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LOCAL SPECTRUM OF A FAMILY OF OPERATORS

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ABSTRACT. Starting from the classic definitions of resolvent set and spectrum of a linear bounded operator on a Banach space, we introduce the local resolvent set and local spectrum, the local spectral space and the single-valued extension property of a family of linear bounded operators on a Banach space. Keeping the analogy with the classic case, we extend some of the known results from the case of a linear bounded operator to the case of a family of linear bounded operators on a Banach space.

1. Introduction

Let X be a complex Banach space and $B(X)$ the Banach algebra of linear bounded operators on X. Let T be a linear bounded operator on X. The *norm* of T is

$$\|T\| = \sup\{\|Tx\| \mid x \in X,\ \|x\| \leq 1\}.$$

The *spectrum* of an operator $T \in B(X)$ is defined as the set

$$\mathrm{sp}\,(T) = \mathbb{C} \backslash r(T),$$

where $r(T)$ is the *resolvent set* of T and consists in all complex numbers $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is bijective on X.
An operator $T \in B(X)$ is said to have the *single-valued extension property* if for any analytic function $f : D_f \to X$, where $D_f \subset \mathbb{C}$ is open, with $(\lambda I - T)\, f(\lambda) \equiv 0$, it results $f(\lambda) \equiv 0$.
For an operator $T \in B(X)$ having the single-valued extension property and for $x \in X$ we can consider the set $r_T(x)$ of elements $\lambda_0 \in \mathbb{C}$ such that there is an analytic function $\lambda \mapsto x(\lambda)$ defined in a neighborhood of λ_0 with values in X, which verifies $(\lambda I - T)\, x(\lambda) \equiv x$. The set $r_T(x)$ is said *the local resolvent set of* T *at* x, and the set $\mathrm{sp}_T(x) = \mathbb{C} \backslash r_T(x)$ is called *the local spectrum of* T *at* x.

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An analytic function $f_x : D_x \to X$, where $D_x \subset \mathbb{C}$ is open, is said the *analytic extension* of function $\lambda \mapsto R(\lambda, T)\, x$ if $r(T) \subset D_x$ and $(\lambda I - T)\, f_x(\lambda) \equiv x$.
If T has the single-valued extension property, then, for any $x \in X$ there is a unique *maximal analytic extension* of function $\lambda \mapsto R(\lambda, T)\, x : r_T(x) \to X$, referred from now as $x(\lambda)$. Moreover, $r_T(x)$ is an open set of C and $r(T) \subset r_T(x)$.
Let

$$X_T(a) = \{x \in X | \mathrm{sp}_T(x) \subset a\}$$

be the *local spectral space* of T for all sets $a \subset \mathbb{C}$. The space $X_T(a)$ is a linear subspace (not necessary closed) of X.
Two operators $T, S \in B(X)$ are *quasinilpotent equivalent* if

$$\lim_{n\to\infty} \left\| (T-S)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\| (S-T)^{[n]} \right\|^{\frac{1}{n}} = 0,$$

where $(T-S)^{[n]} = \sum_{k=0}^{n} (-1)^{n-k} C_k^n T^k S^{n-k}$, for any $n \in \mathbb{N}$.
The quasinilpotent equivalence relation is an equivalence relation (i.e. is reflexive, symmetric and transitive) on $B(X)$.

Theorem 1.1. *Let $T, S \in B(X)$ be two quasinilpotent equivalent operators. Then*
(i) $\mathrm{sp}(T) = \mathrm{sp}(S)$;
(ii)T has the single-valued extension property if an only if S has the single-valued extension property. Moreover, $\mathrm{sp}_T(x) = \mathrm{sp}_S(x)$.

For an easier understanding of the results from this paper, we recall some definitions and results introduced in [4]; see also [1, 2, 3].
We say that two families of operators $\{S_h\}, \{T_h\} \subset B(X)$, with $h \in (0,1]$, are *asymptotically equivalent* if

$$\lim_{h\to 0} \|S_h - T_h\| = 0 .$$

Two families of operators $\{S_h\}, \{T_h\} \subset B(X)$, with $h \in (0,1]$, are *asymptotically quasinilpotent (spectral) equivalent* if

$$\lim_{n\to\infty} \limsup_{h\to 0} \left\| (S_h - T_h)^{[n]} \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \limsup_{h\to 0} \left\| (T_h - S_h)^{[n]} \right\|^{\frac{1}{n}} = 0.$$

The asymptotic (quasinilpotent) equivalence between two families of operators $\{S_h\}$, $\{T_h\} \subset B(X)$ is an equivalence relation (i.e. reflexive, symmetric and transitive) on $L(X)$. Moreover, if $\{S_h\}, \{T_h\}$ are two bounded asymptotically equivalent families, then are asymptotically quasinilpotent equivalent.
Let be the sets

$$C_b((0,1],\ B(X)) =$$
$$= \{\varphi : (0,1] \to B(X) |\, \varphi(h) = T_h \text{ such that } \varphi \text{ } is \text{ countinous and bounded}\} =$$
$$= \left\{ \{T_h\}_{h\in(0,1]} \subset B(X) \middle| \{T_h\}_{h\in(0,1]} \text{ is a bounded family, i.e. } \sup_{h\in(0,1]} \|T_h\| < \infty \right\}.$$

and

$$C_0((0,1],\ B(X)) = \left\{ \varphi \in C_b((0,1],\ B(X)) |\, \lim_{h\to 0} \|\varphi(h)\| = 0 \right\} =$$
$$= \left\{ \{T_h\}_{h\in(0,1]} \subset B(X) \middle| \lim_{h\to 0} \|T_h\| = 0 \right\}.$$

$C_b((0,1],\ B(X))$ is a Banach algebra non-commutative with norm

$$\|\{T_h\}\| = \sup_{h\in(0,1]} \|T_h\|,$$

and $C_0((0,1],\ B(X))$ is a closed bilateral ideal of $C_b((0,1],\ B(X))$. Therefore the quotient algebra $C_b((0,1],\ B(X))/C_0((0,1],\ B(X))$, which will be called from now B_∞, is also a Banach algebra with quotient norm

$$\left\|\dot{\{T_h\}}\right\| = \inf_{\{U_h\}_{h\in(0,1]}\in C_0((0,1],\ B(X))} \|\{T_h\}+\{U_h\}\| = \inf_{\{S_h\}_{h\in(0,1]}\in\dot{\{T_h\}}} \|\{S_h\}\|.$$

Then

$$\left\|\dot{\{T_h\}}\right\| = \inf_{\{S_h\}_{h\in(0,1]}\in\dot{\{T_h\}}} \|\{S_h\}\| \le \|\{S_h\}\| = \sup_{h\in(0,1]} \|S_h\|,$$

for any $\{S_h\}_{h\in(0,1]} \in \dot{\{T_h\}}$. Moreover,

$$\left\|\dot{\{T_h\}}\right\| = \inf_{\{S_h\}_{h\in(0,1]}\in\dot{\{T_h\}}} \|\{S_h\}\| = \inf_{\{S_h\}_{h\in(0,1]}\in\dot{\{T_h\}}} \sup_{h\in(0,1]} \|S_h\|.$$

If two bounded families $\{T_h\}_{h\in(0,1]},\ \{S_h\}_{h\in(0,1]} \subset B(X)$ are asymptotically equivalent, then $\lim_{h\to 0}\|S_h - T_h\| = 0$, i.e. $\{T_h - S_h\}_{h\in(0,1]} \in C_0((0,1],\ B(X))$.
Let $\{T_h\}_{h\in(0,1]},\ \{S_h\}_{h\in(0,1]} \in C_b((0,1],\ B(X))$ be asymptotically equivalent. Then

$$\limsup_{h\to 0}\|S_h\| = \limsup_{h\to 0}\|T_h\|.$$

Since

$$\limsup_{h\to 0}\|S_h\| \le \sup_{h\in(0,1]}\|S_h\|,$$

results that

$$\begin{aligned}\limsup_{h\to 0}\|S_h\| &= \inf_{\{S_h\}_{h\in(0,1]}\in\dot{\{T_h\}}} \limsup_{h\to 0}\|S_h\| \le \\ &\le \inf_{\{S_h\}_{h\in(0,1]}\in\dot{\{T_h\}}} \sup_{h\in(0,1]}\|S_h\| = \left\|\dot{\{T_h\}}\right\|,\end{aligned}$$

for any $\{S_h\}_{h\in(0,1]} \in \dot{\{T_h\}}$.
In particular

$$\lim_{h\to 0}\lim\|T_h\| \le \left\|\dot{\{T_h\}}\right\| \le \|\{T_h\}\| = \sup_{h\in(0,1]}\|T_h\|.$$

Definition 1.2. We say $\dot{\{S_h\}},\ \dot{\{T_h\}} \in B_\infty$ are spectral equivalent if

$$\lim_{\mathbf{n}\to\infty}\left(\left\|\left(\dot{\{S_h\}} - \dot{\{T_h\}}\right)^{[\mathbf{n}]}\right\|\right)^{\frac{1}{\mathbf{n}}} = \lim_{\mathbf{n}\to\infty}\left(\left\|\left(\dot{\{T_h\}} - \dot{\{S_h\}}\right)^{[\mathbf{n}]}\right\|\right)^{\frac{1}{\mathbf{n}}} = 0,$$

where $\left(\dot{\{S_h\}} - \dot{\{T_h\}}\right)^{[n]} = \sum_{k=0}^{n}(-1)^{n-k}C_n^k\dot{\{S_h\}}^k\dot{\{T_h\}}^{n-k}$.

$$\begin{aligned}\left(\dot{\{S_h\}} - \dot{\{T_h\}}\right)^{[n]} &= \sum_{k=0}^{n}(-1)^{n-k}C_n^k\dot{\{S_h\}}^k\dot{\{T_h\}}^{n-k} \\ &= \dot{\left\{\sum_{k=0}^{n}(-1)^{n-k}C_n^k {S_h}^k {T_h}^{n-k}\right\}} = \dot{\left\{(S_h - T_h)^{[n]}\right\}}.\end{aligned}$$

Therefore $\{\dot{S_h}\},\ \{\dot{T_h}\} \in\ B_\infty$ are spectral equivalent if

$$\lim_{\mathbf{n}\to\infty}\left\|\dot{\left\{(S_h - T_h)^{[n]}\right\}}\right\|^{\frac{\mathbf{1}}{\mathbf{n}}} = \lim_{\mathbf{n}\to\infty}\left\|\dot{\left\{(T_h - S_h)^{[n]}\right\}}\right\|^{\frac{\mathbf{1}}{\mathbf{n}}} = 0.$$

Proposition 1.3. *If* $\{\dot{S_h}\},\ \{\dot{T_h}\} \in\ B_\infty$ *are spectral equivalent, then any* $\{S_h\} \in \{\dot{S_h}\}$ *and* $\{T_h\} \in \{\dot{T_h}\}$ *are asymptotically spectral equivalent.*

Proof. Let $\{S_h\} \in \{\dot{S_h}\}$ and $\{T_h\} \in \{\dot{T_h}\}$ be arbitrary. Thus

$$\lim_{n\to\infty}\overline{\lim_{h\to 0}}\left\|(S_h - T_h)^{[n]}\right\|^{\frac{1}{n}} \le \lim_{\mathbf{n}\to\infty}\left\|\dot{\left\{(S_h - T_h)^{[n]}\right\}}\right\|^{\frac{\mathbf{1}}{\mathbf{n}}}.$$

Since $\{\dot{S_h}\},\ \{\dot{T_h}\} \in\ B_\infty$ are spectral equivalent, by Definition 1.2 and above relation, it follows that

$$\lim_{n\to\infty}\overline{\lim_{h\to 0}}\left\|(S_h - T_h)^{[n]}\right\|^{\frac{1}{n}} = 0.$$

Analogously we can prove that $\lim_{n\to\infty}\overline{\lim}_{h\to 0}\left\|(T_h - S_h)^{[n]}\right\|^{\frac{1}{n}} = 0.$ □

Proposition 1.4. *Let* $\{T_h\},\{S_h\}\ \subset B(X)$ *be two continuous bounded families. Then* $\lim_{h\to 0}\|T_hS_h - S_hT_h\| = 0$ *if and only if* $\{\dot{S_h}\}\{\dot{T_h}\} = \{\dot{T_h}\}\{\dot{S_h}\}$.

Proof. $\lim_{h\to 0}\|T_hS_h - S_hT_h\| = 0 \Leftrightarrow \{\dot{T_hS_h}\} = \{\dot{S_hT_h}\} \Leftrightarrow \{\dot{S_h}\}\{\dot{T_h}\} = \{\dot{T_h}\}\{\dot{S_h}\}$. □

Definition 1.5. We call the *resolvent set* of a family of operators $\{S_h\} \in C_b((0,1], B(X))$ the set

$$r(\{S_h\}) = \Big\{\ \lambda \in \mathbb{C} |\ \exists\{\mathcal{R}(\lambda, S_h)\} \in C_b((0,1], B(X)),\ \lim_{h\to 0}\|(\lambda I - S_h)\mathcal{R}(\lambda, S_h) - I\| = \\ = \lim_{h\to 0}\|\mathcal{R}(\lambda, S_h)(\lambda I - S_h) - I\| = 0\Big\}$$

We call the *spectrum* of a family of operators $\{S_h\}\ \in C_b((0,1],\ B(X))$ the set

$$\mathrm{sp}(\{S_h\}) = \mathbb{C}\backslash r(\{S_h\}).$$

$$\mathrm{sp}(\{S_h\}) = \mathbb{C}\backslash r(\{S_h\}).$$

$r(\{S_h\})$ is an open set of C. If $\{S_h\}$ is a bounded family, then $\mathrm{sp}(\{S_h\})$ is a compact set of C.

Remark 1.6. (i) If $\lambda \in r(S_h)$ for any $h \in (0,1]$, then $\lambda \in r(\{S_h\})$. So $\bigcap_{h\in(0,1]} r(S_h) \subseteq r(\{S_h\})$;
(ii) If $\lambda \in \mathrm{sp}(\{S_h\})$, then $|\lambda| \le \limsup_{n\to\infty}\lim_{h\to 0}\|{S_h}^n\|^{\frac{1}{n}}$;
(iii) If $\|S_h\| < |\lambda|$ for any $h \in (0,1]$, then $\lambda \in r(\{S_h\})$;
(iv) If $\{S_h\}$ is bounded, then $\{\mathcal{R}(\lambda, S_h)\}$ is also bounded, for every $\lambda \in r(\{S_h\})$;
(v) If $\{S_h\}$ is bounded, then $\lim_{h\to 0}\|\mathcal{R}(\lambda, S_h)\| \ne 0$, for every $\lambda \in r(\{S_h\})$.

Proposition 1.7. *(resolvent equation - asymptotic) Let* $\{S_h\} \subset B(X)$ *be a bounded family and* $\lambda, \mu \in r(\{S_h\})$*. Then*

$$\lim_{h\to 0} \|\mathcal{R}(\lambda, S_h) - \mathcal{R}(\mu, S_h) - (\mu - \lambda)\mathcal{R}(\lambda, S_h)\mathcal{R}(\mu, S_h)\| = 0.$$

Proposition 1.8. *Let* $\{S_h\} \subset B(X)$ *be a bounded family. If* $\lambda \in r(\{S_h\})$ *and*

$$\left\{\mathcal{R}_i(\lambda, S_h)\right\} \in C_b((0,1], B(X)), \ i = \overline{1,2}\}$$

such that

$$\lim_{h\to 0} \|(\lambda I - S_h)\mathcal{R}_i(\lambda, S_h) - I\| = \lim_{h\to 0} \|\mathcal{R}_i(\lambda, S_h)(\lambda I - S_h) - I\| = 0$$

for $i = \overline{1,2}$*, then*

$$\lim_{h\to 0} \|\mathcal{R}_1(\lambda, S_h) - \mathcal{R}_2(\lambda, S_h)\| = 0.$$

Theorem 1.9. *Let* $\{S_h\} \in C_b((0,1], B(X))$*. Then*

$$\mathrm{sp}\left(\{\dot{S_h}\}\right) = \mathrm{sp}(\{S_h\}).$$

Theorem 1.10. *If two bounded families* $\{S_h\}, \{T_h\} \subset B(X)$ *are asymptotically equivalent, then*

$$\mathrm{sp}(\{S_h\}) = \mathrm{sp}(\{T_h\}).$$

2. Local Spectrum of a Family of Operators

Let $\mathcal{O}$ be the set of analytic functions families $\{f_h\}_{h\in(0,1]}$ defined on an open complex set with values in a Banach space X, having property

$$\overline{\lim_{h\to 0}} \|f_h(\lambda)\| < \infty,$$

for any λ from definition set.

Definition 2.1. A bounded continue family of operators $\{T_h\} \subset B(X)$ we said to have *single-valued extension property*, if for any family of analytic functions $\{f_h\}_{h\in(0,1]} \in \mathcal{O}$, $f_h : D \to X$, where $D \subset \mathbb{C}$ open, with property

$$\lim_{h\to 0} \|(\lambda I - T_h) f_h(\lambda)\| \equiv 0,$$

it results $\lim_{h\to 0} \|f_h(\lambda)\| \equiv 0$.

Remark 2.2. Let $\{S_h\}, \{T_h\} \subset B(X)$ be two bounded continue families of operators asymptotically equivalent . If $\{S_h\}$ has single-valued extension property, then $\{T_h\}$ has also single-valued extension property.

Proof. Let $\{f_h\}_{h\in(0,1]} \in \mathcal{O}$ be a family of functions, $f_h : D \to X$, where $D \subset \mathbb{C}$ open, with $\lim_{h\to 0} \|(\lambda I - T_h) f_h(\lambda)\| \equiv 0$. Then

$$\overline{\lim_{h\to 0}} \|(\lambda I - S_h) f_h(\lambda)\| = \overline{\lim_{h\to 0}} \|(\lambda I - S_h - T_h + T_h) f_h(\lambda)\| \leq$$

$$\lim_{h\to 0} \|(\lambda I - T_h) f_h(\lambda)\| + \overline{\lim_{h\to 0}} \|(S_h - T_h) f_h(\lambda)\| \leq \lim_{h\to 0} \|(S_h - T_h)\| \overline{\lim_{h\to 0}} \|f_h(\lambda)\|,$$

for any $\lambda \in D$.

Raking into account $\{S_h\}, \{T_h\}$ are asymptotically equivalent, it follows

$$\lim_{h\to 0}\|(\lambda I - T_h)\, f_h(\lambda)\| \equiv 0.$$

Since $\{T_h\}$ has single-valued extension property, we obtain $\lim_{h\to 0}\|f_h(\lambda)\| \equiv 0$, thus $\{S_h\}$ has single-valued extension property.

□

Definition 2.3. Let $\{T_h\} \subset B(X)$ be a family with single-valued extension property and $x \in X$. From now we consider $r_{\{T_h\}}(x)$ being the set of elements $\lambda_0 \in \mathbb{C}$ such that there are the analytic functions from $\mathcal{O}\ \lambda \mapsto x_h(\lambda)$ defined on an open neighborhood of λ_0 $D \subset r_{\{T_h\}}(x)$ with values in X for any $h \in (0,1]$, having property

$$\lim_{h\to 0}\|(\lambda I - T_h)\, x_h(\lambda) - x\| \equiv 0.$$

$r_{\{T_h\}}(x)$ is called *the local resolvent set of* $\{T_h\}$ *at* x.
The *local spectrum of* $\{T_h\}$ *at* x is defined as the set

$$\mathrm{sp}_{\{T_h\}}(x) = \mathbb{C} \backslash r_{\{T_h\}}(x).$$

We also define the *local spectral space* of $\{T_h\}$ as

$$X_{\{T_h\}}(a) = \left\{x \in X | \mathrm{sp}_{\{T_h\}}(x) \subset a\right\},$$

for all sets $a \subset \mathbb{C}$.

Let be the set

$$X_b\left((0,1],\ X\right) = \{\varphi : (0,1] \to X|\, \varphi(h) = x_h \text{ such that } \varphi \text{ is continue and bounded}\} =$$

$$= \left\{ \{x_h\}_{h\in(0,1]} \subset X \,\middle|\, \{x_h\}_{h\in(0,1]} \text{ a bounded sequence, i.e. } \sup_{h\in(0,1]}\|x_h\| < \infty \right\}.$$

and

$$X_0\left((0,1],\ X\right) = \left\{\varphi \in X_b\left((0,1],\ X\right) |\, \lim_{h\to 0}\|\varphi(h)\| = 0\right\} =$$

$$= \left\{ \{x_h\}_{h\in(0,1]} \subset X \,\middle|\, \lim_{h\to 0}\|x_h\| = 0 \right\}.$$

$X_b\left((0,1],\ X\right)$ is a Banach space in rapport with norm

$$\|\varphi\| = \sup_{h\in(0,1]}\|\varphi(h)\| \Leftrightarrow \|\{x_h\}\| = \sup_{h\in(0,1]}\|x_h\|,$$

and $X_0\left((0,1],\ X\right)$ is a closed subspace of $X_b\left((0,1],\ X\right)$. Therefore, the quotient space $X_b\left((0,1],\ X\right)/X_0\left((0,1], X\right)$, which will be called from now X_∞, is a Banach space in rapport with quotient norm

$$\left\|\{\dot{x_h}\}\right\| = \inf_{\{u_h\}_{h\in(0,1]}\in X_0((0,1],\ X)}\|\{x_h\} + \{u_h\}\| =$$

$$= \inf_{\{y_h\}_{h\in(0,1]}\in\{\dot{x}\}}\|\{y_h\}\| = \inf_{\{y_h\}_{h\in(0,1]}\in\{\dot{x_h}\}}\ \sup_{h\in(0,1]}\|y_h\|.$$

Thus

$$\left\|\{\dot{x_h}\}\right\| = \inf_{\{y_h\}_{h\in(0,1]}\in\{\dot{x_h}\}}\|\{y_h\}\| \le \|\{y_h\}\| = \sup_{h\in(0,1]}\|y_h\|,$$

for all $\{y_h\}_{h\in(0,1]} \in \{\dot{x_h}\}$.
Let $B_\infty = C_b\left((0,1],\ B(X)\right)/C_0\left((0,1],\ B(X)\right)$ and we consider the application Ψ defines by

$$\left(\{\dot{T_h}\}, \{\dot{x_h}\}\right) \longmapsto \{T_h\dot{x}_h\} : B_\infty \times X_\infty \to X_\infty.$$

Remark 2.4. X_∞ is a $B_\infty -$ Banach module in rapport with the above application.

Proof. Is the application well defined (i.e. not depending by selection of representatives)?
Let $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ and $\{y_h\}_{h\in(0,1]} \in \{\dot{x_h}\}$. Then

$$\overline{\lim_{h\to 0}} \|S_h y_h - T_h x_h\| = \overline{\lim_{h\to 0}} \|S_h y_h - T_h y_h + T_h y_h - T_h x_h\| \leq$$
$$\leq \overline{\lim_{h\to 0}} \|S_h y_h - T_h y_h\| + \overline{\lim_{h\to 0}} \|T_h y_h - T_h x_h\| \leq$$
$$\leq \lim_{h\to 0} \|S_h - T_h\| \overline{\lim_{h\to 0}} \|y_h\| + \overline{\lim_{h\to 0}} \|T_h\| \lim_{h\to 0} \|y_h - x_h\| = 0.$$

Therefore $\{S_h y_h\}_{h\in(0,1]} \in \{T_h \dot{x}_h\}$, for any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$ and $\{y_h\}_{h\in(0,1]} \in \{\dot{x_h}\}$.
Is Ψ a bilinear application?

$$\Psi\left(\alpha\{\dot{T_h}\} + \beta\{\dot{S_h}\}, \{\dot{x_h}\}\right) = \Psi\left(\{\alpha T_h \dot{+} \beta S_h\}, \{\dot{x_h}\}\right) =$$
$$= \{(\alpha T_h \dot{+} \beta S_h) x_h\} = \{\alpha T_h x_h \dot{+} \beta S_h x_h\} =$$
$$= \alpha\{T_h \dot{x}_h\} + \beta\{S_h \dot{x}_h\} = \alpha\Psi\left(\{\dot{T_h}\}, \{\dot{x_h}\}\right) + \beta\Psi\left(\{\dot{S_h}\}, \{\dot{x_h}\}\right),$$

for any $\alpha, \beta \in \mathbb{C}$.
Analogously we can prove that

$$\Psi\left(\{\dot{T_h}\}, \alpha\{\dot{y_h}\} + \beta\{\dot{x_h}\}\right) = \alpha\Psi\left(\{\dot{T_h}\}, \{\dot{y_h}\}\right) + \beta\Psi\left(\{\dot{T_h}\}, \{\dot{x_h}\}\right).$$

Is Ψ a continue application?

$$\left\|\Psi\left(\{\dot{T_h}\}, \{\dot{x_h}\}\right)\right\| = \left\|\{T_h \dot{x}_h\}\right\| =$$
$$= \inf_{\{T_h \dot{x}_h\}} \|\{T_h x_h\}\| = \inf_{\{T_h \dot{x}_h\}} \sup_{h\in(0,1]} \|T_h x_h\| \leq$$
$$\leq \inf_{\{T_h \dot{x}_h\}} \sup_{h\in(0,1]} \|T_h\| \, \|x_h\| \leq \inf_{\{\dot{T_h}\},\{\dot{x_h}\}} \sup_{h\in(0,1]} \|T_h\| \, \|x_h\| \leq$$
$$\leq \inf_{\{\dot{T_h}\}} \sup_{h\in(0,1]} \|T_h\| \inf_{\{\dot{x_h}\}} \sup_{h\in(0,1]} \|x_h\| = \left\|\{\dot{T_h}\}\right\| \left\|\{\dot{x_h}\}\right\|.$$

Thus $\left\|\Psi\left(\{\dot{T_h}\}, \{\dot{x_h}\}\right)\right\| \leq \left\|\{\dot{T_h}\}\right\| \left\|\{\dot{x_h}\}\right\|$.
Let $\{\dot{T_h}\} \in B_\infty$ be fixed. The application $\{\dot{x_h}\} \longmapsto \{T_h \dot{x}_h\}$ is a linear bounded operator on X_∞?

$$\{T_h(\alpha x_h \dot{+} \beta y_h)\} = \{\alpha T_h x_h \dot{+} \beta T_h y_h\} = \alpha\{T_h \dot{x}_h\} + \beta\{T_h \dot{y}_h\}.$$

In addition, since

$$\left\|\{T_h \dot{x}_h\}\right\| \leq \left\|\{\dot{T_h}\}\right\| \left\|\{\dot{x_h}\}\right\|,$$

it follows the application $\{\dot{x_h}\} \longmapsto \{T_h \dot{x}_h\}$ is a bounded operator.
Therefore, $B_\infty \subseteq B(X_\infty)$, where $B(X_\infty)$ is the algebra of linear bounded operators on X_∞. □

Definition 2.5. We say that $\{\dot{T_h}\}_{h\in(0,1]} \in B_\infty$ has *single-valued extension property* if for any analytic function $f : D_0 \to X_\infty$, where D_0 is an open complex set with $\left(\lambda\{\dot{I}\} - \{\dot{T_h}\}\right) f(\lambda) \equiv 0$, we have $f(\lambda) \equiv 0$, where $0 = \{\dot{0}\} = X_0\left((0,1],\ X\right)$.

Since $f(\lambda) \in X_\infty$, it follows there is $\{x_h\dot{(}\lambda)\} \in X_\infty$ such that $f(\lambda) = \{x_h\dot{(}\lambda)\}$. Then

$$0 \equiv \left(\lambda\{\dot{I}\} - \{\dot{T_h}\}\right) f(\lambda) = \{\lambda I \dot{-} T_h\}\{x_h\dot{(}\lambda)\} = \{(\lambda I - \dot{T_h}) x_h(\lambda)\},$$

i.e. $\lim_{h\to 0} \|(\lambda I - T_h)x_h(\lambda)\| = 0$.

Definition 2.6. We say $\{\dot{T_h}\}_{h\in(0,1]} \in B_\infty$ has the *single-valued extension property* if for any analytic function $f : D_0 \to X_\infty$, where D_0 is an open complex set with $\lim_{h\to 0} \|(\lambda I - T_h)x_h(\lambda)\| \equiv 0$ we have $\lim_{h\to 0} \|x_h(\lambda)\| \equiv 0$.
The *resolvent set* of an element $\{\dot{x_h}\} \in X_\infty$ in rapport with $\{\dot{T_h}\}_{h\in(0,1]} \in B_\infty$ is

$$r_{\{\dot{T_h}\}}\left(\{\dot{x_h}\}\right) = \left\{ \lambda_0 \in \mathbb{C} | \, \exists \text{ an analytic function} \left(\lambda\{\dot{I}\} - \{\dot{T_h}\}\right)\{x_h\dot{(}\lambda)\} \equiv \{\dot{x_h}\} \right\} =$$
$$= \{ \lambda_0 \in \mathbb{C} | \, \exists \text{ an analytic function } \lambda \mapsto \{x_h\dot{(}\lambda)\} : V_{\lambda_0} \to X_\infty,$$
$$\lim_{h\to 0} \|(\lambda I - T_h)\, x_h(\lambda) - x_h\| \equiv 0 \},$$

when V_{λ_0} is an open neighborhood of λ_0.
Let $\{\dot{x}\} \in X_\infty$, where $\{\dot{x}\} = \{ \{x_h\} \in X_b((0,1],\ X) | \lim_{h\to 0} \|x_h - x\| = 0\}$.
We will call from now

$$X^0_\infty = \left\{ \{\dot{x}\} \in X_\infty \middle| x \in X \right\} \subset X_\infty.$$

Thus

$$r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right) = \{ \lambda_0 \in \mathbb{C} | \, \exists \text{ an analytic function } \lambda \mapsto \{x_h\dot{(}\lambda)\} : V_{\lambda_0} \to X_\infty,$$
$$\lim_{h\to 0} \|(\lambda I - T_h)\, x_h(\lambda) - x\| \equiv 0 \}.$$

Theorem 2.7. *$\{\dot{T_h}\}_{h\in(0,1]} \in B_\infty$ has the single-valued extension property if and only if there is $\{T_h\} \in \{\dot{T_h}\}$ with single-valued extension property.*

Proof. Let $\{f_h\}_{h\in(0,1]} \in \mathcal{O}$, $f_h : D \to X$, be a family of analytic functions, when $D \subset \mathbb{C}$ open, with $\lim_{h\to 0} \|(\lambda I - T_h)\, f_h(\lambda)\| \equiv 0$.
Since $\{f_h\}_{h\in(0,1]} \in \mathcal{O}$, it follows that $\overline{\lim}_{h\to 0} \|f_h(\lambda)\| < \infty$, for all $\lambda \in D$. Thus $\{f_h(\lambda)\} \in X_b((0,1],\ X)$.
Let $f : D \to X_\infty$ be an application defined by $f(\lambda) = \{f_h\dot{(}\lambda)\}$. We prove that f is an analytic function.
Having in view $\{f_h\}$ are analytic functions on D, for any $\lambda_0 \in D$, we obtain

$$\lim_{\lambda\to\lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda\to\lambda_0} \frac{\{f_h\dot{(}\lambda)\} - \{f_h\dot{(}\lambda_0)\}}{\lambda - \lambda_0} =$$
$$= \lim_{\lambda\to\lambda_0} \left\{\frac{f_h(\lambda) \dot{-} f_h(\lambda_0)}{\lambda - \lambda_0}\right\} = \left\{\lim_{\lambda\to\lambda_0} \frac{f_h(\lambda) \dot{-} f_h(\lambda_0)}{\lambda - \lambda_0}\right\},$$

for any $\lambda \in D$. Therefore, f is analytic function on D.
By relation $\lim_{h\to 0} \|(\lambda I - T_h)\, f_h(\lambda)\| \equiv 0$, i.e. $\left(\lambda\{\dot{I}\} - \{\dot{T_h}\}\right) f(\lambda) \equiv \{\dot{0}\}$, since $\{\dot{T_h}\}$ has the single-valued extension property, it follows that $f() \equiv \{\dot{0}\}$, i.e.

$$\lim_{h\to 0} \|f_h(\lambda)\| \equiv 0.$$

Hence $\{T_h\}$ has the single-valued extension property.

Reciprocal: Let $\{T_h\}$ has the single-valued extension property. We prove $\{\dot{T_h}\}$ has also the single-valued extension property.
Let $f : D \to X_\infty$ be an analytic application defined by $f(\lambda) = \{x_h\dot{(}\lambda)\}$ such that

$$\left(\lambda\{\dot{I}\} - \{\dot{T_h}\}\right) f(\lambda) \equiv \{\dot{0}\}.$$

Then $\lim_{h\to 0} \|(\lambda I - T_h) x_h(\lambda)\| \equiv 0$.
We prove that the applications $\lambda \longmapsto x_h(\lambda) : D \to X$ are analytical for all $h \in (0, 1]$. Since f is analytical function, it follows that

$$f'(\lambda_0) = \lim_{\lambda\to\lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda\to\lambda_0} \frac{\{x_h\dot{(}\lambda)\} - \{x_h\dot{(}\lambda_0)\}}{\lambda - \lambda_0} = \lim_{\lambda\to\lambda_0} \left\{ \frac{x_h(\lambda) - x_h(\lambda_0)}{\lambda - \lambda_0} \dot{\right\}}.$$

Therefore, there is $\left\{\lim_{\lambda\to\lambda_0} \frac{x_h(\lambda)-x_h(\lambda_0)}{\lambda-\lambda_0} \dot{\right\}} \in X_\infty$ and thus there is $\lim_{\lambda\to\lambda_0} \frac{x_h(\lambda)-x_h(\lambda_0)}{\lambda-\lambda_0} \in X$ for all $h \in (0, 1]$.
Since $\left(\lambda\{\dot{I}\} - \{\dot{T_h}\}\right) f(\lambda) \equiv \{\dot{0}\}$, i.e. $\lim_{h\to 0} \|(\lambda I - T_h)x_h(\lambda)\| \equiv 0$, taking into account $\{T_h\}$ has the single-valued extension property, we have $\lim_{h\to 0} \|x_h(\lambda)\| \equiv 0$, i.e. $\{x_h\dot{(}\lambda)\} = \{\dot{0}\}$. Therefore, $\{\dot{T_h}\}$ has the single-valued extension property. □

Proposition 2.8. *Let $\{\dot{T_h}\}_{h\in(0,1]} \in B_\infty$ with the single-valued extension property. Then*

$$r_{\{T_h\}}(x) = r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right),$$

for all $x \in X$.

Proof. If $\{\dot{T_h}\}_{h\in(0,1]} \in B_\infty$ has the single-valued extension property, then $\{T_h\} \in \{\dot{T_h}\}$ has the single-valued extension property (Theorem 2.7).
Let $\lambda_0 \in r_{\{T_h\}}(x)$. Hence there are the analytic functions from $\mathcal{O}$ $\lambda \mapsto x_h(\lambda)$ defined on an open neighborhood of λ_0 $D \subset r_{\{T_h\}}(x)$ with values in X for all $h \in (0, 1]$, having property

$$\lim_{h\to 0} \|(\lambda I - T_h) x_h(\lambda) - x\| \equiv 0.$$

Similar to proof of Theorem 2.7, we prove that the application $f : D \to X_\infty$ defined by $f(\lambda) = \{x_h\dot{(}\lambda)\}$ is analytical. Thus $\lambda_0 \in r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right)$.
Reciprocal: Let

$$\lambda_0 \in r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right) = \{\, \lambda_0 \in \mathbb{C} |\, \exists \text{ an analytic function } \lambda \mapsto \{x_h\dot{(}\lambda)\} : V_{\lambda_0} \to X_\infty,$$
$$\lim_{h\to 0} \|(\lambda I - T_h) x_h(\lambda) - x\| \equiv 0 \,\}.$$

Analog proof of Theorem 2.7, we prove that the applications $\lambda \mapsto x_h(\lambda) : V_{\lambda_0} \to X$ are analytical for all $h \in (0, 1]$. Thus $\lambda_0 \in r_{\{T_h\}}(x)$. □

Remark 2.9. Let $\{T_h\} \subset B(X)$ be a continuous bounded family of operators having the single-valued extension property and $x \in X$. Then
(i) $r(\{T_h\}) \subset r_{\{T_h\}}(x)$.
(ii) $X_{\{T_h\}}(a) = X_{\{T_h\}}(\mathrm{sp}\,\{T_h\} \bigcap a)$, for each $a \subset \mathbb{C}$.

(iii) Let $\lambda_0 \in r_{\{T_h\}}(x)$ and the families of holomorphic function from $\mathcal{O}$ $\lambda \mapsto x_h(\lambda)$ and $\lambda \mapsto y_h(\lambda)$ defined on D, an open neighborhood of λ_0, with values in $X$for all $h \in (0,1]$, having properties

$$\lim_{h\to 0} \|(\lambda I - T_h) x_h(\lambda) - x\| = 0$$

and

$$\lim_{h\to 0} \|(\lambda I - T_h) y_h(\lambda) - x\| = 0,$$

for each $\lambda \in D$. Then

$$\lim_{h\to 0} \|x_h(\lambda) - y_h(\lambda)\| = 0,$$

for each $\lambda \in D$.

(iv) If $\{T_h\}, \{S_h\} \in C_b((0,1], B(X))$ are asymptotically equivalent, then

$$r_{\{T_h\}}(x) = r_{\{S_h\}}(x) \quad (x \in X).$$

Proof. (i) By Proposition 2.8 we have

$$r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right) = r_{\{T_h\}}(x) \quad (x \in X).$$

Moreover, by Theorem 1.9, we know that

$$r\left(\{\dot{T_h}\}\right) = r(\{T_h\}).$$

Combing the above relations, we obtain

$$r(\{T_h\}) = r\left(\{\dot{T_h}\}\right) \subset r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right) = r_{\{T_h\}}(x) \quad (x \in X).$$

(ii) By i) it results

$$\mathrm{sp}_{\{T_h\}}(x) \subset \mathrm{sp}(\{T_h\}).$$

Therefore $x \in X_{\{T_h\}}(a)$ if and only if

$$\mathrm{sp}_{\{T_h\}}(x) \subset a \bigcap \mathrm{sp}(\{T_h\}),$$

i.e. $x \in X_{\{T_h\}}(a \bigcap \mathrm{sp}(\{T_h\}))$.

(iii) By Definition 2.3., it results that the analytic functions $\lambda \mapsto x_h(\lambda)$ are defined on an open neighborhood of λ_0 $D_1 \subset r(\{T_h\})$ with values in X and the analytic functions $\lambda \mapsto y_h(\lambda)$ are defined on an open neighborhood of λ_0 $D_2 \subset r(\{T_h\})$ on X.

Let $D \subset D_1 \bigcap D_2 \subset r(\{T_h\})$ be an open neighborhood of λ_0.

Since

$$\lim_{h\to 0} \|(\lambda I - T_h) x_h(\lambda) - x\| = 0$$

and

$$\lim_{h\to 0} \|(\lambda I - T_h) y_h(\lambda) - x\| = 0,$$

for each $\lambda \in D$, thus

$$\lim_{h\to 0} \|(\lambda I - T_h) x_h(\lambda) - (\lambda I - T_h) y_h(\lambda)\| = \lim_{h\to 0} \|(\lambda I - T_h)(x_h(\lambda) - y_h(\lambda))\| = 0,$$

for each $\lambda \in D$.

Having in view that the families of functions $\mapsto x_h(\lambda)$ and $\lambda \mapsto y_h(\lambda)$ are analytical on D, hence the functions $\lambda \mapsto x_h(\lambda) - y_h(\lambda)$ are analytical. Since $\{T_h\}$ has the single-valued extension property, it follows that

$$\lim_{h\to 0} \|x_h(\lambda) - y_h(\lambda)\| = 0,$$

for all $\lambda \in D$.

(iv) Let $\lambda_0 \in \ r_{\{T_h\}}(x)$. Then there is a family of functions $\{x_h\}$ from $\mathcal{O}$, with the property

$$\lim_{h\to 0} \|(\lambda I - T_h)\, x_h(\lambda) - x\| \equiv 0.$$

Thus

$$\overline{\lim_{h\to 0}} \|(\lambda I - S_h)\, x_h(\lambda) - x\| = \overline{\lim_{h\to 0}} \|(\lambda I - S_h - T_h + T_h)\, x_h(\lambda) - x\| \leq$$
$$\leq \lim_{h\to 0} \|(\lambda I - T_h)\, x_h(\lambda) - x\| + \overline{\lim_{h\to 0}} \|(S_h - T_h)\, x_h(\lambda)\| \leq$$
$$\leq \lim_{h\to 0} \|S_h - T_h\| \, \overline{\lim_{h\to 0}} \|x_h(\lambda)\|.$$

Since $\{T_h\}$, $\{S_h\}$ are asymptotically equivalent, by above relation it follows that

$$\lim_{h\to 0} \|(\lambda I - S_h)\, x_h(\lambda) - x\| \equiv 0.$$

Therefore $\lambda_0 \in \ r_{\{S_h\}}(x)$. □

Proposition 2.10. *Let $\{T_h\} \subset B(X)$ be a continuous bounded family of operators having the single-valued extension property. Then*
(i) For any $a \subset b$ we have $X_{\{T_h\}}(a) \subset X_{\{T_h\}}(b)$;
(ii) $X_{\{T_h\}}(a)$ is a linear sub-space of X for all $a \subset \mathbb{C}$;
(iii) $\left\{ \{\dot{x}\} \in X_\infty \middle| x \in X_{\{T_h\}}(a) \right\} = X^0_\infty \bigcap X_{\{\dot{T_h}\}}(a)$ for all $a \subset \mathbb{C}$.

Proof. (i) Let $a, b \subset \mathbb{C}$ such that $a \subset b$ and $x \in X_{\{T_h\}}(a)$. Then $\mathrm{sp}_{\{T_h\}}(x) \subset a$, and thus $\mathrm{sp}_{\{T_h\}}(x) \subset b$. Therefore $x \in X_{\{T_h\}}(b)$.
(ii) Let $x, y \in X_{\{T_h\}}(a)$ and $\alpha, \beta \in \mathbb{C}$. In addition, for any $\lambda_0 \in r_{\{T_h\}}(x) \bigcap r_{\{T_h\}}(y)$ there are the analytic functions families $\{x_h\}$ and $\{y_h\}$ defined on an open neighborhood D of λ_0 such that

$$\lim_{h\to 0} \|(\lambda I - T_h)\, x_h(\lambda) - x\| = 0$$

and

$$\lim_{h\to 0} \|(\lambda I - T_h)\, y_h(\lambda) - y\| = 0,$$

for each $\lambda \in D$.
Let $z_h(\lambda) = \alpha x_h(\lambda) + \beta y_h(\lambda)$, for any $\lambda \in D$ and $h \in (0, 1]$. Since $\{x_h\}$ and $\{y_h\}$ are analytic functions families on D, it follows that $\{z_h\}$ is also an analytic functions family on D and more

$$\lim_{h\to 0} \|(\lambda I - T_h)\, z_h(\lambda) - (\alpha x + \beta y)\| \leq$$
$$\leq |\alpha| \lim_{h\to 0} \|(\lambda I - T_h)\, x_h(\lambda) - x\| + |\beta| \lim_{h\to 0} \|(\lambda I - T_h)\, y_h(\lambda) - y\| = 0,$$

for each $\lambda \in D$.
Therefor $\lambda_0 \in r_{\{T_h\}}(\alpha x + \beta y)$ and

$$r_{\{T_h\}}(x) \bigcap r_{\{T_h\}}(y) \subset r_{\{T_h\}}(\alpha x + \beta y).$$

Moreover

$$\mathrm{sp}_{\{T_h\}}(\alpha x + \beta y) \subset \mathrm{sp}_{\{T_h\}}(x) \bigcup \mathrm{sp}_{\{T_h\}}(y).$$

Since $x, y \in X_{\{T_h\}}(a)$, i.e. $\mathrm{sp}_{\{T_h\}}(x) \subset a$ and $\mathrm{sp}_{\{T_h\}}(y) \subset a$, by above relation, it follows that

$$\mathrm{sp}_{\{T_h\}}(\alpha x + \beta y) \subset a,$$

hence $\alpha x+\beta y\in X_{\{T_h\}}(a)$.
(iii) By Proposition 2.8 we have ($r_{\{T_h\}}(x)=r_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right)$), it follows that $x\in X_{\{T_h\}}(a)$ if and only if $\{\dot{x}\}\in X_{\{\dot{T_h}\}}(a)$. Hence

$$\left\{\{\dot{x}\}\in X_\infty \middle| x\in X_{\{T_h\}}(a)\right\}=\left\{\{\dot{x}\}\in X_\infty \middle| \mathrm{sp}_{\{T_h\}}(x)\subset a\right\}=$$
$$=\left\{\{\dot{x}\}\in X_\infty \middle| \mathrm{sp}_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right)\subset a\right\}=X_\infty^0\bigcap X_{\{\dot{T_h}\}}(a).$$

□

Theorem 2.11. *Let $\{S_h\}$, $\{T_h\}\subset B(X)$ be two continuous bounded families of operators having the single-valued extension property, such that* $\lim_{h\to 0}\|T_hS_h-S_hT_h\|=0$. *If $\{S_h\}$, $\{T_h\}$ are asymptotically spectral equivalent, then*

$$\mathrm{sp}_{\{T_h\}}(x)=\mathrm{sp}_{\{S_h\}}(x)\quad (x\in X).$$

Proof. Since $\{S_h\}$, $\{T_h\}$ have the single-valued extension property, by Theorem 2.7 it results that $\{\dot{T_h}\}_{h\in(0,1]}$, $\{\dot{S_h}\}_{h\in(0,1]}\in B_\infty$ have the single-valued extension property.
If $\{S_h\}$, $\{T_h\}$ are asymptotically spectral equivalent, by Proposition 1.3 have that $\{\dot{T_h}\}_{h\in(0,1]}$, $\{\dot{S_h}\}_{h\in(0,1]}$ are spectral equivalent. Moreover, we obtain that for any $\{\dot{T_h}\}_{h\in(0,1]}$, $\{\dot{S_h}\}_{h\in(0,1]}\in B_\infty$ have the single-valued extension property and being spectral equivalent, it follows that

$$\mathrm{sp}_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right)=\mathrm{sp}_{\{\dot{S_h}\}}\left(\{\dot{x}\}\right),$$

for any $x\in X$.
Therefore, applying Proposition 2.8, we have

$$\mathrm{sp}_{\{T_h\}}(x)=\mathrm{sp}_{\{\dot{T_h}\}}\left(\{\dot{x}\}\right)=\mathrm{sp}_{\{\dot{S_h}\}}\left(\{\dot{x}\}\right)=\mathrm{sp}_{\{S_h\}}(x)\quad (x\in X).$$

□

Remark 2.12. Let $\{S_h\}$, $\{T_h\}\subset B(X)$ be two continuous bounded families of operators having the single-valued extension property, such that $\lim_{h\to 0}\|T_hS_h-S_hT_h\|=0$. If $\{S_h\}$, $\{T_h\}$ are asymptotically spectral equivalent, then

$$X_{\{T_h\}}(a)=X_{\{S_h\}}(a),$$

for any $a\subset\mathbb{C}$.

Proof. Since $\{S_h\}$, $\{T_h\}$ are asymptotically spectral equivalent, by Theorem 2.11, it follows that $\mathrm{sp}_{\{T_h\}}(x)=\mathrm{sp}_{\{S_h\}}(x)$, for all $x\in X$. Then, for any $x\in X_{\{T_h\}}(a)$, i.e. $\mathrm{sp}_{\{T_h\}}(x)\subset a$, it results that $x\in X_{\{S_h\}}(a)$, thus

$$X_{\{T_h\}}(a)\subseteq X_{\{S_h\}}(a).$$

Analog, we can show that $X_{\{S_h\}}(a)\subseteq X_{\{T_h\}}(a)$. □

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