = M_1 i + M_2 j + M_3 k$, and denote the Moore-Penrose inverse of M as $M^\dagger = G_0 + G_1 i + G_2 j + G_3 k$,*

$$\widetilde{\overline{M}} = \begin{bmatrix} 0 & -M_1 & -M_2 & -M_3 \\ M_1 & 0 & M_3 & -M_2 \\ M_2 & -M_3 & 0 & M_1 \\ M_3 & M_2 & -M_1 & 0 \end{bmatrix},$$

Then

$$r(G_0) = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A, B] - r\left(\widetilde{\overline{M}}\right),$$
$$r(G_1) = r(C), \; r(G_2) = r(B),$$
$$r(G_3) = r\begin{bmatrix} AA^*A & AA^*B & B \\ CA^*A & CA^*B & 0 \\ C & 0 & 0 \end{bmatrix} - r\left(\widetilde{\overline{M}}\right),$$

where

$$A = \begin{bmatrix} 0 & -M_1 & -M_2 \\ M_1 & 0 & M_3 \\ M_2 & -M_3 & 0 \end{bmatrix}, B = \begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \end{bmatrix}, C = [M_3, M_2, -M_1].$$

Let $M_2 = M_3 = 0$, we get a complex matrix $\widehat{M} = M_0 + M_1 i$. As a special case of Theorem 2.1, we have the following corollary.

Corollary 2.7. *Suppose that $\widehat{M} = M_0 + M_1 i$ and $\widehat{M}^\dagger = G_0 + G_1 i$. Then the ranks of G_0, G_1 can be determined by the following formulas*

$$r(G_0) = r\begin{bmatrix} \widehat{V}_0 & V_0 \\ W & 0 \end{bmatrix} - r\begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix},$$
$$r(G_1) = r\begin{bmatrix} \widehat{V}_1 & V_1 \\ W & 0 \end{bmatrix} - r\begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix},$$

where

$$V_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix}, \widehat{V}_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix} M_0^* [M_1, M_0],$$

$$V_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix}, \widehat{V}_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix} M_1^* [M_1, M_0], W = [M_0, -M_1].$$

Now we give a group of rank inequalities derived from (2.2) and (2.3).

Corollary 2.8. *Let $M, \overline{M}$ and $M^{\dagger}$ be given by (1.1), (1.2) and (1.3). Then the ranks of G_0 in $M^{\dagger}$ satisfies the rank inequalities*

$$r(G_0) \le r\begin{bmatrix} M_0 & -M_3 & M_2 \\ M_3 & M_0 & -M_1 \\ -M_2 & M_1 & M_0 \end{bmatrix} + r[M_0, -M_1, -M_2, -M_3] + r\begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \\ M_0 \end{bmatrix} - r\left(\overline{M}\right), \tag{2.5}$$

$$r(G_0) \ge r[M_0, -M_1, -M_2, -M_3] + r\begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \\ M_0 \end{bmatrix} - r\left(\overline{M}\right), \tag{2.6}$$

$$r(G_0) \ge r\begin{bmatrix} M_0 & -M_3 & M_2 \\ M_3 & M_0 & -M_1 \\ -M_2 & M_1 & M_0 \end{bmatrix} - r\begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} - r\begin{bmatrix} -M_1 & -M_2 & -M_3 \\ M_0 & M_3 & -M_2 \\ -M_3 & M_0 & M_1 \\ M_2 & -M_1 & M_0 \end{bmatrix} + r\left(\overline{M}\right). \tag{2.7}$$

Proof. It is clearly that

$$r\left(\widetilde{M_0}\right) + r\left(\widetilde{M}\right) \le r\begin{bmatrix} \widehat{M_0} & \widetilde{M_0} \\ \widetilde{M} & 0 \end{bmatrix} \le r\begin{bmatrix} M_0^* & -M_3^* & M_2^* \\ M_3^* & M_0^* & -M_1^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* + r\left(\widetilde{M}\right) + r\left(\widetilde{M_0}\right),$$

where $\widetilde{M_0}$, $\widehat{M_0}$ and $\widetilde{M}$ are defined as same as Theorem 2.1.
Putting them in the first rank equality in (2.2), we obtain (2.5) and (2.6). To show (2.7), we need the following rank equality

$$r\left(CA^{\dagger}B\right) \ge r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix} - r[A, B] + r(A),$$

Now applying above inequality to $P\overline{M}^{\dagger}Q$ in (2.4), we have

$$r(G_0) = r\left(P\overline{M}^{\dagger}Q\right) \ge r\begin{bmatrix} \overline{M} & Q \\ P & 0 \end{bmatrix} - r\begin{bmatrix} \overline{M} \\ P \end{bmatrix} - r\left[\overline{M}, Q\right] + r\left(\overline{M}\right),$$

which is (2.7). □

Rank inequalities for the G_1, G_2 and G_3 in $M^{\dagger}$ can also be derived in the similar way shown above. We omit them here for simplicity.

Using the result of Theorem 2.1 and Corollary 2.2, we give a necessary and sufficient condition for an arbitrary quaternion matrix M to have a pure real or pure imaginary Moore-Penrose inverse. As a special case, a necessary and sufficient condition for an arbitrary complex matrix to have a pure real or pure imaginary Moore-Penrose inverse is also presented.

Theorem 2.9. *Let $M, \overline{M}$ and $M^{\dagger}$be given by (1.1), (1.2) and (1.3). Then*
(a) the Moore-Penrose inverse of M is a pure real matrix if and only if

$$r\left(\overline{M}\right) = r\begin{bmatrix} \widehat{M_1} & M_1 \\ M & 0 \end{bmatrix} = r\begin{bmatrix} \widehat{M_2} & M_2 \\ M & 0 \end{bmatrix} = r\begin{bmatrix} \widehat{M_3} & M_3 \\ M & 0 \end{bmatrix},$$

(b) the Moore-Penrose inverse of M is a pure imaginary matrix if and only if

$$r\begin{bmatrix} \widehat{M_0} & M_0 \\ M & 0 \end{bmatrix} = r\left(\overline{M}\right)$$

where M, $\widehat{M_i}$ and $M_i\,(i = 0, 1, 2, 3)$ are defined as Theorem 2.1.

Proof. From Theorem 2.1, the Moore-Penrose inverse of M is a pure real matrix if and only if

$$r\left(G_1\right) = r\left(G_2\right) = r\left(G_1\right) = 0.$$

That is

$$r\begin{bmatrix} \widehat{M_1} & M_1 \\ M & 0 \end{bmatrix} - r\left(\overline{M}\right) = 0, r\begin{bmatrix} \widehat{M_2} & M_2 \\ M & 0 \end{bmatrix} - r\left(\overline{M}\right) = 0, r\begin{bmatrix} \widehat{M_3} & M_3 \\ M & 0 \end{bmatrix} - r\left(\overline{M}\right) = 0.$$

Thus we have part (a). By the same manner, we can get part (b). □

Corollary 2.10. *Suppose that $\widehat{M} = M_0 + M_1 i$ and $\widehat{M}^{\dagger} = G_0 + G_1 i$. Then*
(a) the Moore-Penrose inverse of $\widehat{M}$ is a pure real matrix if and only if

$$r\begin{bmatrix} \widehat{V_0} & V_0 \\ W & 0 \end{bmatrix} = r\begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix}$$

(b) the Moore-Penrose inverse of $\widehat{M}$ is a pure imaginary matrix if and only if

$$r\begin{bmatrix} \widehat{V_1} & V_1 \\ W & 0 \end{bmatrix} = r\begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix},$$

where

$$V_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix}, \widehat{V_0} = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix} M_0^{*}\left[M_1, M_0\right],$$

and

$$V_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix}, \widehat{V_1} = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix} M_1^{*}\left[M_1, M_0\right], W = \left[M_0, -M_1\right].$$

3. Rank equality for $D_i\,(i = 0, 1, 2, 3)$ in N^D

In this section, we derive the formulas of rank equalities of four real matrices D_0, D_1, D_2 and D_3 in $N^D = D_0 + D_1 i + D_2 j + D_3 k$. Moreover, we established the necessary and sufficient conditions for N have a pure real or pure imaginary Drazin inverse.

Theorem 3.1. *Let $N, \overline{N}$ and N^D be given by (1.1), (1.2) and (1.3) with $IndM \geq 1$. Then the ranks of in (1.3) can be determined by the following formulas*

$$r\left(D_0\right) = r\begin{bmatrix} \overline{N}^k \widehat{N_0} \overline{N}^k & \overline{N}^{k-1}\widetilde{N} \\ \widetilde{N_0}\overline{N}^k & 0 \end{bmatrix} - r\left(\overline{N}^k\right), \tag{3.1}$$

$$r\left(D_1\right) = r\begin{bmatrix} \overline{N}^k \widehat{N_1} \overline{N}^k & \overline{N}^{k-1}\widetilde{N} \\ \widetilde{N_1}\overline{N}^k & 0 \end{bmatrix} - r\left(\overline{N}^k\right), \tag{3.2}$$

$$r\left(D_2\right) = r\begin{bmatrix} \overline{N}^k \widehat{N_2} \overline{N}^k & \overline{N}^{k-1}\widetilde{N} \\ \widetilde{N_2}\overline{N}^k & 0 \end{bmatrix} - r\left(\overline{N}^k\right), \tag{3.3}$$

$$r\left(D_1\right) = r\begin{bmatrix} \overline{N}^k \widehat{N_3} \overline{N}^k & \overline{N}^{k-1}\widetilde{N} \\ \widetilde{N_3}\overline{N}^k & 0 \end{bmatrix} - r\left(\overline{N}^k\right), \tag{3.4}$$

where

$$\widetilde{N} = \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \widehat{N_0} = \begin{bmatrix} N_0 & 0 & 0 & 0 \\ 0 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & N_0 & N_1 \\ 0 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N_0} = \left[N_0, -N_1, -N_2, -N_3\right],$$

$$\widehat{N_1} = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ 0 & N_0 & 0 & 0 \\ N_2 & -N_3 & N_0 & N_1 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N_1} = \begin{bmatrix} N_1 & N_0 & N_3 & -N_2 \end{bmatrix},$$

$$\widehat{N_2} = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & 0 & 0 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N_2} = \begin{bmatrix} N_2 & -N_3 & N_0 & N_1 \end{bmatrix},$$

and

$$\widehat{N_3} = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & 0 & 0 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N_3} = \begin{bmatrix} N_3 & N_2 & -N_1 & N_0 \end{bmatrix}.$$

Proof. According to Lemma 1, we have

$$\begin{aligned}
&(N_0+N_1i+N_2j+N_3k)^D\\
&=\frac{1}{4}[I_m,I_m,I_m,I_m]\begin{bmatrix} N_0 & N_1i & N_2j & N_3k\\ N_1i & N_0 & N_3k & N_2j\\ N_2j & N_3k & N_0 & N_1i\\ N_3k & N_2j & N_1i & N_0\end{bmatrix}^D\begin{bmatrix} I_m\\ I_m\\ I_m\\ I_m\end{bmatrix}\\
&=\frac{1}{4}[I_m,iI_m,jI_m,kI_m]\begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3\\ N_1 & N_0 & N_3 & -N_2\\ N_2 & -N_3 & N_0 & N_1\\ N_3 & N_2 & -N_1 & N_0\end{bmatrix}^D\begin{bmatrix} I_m\\ -iI_m\\ -jI_m\\ -kI_m\end{bmatrix}\\
&=\frac{1}{4}[I_m,iI_m,jI_m,kI_m]\begin{bmatrix} D_0 & -D_1 & -D_2 & -D_3\\ D_1 & D_0 & D_3 & -D_2\\ D_2 & -D_3 & D_0 & D_1\\ D_3 & D_2 & -D_1 & D_0\end{bmatrix}\begin{bmatrix} I_m\\ -iI_m\\ -jI_m\\ -kI_m\end{bmatrix}.
\end{aligned}$$

Obviously, G_0 can be written as

$$G_0=[I_m,0,0,0]\,\overline{N}^D\begin{bmatrix} I_m\\ 0\\ 0\\ 0\end{bmatrix}=P\overline{N}^DQ=P\overline{N}^k\left(\overline{N}^{2k+1}\right)^D\overline{N}^kQ,$$

where $P=[I_m,0,0,0]$ and $Q=\begin{bmatrix} I_m\\ 0\\ 0\\ 0\end{bmatrix}$.

Then it follows by Lemma 2, Lemma 3, (1.4) and (2.4) we get

$$\begin{aligned}
r(D_0)&=\begin{bmatrix} \left(\overline{N}^{2k+1}\right)^*\overline{N}^{2k+1}\left(\overline{N}^{2k+1}\right)^* & \left(\overline{N}^{2k+1}\right)^*\overline{N}^kQ\\ P\overline{N}^k\left(\overline{N}^{2k+1}\right)^* & 0\end{bmatrix}-r\left(\overline{N}^{2k+1}\right)\\
&=\begin{bmatrix} \overline{N}^{2k+1} & \overline{N}^kQ\\ P\overline{N}^k & 0\end{bmatrix}-r\left(\overline{N}^{2k+1}\right)\\
&=\begin{bmatrix} \overline{N}^{2k+1}-\overline{N}^kQP\overline{N}\overline{N}^k-\overline{N}^k\overline{N}QP\overline{N}^k & \overline{N}^kQ\\ P\overline{M}^k & 0\end{bmatrix}-r\left(\overline{N}^k\right)\\
&=\begin{bmatrix} \overline{N}^k\left(\overline{N}-QP\overline{N}-\overline{N}QP\right)\overline{N}^k & \overline{N}^kQ\\ P\overline{N}^k & 0\end{bmatrix}-r\left(\overline{N}^k\right)\\
&=\begin{bmatrix} \overline{N}^k\begin{bmatrix} N_0 & 0 & 0 & 0\\ 0 & N_0 & N_3 & -N_2\\ 0 & -N_3 & N_0 & N_1\\ 0 & N_2 & -N_1 & N_0\end{bmatrix}\overline{N}^k & \overline{N}^{k-1}\begin{bmatrix} N_0\\ N_1\\ N_2\\ N_3\end{bmatrix}\\ [N_0,-N_1,-N_2,-N_3]\,\overline{N}^k & 0\end{bmatrix}-r\left(\overline{N}^k\right),
\end{aligned}$$

which is the equality in (3.1). The equalities (3.2-3.4) can also be derived by the similar approach.

□

Let $N_2 = N_3 = 0$, we get a complex matrix $\widehat{N} = N_0 + N_1 i$. As a special case of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Suppose that $\widehat{N} = N_0 + N_1 i$ and $\widehat{N}^+ = D_0 + D_1 i$. Then the ranks of D_0, D_1 can be determined by the following formulas*

$$r(D_0) = r\begin{bmatrix} \widetilde{W}^k \widehat{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widehat{V} \widetilde{W}^k & 0 \end{bmatrix} - r\left(\widetilde{W}^k\right),$$

$$r(D_1) = r\begin{bmatrix} \widetilde{W}^k \widetilde{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widetilde{V} \widetilde{W}^k & 0 \end{bmatrix} - r\left(\widetilde{W}^k\right),$$

where

$$V_1 = \begin{bmatrix} N_0 \\ N_1 \end{bmatrix}, \widehat{V} = \begin{bmatrix} N_0 & 0 \\ 0 & N_0 \end{bmatrix}, \widetilde{V} = \begin{bmatrix} 0 & -N_1 \\ N_1 & 0 \end{bmatrix},$$

$$\widetilde{W} = \begin{bmatrix} N_0 & -N_1 \\ N_1 & N_0 \end{bmatrix}, W_1 = [N_0, -N_1], W_2 = [N_1, N_0].$$

Using the result of Theorem 3.1 and Corollary 3.2, we give a necessary and sufficient condition for an arbitrary quaternion matrix N to have a pure real or pure imaginary Drazin inverse. As a special case, a necessary and sufficient condition for an arbitrary square complex matrix to have a pure real or pure imaginary Drazin inverse is also presented.

Theorem 3.3. *Let $N, \overline{N}$ and N^D be given by (1.1), (1.2) and (1.3) with $IndM \geq 1$. Then*
(a) the Drazin inverse of N is a pure real matrix if and only if

$$r\left(\overline{N}^k\right) = r\begin{bmatrix} \overline{N}^k \widehat{N_1} \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N_1} \overline{M}^k & 0 \end{bmatrix} = r\begin{bmatrix} \overline{N}^k \widehat{N_2} \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N_2} \overline{N}^k & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} \overline{N}^k \widehat{N_3} \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N_3} \overline{N}^k & 0 \end{bmatrix},$$

(b) the Drazin inverse of N is a pure imaginary matrix if and only if

$$r\begin{bmatrix} \overline{N}^k \widehat{N_0} \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N_0} \overline{N}^k & 0 \end{bmatrix} = r\left(\overline{N}^k\right),$$

where $\widetilde{N}$, $\widehat{N_i}$ and $\widetilde{N_1}$ $(i = 0, 1, 2, 3)$ are defined as Theorem 3.1.

Corollary 3.4. *Suppose that $\widehat{N} = N_0 + N_1 i$ and $\widehat{N}^D = D_0 + D_1 i$. Then*
(a) the Drazin inverse of $\widehat{N}$ is a pure real matrix if and only if

$$r\begin{bmatrix} \widetilde{W}^k \widetilde{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widetilde{V} \widetilde{W}^k & 0 \end{bmatrix} = r\left(\widetilde{W}^k\right),$$

(b) the Drazin inverse of $\widehat{N}$ *is a pure imaginary matrix if and only if*

$$r\begin{bmatrix} \widetilde{W}^k\widehat{V}\widetilde{W}^k & \widetilde{W}^{k-1}V_1 \\ \widehat{V}\widetilde{W}^k & 0 \end{bmatrix} = r\left(\widetilde{W}^k\right),$$

where

$$V_1 = \begin{bmatrix} N_0 \\ N_1 \end{bmatrix}, \widehat{V} = \begin{bmatrix} N_0 & 0 \\ 0 & N_0 \end{bmatrix}, \widetilde{V} = \begin{bmatrix} 0 & -N_1 \\ N_1 & 0 \end{bmatrix},$$
$$\widetilde{W} = \begin{bmatrix} N_0 & -N_1 \\ N_1 & N_0 \end{bmatrix}, W_1 = [N_0, -N_1], W_2 = [N_1, N_0].$$

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References

1. S. L. Campbell and C.D. Meyer, *Generalized inverse of linear transformations*, Corrected reprint of the 1979 original. Dover Publications, Inc., New York, 1991.
2. A. Ben-Israel and T. N. E. Greville, *Generalized inverses: Theory and Applications*, second ed., Springer, New York, 2003.
3. C. H. Hung and T.L. Markham, *The Moore-Penrose inverse of a partitioned matrix*, Linear Algebra Appl. **11** (1975), 73–86.
4. C.D. Meyer Jr., *Generalized inverses and ranks of block matrices*, SIAM J. Appl. Math. **25** (1973), 597–602.
5. J. Miao, *General expression for Moore-Penrose invers of a* 2×2 *block matrix*, Linear Algebra Appl. **151** (1990) 1–15.
6. Y. Tian, *The Moore-Penrose inverses of a triple matrix product*, Math. In Theory and Practice **1** (1992), 64–67 .
7. Y. Tian, *How to characterize equalities for the Moore-Penrose inverses of a matrix*, Kyungpook Math. J. **41** (2001), 125–131.
8. G. Marsaglia and G.P.H. Styan, *Equalities and inequalities for ranks of matrices*, Linear Multilinear Algebra **2** (1974), 269–292 .
9. P. Patricio, *The Moore-Penrose inverse of von Neumann regular matrices over a ring*, Linear Algebra Appl. **332** (2001), 469–483.
10. P. Patricio, *The Moore-Penrose inverse of a factorization*, Linear Algebra Appl. **370** (2003), 227–236.
11. D.W. Robinson, *Nullities of submatrices of the Moore-Penrose inverse*, Linear Algebra Appl. **94** (1987), 127–132.
12. Y. Tian, *Rank and inertia of submatrices of the Moore-Penrose inverse of a Hermitian Matrix*, Electron. J. Linear Algebra. **20** (2010), 226–240.
13. L. Zhang, *A characterization of the Drazin inverse*, Linear Algebra Appl. **335** (2001), 183–188.
14. N. Castro-Gonzalez and E. Dopazo, *Representations of the Drazin inverse for a class of block matrices*, Linear Algebra Appl. **400** (2005), 253–269 .
15. R. E. Harwig, E. Li and Y. Wei, *Representations for the Drazin inverse of a block matrix*, SIAM J. Matrix Anal. Appl. **27** (2006), 757–771.
16. X. Li and M. Wei, *A note on the representations for the Drazin inverse of* 2×2 *block matrices*, Linear Algebra Appl. **423** (2007), 332–338.

17. C. Deng and Y. Wei, *New additive results for the generalized Drazin inverse*, J. Math. Anal. Appl. **370** (2010), 313–321.
18. S. Dragana and S. Cvetković-Ilić, *New additive results on Drazin inverse and its applications*, Appl. Math. Comput. **218** (2011), 3019–3024.

Department of Mathematics, Liaocheng University, Shandong252059, P.R. China.
E-mail address: zhsh0510@163.com; zhsh0510@yahoo.com.cn