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A NOTE ON THE SPECTRA OF A 3×3 OPERATOR MATRIX AND APPLICATION

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ABSTRACT. In this paper, we investigate the essential approximate point spectrum and the essential defect spectrum of a 3×3 block operator matrix with unbounded entries and with domain consisting of vectors which satisfy certain relations between their components by means of the Browder resolvent set. Furthermore, we apply the obtained results to three-Group transport operators in the Banach space $L_p([-a,a] \times [-1,1])$ where $a > 0$ and $1 \leq p < +\infty$.

1. Introduction

In this work we are concerned with the essential spectra of operators defined by a 3×3 block operator matrix

$$\mathcal{A}_0 := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & L \end{pmatrix} \tag{1.1}$$

where the entries of the matrix are in general unbounded operators. The operator (1.1) is defined on $\big(\mathcal{D}(A) \cap \mathcal{D}(D) \cap \mathcal{D}(G)\big) \times \big(\mathcal{D}(B) \cap \mathcal{D}(E) \cap \mathcal{D}(H)\big) \times \big(\mathcal{D}(C) \cap \mathcal{D}(F) \cap \mathcal{D}(L)\big)$. Note that, the operator $\mathcal{A}_0$ need to be closed, or the domain of this operator can be determined by an additional relation between the components $x \in \mathcal{D}(A)$, $y \in \mathcal{D}(B) \cap \mathcal{D}(E)$ and $z \in \mathcal{D}(C) \cap D(F) \cap \mathcal{D}(L)$ of its elements.

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (resp. $\mathcal{C}(X,Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of compact operators from X into Y.

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For $T \in \mathcal{C}(X,Y)$, we write $\mathcal{D}(T) \subset X$ for the domain, $\mathcal{N}(T) \subset X$ for the null space and $\mathcal{R}(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathcal{R}(T)$ in Y.
Let $\sigma(T)$ (resp. $\rho(T)$) denote the spectrum (resp. the resolvent set) of T. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) := \big\{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in Y}\big\}$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) := \big\{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in Y}\big\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denote the set of Fredholm operators from X into Y and $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ the set of semi-Fredholm operators from X into Y. While the number $i(T) := \alpha(T) - \beta(T)$ is called the index of T, for $T \in \Phi_\pm(X,Y)$.
If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$ respectively. A complex number λ is in Φ_{+T}, Φ_{-T}, $\Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ or $\Phi(X)$, respectively.

In this work, we are concerned with the following essential spectra:

$$\begin{array}{ll} \sigma_{ess}(T) & := \mathbb{C} \setminus \rho_{ess}(T), \\ \sigma_b(T) & := \sigma(T) \setminus \sigma_d(T), \\ \sigma_{eap}(T) & := \mathbb{C} \setminus \rho_{eap}(T), \\ \sigma_{e\delta}(T) & := \mathbb{C} \setminus \rho_{e\delta}(T), \end{array}$$

where $\rho_{ess}(T) := \big\{\lambda \in \Phi_T \text{ such that } i(\lambda - T) = 0\big\}$ and $\sigma_d(T)$ is the set of isolated points λ of the spectrum such that the corresponding Riesz projectors P_λ are finite dimensional. The characterization of the sets $\rho_{eap}(.)$ and $\rho_{e\delta}(.)$ is given by Jeribi and Moalla in [14] as follows

$$\rho_{eap}(T) := \big\{\lambda \in \mathbb{C} \text{ such that } \lambda - T \in \Phi_+(X) \text{ and } i(\lambda - T) \leq 0\big\},$$

and

$$\rho_{e\delta}(T) := \big\{\lambda \in \mathbb{C} \text{ such that } \lambda - T \in \Phi_-(X) \text{ and } i(\lambda - T) \geq 0\big\}.$$

We call $\sigma_{ess}(.)$ is the Schechter essential spectrum [1, 12, 11, 17, 24], $\sigma_{eap}(.)$ and $\sigma_{e\delta}(.)$ the Rakočević and Schmoeger essential spectrum, (see for instance [22, 23, 25]). $\sigma_b(.)$ is the Browder spectrum [7, 21]. In the next, we will denote by $\rho_b(.) := \mathbb{C} \setminus \sigma_b(.)$ the Browder resolvent set. For an operator $T \in \mathcal{C}(X)$, it holds

$$\sigma_{ess}(T) = \sigma_{eap}(T) \cup \sigma_{e\delta}(T).$$

During the last years, e.g. the papers [2, 8, 16, 17, 18, 26] were devoted to the study of the essential spectra of operators defined by a 2×2 block operator matrix acts on the product $X \times Y$ of Banach spaces. An account of the research and a wide panorama of methods to investigate the spectrum of block operator matrices are presented by Tretter in [27, 28]. In general, the operators occurring in $\mathcal{A}_0$ are unbounded and $\mathcal{A}_0$ need not be a closed nor a closable operator, even if its entries needs to be closed. However, under some conditions $\mathcal{A}_0$ is closable and its closure $\mathcal{A}$ can be determined. In [19], Moalla, Damak and Jeribi extend the obtained results into a large class of operators and describe many essential spectra of $\mathcal{A}$ and they apply their results to describe the essential spectra of two-group transport operators with general boundary conditions in L_p-spaces. But, to determine the essential spectra of $\mathcal{A}$, they must absolutely know the one of the entry A of the matrix (1.1). In [3], Bátkai, Binding, Dijksma, Hryniv and Langer considered a 2×2 block operator matrix and describe its essential spectrum under the assumption that $\mathcal{D}(A) \subset \mathcal{D}(C)$, that the intersection of the domains of the operators B and D is sufficiently large, and that the domain of the operator matrix is defined by an additional relation of the form $\Gamma_X x = \Gamma_Y y$ between the two components of its elements. In [6] Amar, Jeribi and Krichen considered the case of 3×3 block matrix and they proposed on abstract approach to study some essential spectra of operator in the form (1.1). Recently in [4], Charfi and Jeribi generalized the results of [3]; by using the notion of Browder resolvent set $\rho_b(.)$ given in [21] they are concerned with the investigation of the Rakočević essential spectrum $\sigma_{eap}(.)$ and the Schmoeger essential spectrum $\sigma_{e\delta}(.)$ of $\mathcal{A}$.

In the present paper we extend these results to 3×3 block operator matrices (1.1), where the domain is defined by additional relations of the form $\Gamma_X x = \Gamma_Y y = \Gamma_Z z$ between the three components of its elements by using the notion of Browder resolvent. We focus on the investigation of the closability and the description of the essential spectra. Compared with the papers [15, 19], we can determine the essential spectra of the closure of (1.1) without knowing the essential spectra of the operator A but only that of one of its operator A_1, where $A_1 := A|_{\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)}$.

This paper is divided into four sections. In the next section we give some preliminary results and notations used in the sequel of the paper. In Section 3 we introduce the assumptions $(H1)$-$(H14)$ to be imposed on the entries of the matrix (1.1) and we give a characterization of its essential approximate point spectrum and its essential defect spectrum. In the last section, we apply the obtained results to describe $\sigma_{eap}(.)$ and $\sigma_{e\delta}(.)$ of a class of transport equations acting in the Banach space $X_p \times X_p \times X_p$, $1 \leq p < \infty$, where

$$X_p = L_p([-a, a] \times [-1, 1], dx\, d\xi), a > 0,$$

we will consider the operator

$$\mathcal{A} = T_H + K,$$

where T_H and K are defined by

$$T_H\psi := \begin{pmatrix} -v\dfrac{\partial \psi_1}{\partial x} - \sigma_1(v)\psi_1 & 0 & 0 \\ 0 & -v\dfrac{\partial \psi_2}{\partial x} - \sigma_2(v)\psi_2 & 0 \\ 0 & 0 & -v\dfrac{\partial \psi_3}{\partial x} - \sigma_3(v)\psi_3 \end{pmatrix} \tag{1.2}$$

$$:= \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_Q \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0 & K_{12} & K_{13} \\ K_{21} & K_{22} & 0 \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \tag{1.3}$$

where $\mathcal{K}_{ij},\ i,j = 1,2,3,\ (i,j) \neq (1,1),\ (2,3)$, are bounded linear operators defined by

$$\begin{cases} K_{ij} : X_p \longrightarrow X_p \\ \psi \mapsto \displaystyle\int_{-1}^{1} \kappa_{ij}(x,\xi,\xi')\psi(x,\xi')\, d\xi' \end{cases} \tag{1.4}$$

The kernels $\kappa_{ij} : [-a,a] \times [-1,1] \times [-1,1] \longrightarrow \mathbb{R}$ are assumed to be measurable. The operators $T_i,\ i = 1,2$, are the so-called streaming operators in X_p, defined by

$$\begin{cases} T_i : \mathcal{D}(T_i) \subseteq X_p \longrightarrow X_p \\ \psi \mapsto T_i\psi(x,\xi) = -\xi \dfrac{\partial \psi}{\partial x}(x,\xi) - \sigma_i(\xi)\psi(x,\xi) \\ \\ \mathcal{D}(T_i) = \mathcal{W}_p \end{cases}$$

where $\mathcal{W}_p$ is the partial Sobolev space $\mathcal{W}_p = \left\{\varphi \in X_p \text{ such that } \xi \dfrac{\partial \varphi}{\partial x} \in X_p\right\}$ and T_Q is defined on $\mathcal{D}(T_Q) = \{\varphi \in \mathcal{W}_p \text{ such that } \varphi^i = Q\varphi^o\}$ by

$$T_Q\varphi(x,\xi) = -\xi \frac{\partial \varphi}{\partial x}(x,\xi) - \sigma_3(\xi)\varphi(x,\xi);$$

where $\sigma(.) \in L^\infty(-1,1)$, ψ^0, ψ^i represent the outgoing and the incoming fluxes related by the boundary operator H. The function $\psi(x,\xi)$ represents the number density of gas particles having the position x and the direction cosine of propagation ξ. The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x- direction. The function $\sigma_j(.)$, $j = 1,2$, is a measurable function called the collision frequency.

2. Notations and Preliminaries Results

In this section we recall some definitions and we give some lemmas that we will need in the sequel.

Definition 2.1. *Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.*
(i) F is called a Fredholm perturbation if $T+F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$.
(ii) F is called an upper (resp. lower) semi-Fredholm perturbation if $T + F \in \Phi_+(X,Y)$ (resp. $\Phi_-(X,Y)$) whenever $T \in \Phi_+(X,Y)$ (resp. $\Phi_-(X,Y)$).

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively. The intersections, $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$, are

denoted by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$, $\Phi^b_-(X,Y)$, respectively. If we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by the sets $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$, we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$. These classes of operators were introduced and investigated by Gohberg, Markus and Fel'dman in [9]. Recently, it was shown in [5], that $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ are closed subsets of $\mathcal{L}(X,Y)$.
An operator $T \in \mathcal{L}(X,Y)$ is said to be weakly compact if $T(M)$ is relatively weakly compact in Y for every bounded subset $M \subset X$. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, this family of operators, denoted by $\mathcal{W}(X) := \mathcal{W}(X,X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing that of compact operators on X (see [10]).

Definition 2.2. *Let X be a Banach space. An operator $S \in \mathcal{L}(X)$ is called strictly singular if, for every infinite-dimensional subspace M of X, the restriction of S to M is not a homeomorphism.*

Let $\mathcal{S}(X)$ denote the set of strictly singular operators on X.

Proposition 2.1. [5, Theorem 2.1] *Let X, Y and Z be three Banach spaces.*
(i) If the set $\Phi^b(Y,Z)$ is not empty, then
$F \in \mathcal{F}^b_+(X,Y)$, $T \in \mathcal{L}(Y,Z)$ *imply* $TF \in \mathcal{F}^b_+(X,Z)$.
$F \in \mathcal{F}^b_-(X,Y)$, $T \in \mathcal{L}(Y,Z)$ *imply* $TF \in \mathcal{F}^b_-(X,Z)$.
(ii) If the set $\Phi^b(X,Y)$ is not empty, then
$F \in \mathcal{F}^b_+(Y,Z)$, $T \in \mathcal{L}(X,Y)$ *imply* $FT \in \mathcal{F}^b_+(X,Z)$.
$F \in \mathcal{F}^b_-(Y,Z)$, $T \in \mathcal{L}(X,Y)$ *imply* $FT \in \mathcal{F}^b_-(X,Z)$. $\diamondsuit$

Lemma 2.1. [21, Lemma 1] *Let A be a closed operator in a complex Banach space X with nonempty resolvent set. For λ, $\mu \in \rho_b(A)$, we have the resolvent identity*

$$R_b(A,\lambda) - R_b(A,\mu) = (\lambda-\mu)R_b(A,\lambda)R_b(A,\mu) + R_b(A,\lambda)M_A(\lambda,\mu)R_b(A,\mu)$$

where $M_A(.,.)$ is a finite rank operator with the following expression

$$M_A(\lambda,\mu) := \Big[\,(A-(\lambda+1))\,P_\lambda - (A-(\mu+1))\,P_\mu\Big].$$

Lemma 2.2. [21, Lemma 2] *Let X and Y be two complex Banach space, $B : Y \longrightarrow X$ and $C : X \longrightarrow Y$ linear operators. Then,*
(i) $R_b(A,\mu)B$ is closable for some $\mu \in \rho_b(A)$ if and only if it is closable for all such μ.
(ii) C is A-bounded if and only if $CR_b(A,\mu)$ is bounded for some $($or every$)$ $\mu \in \rho_b(A)$.
(iii) If B and C satisfy the conditions (i) and (ii), respectively, and B is densely defined, then $CR_b(A,\lambda)M_A(\lambda,\mu)R_b(A,\mu)$, $\overline{R_b(A,\lambda)M_A(\lambda,\mu)R_b(A,\mu)B}$ and $\overline{CR_b(A,\lambda)M_A(\lambda,\mu)R_b(A,\mu)B}$ are operators of finite rank for any λ, $\mu \in \rho_b(A)$.$\diamondsuit$

3. The operator $\mathcal{A}_0$ and its closure

Let X, Y, Z and W be Banach spaces. In this paper, we consider the linear operators Γ_X, Γ_Y, Γ_Z acting from X, Y, Z into W, therefore we define in the

Banach space $X \times Y \times Z$ the operator $\mathcal{A}_0$ as follows:

$$\mathcal{A}_0 := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & L \end{pmatrix}$$

$$\mathcal{D}(\mathcal{A}_0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} x \in \mathcal{D}(A) \\ y \in \mathcal{D}(B) \cap \mathcal{D}(E) \\ z \in \mathcal{D}(C) \cap D(F) \cap \mathcal{D}(L) \end{array} , \ \Gamma_X x = \Gamma_Y y = \Gamma_Z z \right\}.$$

In what follows, we will assume that the following conditions hold:

$(H1)$ The operator A is densely defined and closable.
It follows that $\mathcal{D}(A)$, equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ can be completed to a Banach space X_A which coincides with $\mathcal{D}(\overline{A})$, the domain of the closure of A in X.

$(H2)$ $\mathcal{D}(A) \subset \mathcal{D}(\Gamma_X) \subset X_A$ and $\Gamma_X : X_A \longrightarrow W$ is a bounded mapping. Denote by $\overline{\Gamma}_X$ the extension by continuity which is a bounded operator from X_A into W.

$(H3)$ The set $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$ is dense in X and the resolvent set of the restriction $A_1 := A|_{\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)}$ is not empty, i.e. $\rho(A_1) \neq \emptyset$.

Remark 3.1. *It follows from* $(H3)$ *that* A_1 *is a closed operator in the Banach space* X_A *with nonempty resolvent set.*

$(H4)$ $\mathcal{D}(A) \subset \mathcal{D}(D) \subset X_A$ and D is a closable operator from X_A into Y.

$(H5)$ $\mathcal{D}(A) \subset \mathcal{D}(G) \subset X_A$ and G is a closable operator from X_A into Z.

The closed graph theorem and the assumptions $(H5)$, $(H6)$ imply that for $\lambda \in \rho_b(A_1)$ the operators $F_1(\lambda) := DR_{b,}(A_1, \lambda)$ and $F_2(\lambda) := GR_b(A_1, \lambda)$ are bounded from X into Y and X into Z, respectively.

Under the assumptions $(H1)$-$(H3)$, Charfi and Jeribi in [4], have proved the decomposition

$$\mathcal{D}(A) = \mathcal{D}(A_1) \oplus \mathcal{N}(A_\mu)$$

for $\mu \in \rho_b(A_1)$, we denote the inverse of $\Gamma_X|_{\mathcal{N}(A_\mu)}$ by $K_\mu := (\Gamma_X|_{\mathcal{N}(A_\mu)})^{-1}$. We can write

$$K_\mu : \Gamma_X(\mathcal{D}(A)) \longrightarrow \mathcal{N}(A_\mu) \subset \mathcal{D}(A).$$

For $\mu \in \rho_b(A_1)$, and if assumptions $(H1)$-$(H3)$ are satisfied, then

$$A_\mu x = A_{1\mu}(I - K_\mu \Gamma_X)x.$$

$(H6)$ For some (hence for all) $\lambda \in \rho_b(A_1)$, the operator K_λ is bounded on its domain.

Lemma 3.1. [4, Lemma 3.3] *If* $\lambda, \ \mu \in \rho_b(A_1)$, *then*

$$K_\lambda - K_\mu = R_b(A_1, \lambda)\Big[(\lambda - \mu) + M_{A_1}(\lambda, \mu)\Big]K_\mu,$$

where $M_{A_1}(.,.)$ *is the finite rank operator defined by*

$$M_{A_1}(\lambda_1, \lambda_2) := \Big[(A_1 - (\lambda + 1))\, P_\lambda - (A_1 - (\mu + 1))\, P_\mu \Big].$$

If K_λ is closable for at least one $\lambda \in \rho_b(A_1)$, then it is closable for all such λ.

$(H7)$ E is closable and densely defined linear operator. We denote by Y_E the following Banach space:

$$Y_E := \big(\mathcal{D}(E), \|.\|_E\big).$$

$(H8)$ $\mathcal{D}(B) \cap \mathcal{D}(E) \subset \mathcal{D}(\Gamma_Y)$

$$Y_1 = \Big\{ y \in \mathcal{D}(B) \cap \mathcal{D}(E) \text{ such that } \Gamma_Y y \in \Gamma_X(\mathcal{D}(A) \Big\}$$

the set Y_1 is dense in Y and the restriction of Γ_Y to Y_1 is bounded as an operator from Y into W. We denote the extension by continuity of $\Gamma_Y|_{Y_1}$ to Y by $\overline{\Gamma}^0_Y$.
In the following, we denote by $S(\lambda)$ the following operator:

$$S(\lambda) := E + DK_\lambda\Gamma_Y - DR_b(A_1, \lambda)B.$$

For $\lambda \in \rho_b(A_1)$ the operator $S(\lambda)$ is defined on Y_1 and its restriction to $\mathcal{N}(\Gamma_Y) \cap Y_1$ will be denoted by $S_1(\lambda)$, i.e., $S_1(\lambda) := S(\lambda)|_{\mathcal{N}(\Gamma_Y)\cap Y_1}$.

Lemma 3.2. *Let $\lambda,\ \mu \in \rho_b(A_1)$*

$$S_1(\lambda) - S_1(\mu) = (\mu - \lambda) DR_b(A_1, \lambda)R_b(A_1, \mu)B - DR_b(A_1, \lambda)M_{A_1}(\lambda, \mu)R_b(A_1, \mu)B \tag{3.1}$$

Proof. Let $\lambda,\ \mu \in \rho_b(A_1)$.

$$\begin{aligned}
S(\lambda) - S(\mu) &= D(K_\lambda - K_\mu)\Gamma_Y - D\left[R_b(A_1, \lambda) - R_b(A_1, \mu)\right] B \\
&= DR_b(A_1, \lambda)\Big[(\lambda - \mu) + M_{A_1}(\lambda, \mu)\Big] K_\mu\Gamma_Y \\
&- D\Big[(\lambda - \mu)R_b(A_1, \lambda)R_b(A_1, \mu) + R_b(A_1, \lambda)M_{A_1}(\lambda, \mu)R_b(A_1, \mu)\Big] B \\
&= (\lambda - \mu)DR_b(A_1, \lambda)\Big[K_\mu\Gamma_Y - R_b(A_1, \mu)B\Big] \\
&- \Big[DR_b(A_1, \lambda)M_{A_1}(\lambda, \mu)\Big]\Big[- K_\mu\Gamma_Y + R_b(A_1, \mu)B\Big].
\end{aligned}$$

For $y \in \mathcal{D}(S_1(\mu))$, we have $\Gamma_Y y = 0$ and the relation (3.1) holds. □

Remark 3.2. (i) *By assumptions $(H4)$ and $(H5)$, the operator $F_1(\mu)R_b(A_1, \mu)B$ is bounded on its domain. On the other hand $M_{A_1}(\lambda, \mu)$ is a finite rank operator, so if $S_1(\mu)$ is closed for some $\mu \in \rho_b(A_1)$ then it is closed for all such μ.*
(ii) *If the operators A and E generate C_0-semi groups and the operators D and B are bounded using* [15, Remark 3.1]*, there exists $\mu \in \mathbb{C}$ such that $\mu \in \rho(A_1) \cap \rho(S_1(\mu))$, this implies that $\mu \in \rho_b(A_1) \cap \rho_b(S_1(\mu))$.*

$(H9)$ The operator $S_1(\lambda)$ is closed, densely defined in $\mathcal{N}(\Gamma_Y) \cap Y_1$ with a non-empty resolvent set, i.e., $\rho(S_1(\lambda)) \neq \emptyset$.
$(H10)$ The operator H satisfies that $\mathcal{D}(B) \subset \mathcal{D}(H) \subset Y_E$ and for some (hence for all) $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\lambda))$ the operator $\big[H + GK_\lambda\Gamma_Y - GR_b(A_1, \lambda)B\big] R_b(S_1(\lambda), \lambda)$ is bounded. Set $\Psi(\lambda) := H + GK_\lambda\Gamma_Y - GR_b(A_1, \lambda)B$ and denote by

$$F_3(\lambda) := \Psi(\lambda)R_b(S_1(\lambda), \lambda).$$

$(H11)$ The operator B (resp. C) is densely defined and for some (hence for all) $\lambda \in \rho_b(A_1)$ the operator $R_{b,}(A_1,\lambda)B$ (resp. $R_b(A_1,\lambda)C$) is bounded on its domain. We will denote by:

$$G_1(\lambda) := -K_\lambda \Gamma_Y + R_b(A_1,\lambda)B$$

and
$$G_2(\lambda) := R_b(A_1,\lambda)C.$$

Lemma 3.3. *If the operator $\Psi(\mu)$ is closable for some $\mu \in \rho_b(A_1)$, then it is closable for all such μ and for all $\lambda \in \rho_b(A_1)$ its closure satisfy:*

$$\begin{aligned}&\overline{\Psi(\lambda)} - \overline{\Psi(\mu)} \\ &\quad = (\lambda-\mu)GR_b(A_1,\lambda)\Big[\overline{K}_\lambda \overline{\Gamma}_Y^0 - \overline{R_b(A_1,\mu)B}\Big] - \Big[GR_b(A_1,\lambda)M_{A_1}(\lambda,\mu)\Big]\overline{G_1(\mu)}.\end{aligned}$$

Proof. Let $\lambda,\ \mu \in \rho_b(A_1)$.

$$\begin{aligned}&\Psi(\lambda) - \Psi(\mu) \\ &= G(K_\lambda - K_\mu)\Gamma_Y - G\Big[R_b(A_1,\lambda) - (R_b(A_1,\mu)\Big]B \\ &= GR_b(A_1,\lambda)\Big[(\lambda-\mu) + M_{A_1}(\lambda,\mu)\Big]K_\mu\Gamma_Y \\ &- G\Big[(\lambda-\mu)R_b(A_1,\lambda)R_b(A_1,\mu) + R_b(A_1,\lambda)M_{A_1}(\lambda,\mu)R_b(A_1,\mu)\Big]B \\ &= (\lambda-\mu)GR_b(A_1,\lambda)\Big[K_\mu\Gamma_Y - R_b(A_1,\mu)B\Big] - \Big[GR_b(A_1,\lambda)M_{A_1}(\lambda,\mu)\Big]G_1(\mu),\end{aligned}$$

here Γ_Y is bounded on Y_1 by assumption $(H8)$. From $(H6)$, $(H11)$ and $(H5)$ it follows that the operators K_λ, $R_b(A_1,\mu)B$ and $GR_b(A_1,\mu)$ are bounded. Then

$$\begin{aligned}&\overline{\Psi(\lambda)} - \overline{\Psi(\mu)} \\ &\quad = (\lambda-\mu)GR_b(A_1,\lambda)\Big[\overline{K}_\lambda \overline{\Gamma}_Y^0 - \overline{R_b(A_1,\mu)B}\Big] - \Big[GR_b(A_1,\lambda)M_{A_1}(\lambda,\mu)\Big]\overline{G_1(\mu)}\end{aligned}$$

□

Lemma 3.4. *For some $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\lambda))$ and under the assumptions $(H8)$ and $(H9)$, we have $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\lambda))$ the following decomposition holds:*

$$Y_1 = \mathcal{D}(S_1(\lambda)) \oplus \mathcal{N}(S_\lambda(\lambda))$$

where $S_\lambda(\lambda) = (S(\lambda)-\lambda)\,(I - P'_\lambda) + P'_\lambda$ and P'_λ is the finite rank Riesz projection of $S(\lambda)$ corresponding to λ.

Proof. Let $\lambda \in \rho_b(A_1)\cap\rho_b(S_1(\lambda))$. The operator $S_{1\lambda}(\lambda)$ is invertible, so $\mathcal{N}(S_\lambda(\lambda)) = \{0\}$ and we get

$$\mathcal{D}(S_1(\lambda)) \cap \mathcal{N}(S_\lambda(\lambda)) = \{0\}.$$

Now, for any $f \in Y_1$, we set $g = R_b(S_1(\lambda),\lambda)S_\lambda(\lambda)f \in \mathcal{D}(S_1(\lambda))$. We can easily see that $f - g \in \mathcal{N}(S_\lambda(\lambda))$ and $f = g + f - g \ \in \mathcal{D}(S_1(\lambda)) + \mathcal{N}(S_\lambda(\lambda))$. □

For $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\lambda))$, we define the inverse of Γ_Y by:

$$J_\lambda := (\Gamma_Y|_{\mathcal{N}(S_\lambda(\lambda))})^{-1} : \Gamma_Y(Y_1) \longrightarrow \mathcal{N}(S_\lambda(\lambda) \subset Y_1.$$

In other words $J_\lambda w = y$ means that $y \in \mathcal{D}(S_1(\lambda)), S_\lambda(\lambda)y = 0$ and $\Gamma_Y y = w$. Assume that for some $\mu \in \rho_b(A_1)\cap\rho_b(S_1(\lambda))$, J_μ is bounded from $\Gamma_Y(Y_1)$ into Y and its extension by continuity to $\overline{\Gamma_Y(Y_1)}$ is denoted by $\overline{J}_\mu$.

Lemma 3.5. *If $\lambda \in \rho_b(A_1)\cap\rho_b(S_1(\lambda))$ and $\mu \in \rho_b(A_1)\cap\rho_b(S_1(\mu))$, then*

$$\overline{J}_\lambda - \overline{J}_\mu = R_b(S_1(\lambda),\lambda)\Big[(\lambda-\mu)+\mathbb{U}(\lambda,\mu)+\mathbb{V}(\lambda,\mu)\Big]\overline{J}_\mu.$$

where we define the finite rank operator $\mathbb{U}(.,.)$ as

$$\mathbb{U}(\lambda,\mu) := \Big[S_1(\lambda)-(\lambda+1)\Big]P'_\lambda - \Big[S_1(\mu)-(\mu+1)\Big]P'_\mu,$$

and the bounded operator $\mathbb{V}(.,.)$ as

$$\mathbb{V}(\lambda,\mu) = (\lambda-\mu)\Big[F_1(\lambda)\overline{R_b(A_1,\mu)B}\Big] + F_1(\lambda)\overline{M_{A_1}(\lambda,\mu)R_b(A_1,\mu)B}.$$

Proof. Let $w \in \Gamma_Y(Y_1)$. Set $y = y_1 - y_2$ such that $y_1 = J_\lambda w$ and $y_2 = J_\mu w$. Then, we have

$$\begin{aligned} S_{1\lambda}(\lambda)y &= -S_{1\lambda}(\lambda)y_2 \\ &= \Big[-S_1(\lambda)+\lambda+\Big(S_1(\lambda)-(\lambda+1)\Big)P'_\lambda\Big]y_2. \end{aligned}$$

Using Lemmas 2.1 and 2.2, we infer that

$$\begin{aligned} S_{1\lambda}(\lambda)y &= \Big[-S_1(\mu)+\lambda+(\lambda-\mu)F_1(\lambda)R_b(A_1,\mu)B \\ &- F_1(\lambda)M_{A_1}(\lambda,\mu)R_b(A_1,\mu)B+\Big(S_1(\lambda)-(\lambda+1)\Big)P'_\lambda\Big]y_2. \end{aligned}$$

On the other hand, $S_{1\mu}(\mu)y_2 = 0$, then

$$S_1(\mu)y_2 = \Big[\mu+\Big(S_1(\mu)-(\mu+1)\Big)P'_\mu\Big]y_2.$$

A short computation, shows that:

$$\begin{aligned} S_{1\lambda}(\lambda)y &= \Big[(\lambda-\mu)+(\lambda-\mu)F_1(\lambda)R_b(A_1,\mu)B \\ &- F_1(\lambda)M_{A_1}(\lambda,\mu)R_b(A_1,\mu)B \\ &+ \Big(S_1(\lambda)-(\lambda+1)\Big)P'_\lambda-\Big(S_1(\mu)-(\mu+1)\Big)P'_\mu\Big]y_2. \end{aligned}$$

Since $y \in \mathcal{D}(S_1(\lambda))$, then this allow us to conclude that:

$$\begin{aligned} J_\lambda - J_\mu &= R_b(S_1(\lambda),\lambda)\Big[(\lambda-\mu)+(\lambda-\mu)F_1(\lambda)R_b(A_1,\mu)B \\ &- F_1(\lambda)M_{A_1}(\lambda,\mu)R_b(A_1,\mu)B \\ &+ \Big(S_1(\lambda)-(\lambda+1)\Big)P'_\lambda-\Big(S_1(\mu)-(\mu+1)\Big)P'_\mu\Big]. \end{aligned}$$

From the above expression of $J_\lambda - J_\mu$, we get:

$$J_\lambda = R_b(S_1(\lambda),\lambda)S_{1,\mu}(\mu)J_\mu = S_{1,\mu}(\mu)R_b(S_1(\lambda),\lambda)J_\mu.$$

Since $S_{1\mu}(\mu)R_b(S_1(\lambda),\lambda)$ is bounded and boundedly invertible, then J_λ is closable for each λ if J_μ is too and its closure satisfy:

$$\overline{J}_\lambda - \overline{J}_\mu = R_b(S_1(\lambda),\lambda)\Big[(\lambda-\mu) + \mathbb{U}(\lambda,\mu) + \mathbb{V}(\lambda,\mu)\Big]\overline{J}_\mu$$

where the finite rank operator $\mathbb{U}(\lambda,\mu)$ and the bounded operator $\mathbb{V}(\lambda,\mu)$ are given by:

$$\mathbb{U}(\lambda,\mu) := \Big[S_1(\lambda) - (\lambda+1)\Big]P'_\lambda - \Big[S_1(\mu) - (\mu+1)\Big]P'_\mu,$$

and

$$\mathbb{V}(\lambda,\mu) = (\lambda-\mu)\Big[F_1(\lambda)\overline{R_b(A_1,\mu)B}\Big] + F_1(\lambda)\overline{M_{A_1}(\lambda,\mu)R_b(A_1,\mu)B}. \qquad \square$$

$(H11)$ L is densely defined and closed with non-empty resolvent set, i.e., $\rho(L)\neq\emptyset$.

$(H12)$ $\mathcal{D}(C)\cap\mathcal{D}(F)\cap\mathcal{D}(L)\subset\mathcal{D}(\Gamma_Z)$, the set

$$Z_1 := \Big\{z\in\mathcal{D}(C)\cap\mathcal{D}(F)\cap\mathcal{D}(L) \text{ such that } \Gamma_Z z\in\Gamma_Y(Y_1)\Big\}$$

is dense in Z and the restriction of Γ_Z to Z_1 is bounded as an operator from Z into W. Denote the extension by continuity of $\Gamma_Z|_{Z_1}$ to Z by $\overline{\Gamma}^0_Z$.

$(H13)$ For some (and hence for all) $\lambda\in\rho_b(A_1)$, the operator $F - DR_b(A_1,\lambda)C$ is closable and its closure $\overline{F - DR_b(A_1,\lambda)C}$ is bounded and for $\lambda\in\rho_b(A_1)\cap\rho_b(S_1(\lambda))$, set

$$G_3(\lambda) := -J_\lambda\Gamma_Z + R_b(S_1(\lambda),\lambda)(F - F_1(\lambda)C).$$

$(H14)$ For some (hence for all) $\lambda\in\rho_b(A_1)\cap\rho_b(S_1(\lambda))$, the operator

$$S_2(\lambda) = L - F_2(\lambda)C + \Psi(\lambda)\Big[J_\lambda\Gamma_Z - R_b(S_1(\lambda),\lambda)(F - F_1(\lambda)C)\Big]$$

is closable.

Lemma 3.6. *If for some $\mu\in\rho_b(A_1)\cap\rho_b(S_1(\mu))$ the operator $S_2(\mu)$ is closable, then it is closable for all such μ.* $\diamondsuit$

Proof. Let $\mu \in \rho_b(A_1) \cap \rho_b(S_1(\mu))$ and $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\lambda))$. Using the resolvent identity we find

$$\begin{aligned}
&S_2(\mu) - S_2(\lambda)\\
&\quad= \big[F_2(\lambda) - F_2(\mu)\big]C + \big[\Psi(\lambda)G_3(\lambda) - \Psi(\mu)G_3(\mu)\big]\\
&\quad= (\lambda-\mu)F_2(\lambda)G_2(\mu) + R_b(A,\lambda)M_A(\lambda,\mu)G_2(\mu)\\
&\qquad+\Psi(\lambda)G_3(\mu) - \Psi(\mu)G_3(\mu) + (\lambda-\mu)F_2(\lambda)[K_\mu\Gamma_Y - R_b(A_1,\mu)]G_3(\mu)\\
&\qquad+F_2(\lambda)M_A(\lambda,\mu)G_1(\mu)\\
&\quad= (\lambda-\mu)\Big(F_2(\lambda)G_2(\mu) + F_2(\lambda)[K_\mu\Gamma_Y - R_b(A_1,\mu)]G_3(\mu)\Big)\\
&\qquad+\Big(F_3(\lambda)S_{1\lambda}(\lambda) - F_3(\mu)S_{1\mu}(\mu)\Big)G_3(\mu) + R_b(A,\lambda)M_A(\lambda,\mu)G_2(\mu)\\
&\qquad+F_2(\lambda)M_A(\lambda,\mu)G_1(\mu)
\end{aligned}$$

Since the operators F_i, $i = 1,\ 2,\ 3$ are bounded everywhere and the operators G_i, $i = 1,\ 2,\ 3$ are bounded on its domains and by assumptions $(H13)$ the operator $S_2(\lambda)$ is closed and the closure does not depend on the choice of μ. $\square$

Denote the closure of $S_2(\mu)$ by $\overline{S}_2(\mu)$. Then we have

$$\begin{aligned}
\overline{S}_2(\mu) - \overline{S}_2(\lambda) &= (\lambda-\mu)\Big(F_2(\lambda)\overline{G}_2(\mu) + F_2(\lambda)[\overline{K}_\lambda\overline{\Gamma}^0_Y - \overline{R_b(A_1,\mu)}]\overline{G}_3(\mu)\Big)\\
&+ \Big(F_3(\lambda)S_{1\lambda}(\lambda) - F_3(\mu)S_{1\mu}(\mu)\Big)\overline{G}_3(\mu) + R_b(A,\lambda)\overline{M_A(\lambda,\mu)G_2}(\mu)\\
&\quad+F_2(\lambda)\overline{M_A(\lambda,\mu)G_1(\mu)}
\end{aligned}$$

Lemma 3.7. *For some $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\lambda))$ and $y \in Y_1$, we have*

$$S_\lambda(\lambda)y = S_{1,\lambda}(\lambda)(I - J_\lambda\Gamma_Y)y,$$

where the operator $I - J_\lambda\Gamma_Y$ is the projection from Y_1 on $\mathcal{D}(S_{1\lambda}(\lambda))$ parallel to $\mathcal{N}(S_\lambda(\lambda))$.

Proof. Let $y \in Y_1$ then we have

$$y = (I - J_\mu\Gamma_Y)y + J_\mu\Gamma_Y y.$$

The first summand belongs to $\mathcal{D}(S_1(\mu))$ because $y_1 = (I - J_\mu\Gamma_Y)y \in \mathcal{D}(S_1(\mu))$ and $y_2 = J_\mu\Gamma_Y y \in \mathcal{N}(S(\mu)_\mu)$, then

$$\begin{aligned}
(S(\mu)_\mu)y &= (S_1(\mu)_\mu)y_1\\
&= (S_1(\mu)_\mu)(y - y_2)\\
&= (S_1(\mu)_\mu)(I - J_\mu\Gamma_Y)y.
\end{aligned}$$ $\square$

In the following we use these assumptions to show the closeness of the operator $\mathcal{A}_0$ and to describe the closure. The main idea is, as in the 2×2 case, a factorization of the 3×3 matrix with a diagonal matrix of Schur complements in the middle and invertible factors to the right and to the left (see for example[3, 4, 19, 27]). In the following we consider the operators $\widehat{G}_i(\mu) = \overline{G}_i(\mu),\ i = 1,\ 2, 3$.

Theorem 3.1. *Under assumptions $(H1)$-$(H13)$, the operator $\mathcal{A}_0$ is closable if and only if $S_2(\mu)$ is closable for some $\mu \in \rho_b(A_1) \cap \rho_b(S_1(\mu))$. In this case the closure $\mathcal{A}$ of $\mathcal{A}_0$ is given by*

$$\mathcal{A} = \mu\mathbb{I} + \mathcal{G}_1(\mu)\begin{pmatrix} A_{1\mu} & 0 & 0 \\ 0 & S_{1\mu}(\mu) & 0 \\ 0 & 0 & \overline{S}_2(\mu)-\mu \end{pmatrix}\mathcal{G}_2(\mu) + \mathbb{N}(\mu)$$

where $\mathcal{G}_1(\mu) := \begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix}$, $\mathcal{G}_2(\mu) = \begin{pmatrix} I & \widehat{G}_1(\mu) & \widehat{G}_2(\mu) \\ 0 & I & \widehat{G}_3(\mu) \\ 0 & 0 & I \end{pmatrix}$

and $\mathbb{N}(\mu) = \begin{pmatrix} \big[A-(\mu+1)\big]P_\mu & 0 & 0 \\ 0 & \big(S_1(\mu)-(\mu+1)\big)P'_\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}$

or, spelled out,

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} I & -\widehat{G}_1(\mu) & \widehat{G}_1(\mu)\widehat{G}_3(\mu)-\widehat{G}_2(\mu) \\ 0 & I & -\widehat{G}_3(\mu) \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{array}{c} x \in \mathcal{D}(A_1) \\ y \in Y_1 \cap \mathcal{N}(\Gamma(Y)) \\ z \in Y_2 \end{array} \right\}.$$

$$\begin{aligned} &\mathcal{A}\begin{pmatrix} x-\widehat{G}_1(\mu)y+(\widehat{G}_1(\mu)\widehat{G}_3(\mu)-\widehat{G}_2(\mu))z \\ y-\widehat{G}_3(\mu)z \\ z \end{pmatrix} \\ &\qquad = \begin{pmatrix} A_{1\mu}x-\mu\widehat{G}_1(\mu)y+\mu(\widehat{G}_1(\mu)\widehat{G}_3(\mu)-\widehat{G}_2(\mu))z \\ Dx+S_{1\mu}(\mu)y-\mu\widehat{G}_3(\mu)z \\ Gx+\Psi(\mu)y+\overline{S}_2(\mu)z \end{pmatrix}. \end{aligned}$$

Proof. Let $\mu \in \rho_b(A_1)\cap\rho_b(S_1(\mu))$ the lower-upper factorization sense

$$\begin{aligned} \mathcal{A}_0 = \mu\mathbb{I} + &\begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix}\begin{pmatrix} A_{1\mu} & 0 & 0 \\ 0 & S_{1\mu}(\mu) & 0 \\ 0 & 0 & S_2(\mu)-\mu \end{pmatrix}\begin{pmatrix} I & \widehat{G}_1(\mu) & \widehat{G}_2(\mu) \\ 0 & I & \widehat{G}_3(\mu) \\ 0 & 0 & I \end{pmatrix} \\ &+\begin{pmatrix} [A-(\mu+1)]P_\mu & 0 & 0 \\ 0 & (S_1(\mu)-(\mu+1))P'_\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The external operators $\mathcal{G}_1(\mu)$ and $\mathcal{G}_2(\mu)$ are boundedly invertible and

$$\begin{pmatrix} [A-(\mu+1)]P_\mu & 0 & 0 \\ 0 & (S_1(\mu)-(\mu+1))P'_\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a finite rank operator, hence $\mathcal{A}_0-\mu$ is closable if and only if $S_2(\mu)$ is closable. □

4. Rakočević and Schmoeger essential spectra of $\mathcal{A}$

Having obtained the closure $\mathcal{A}$ of the operator $\mathcal{L}_0$, in this section we discuss its essential spectra. As a first step we prove the following stability lemma.

Lemma 4.1. *(i) If $F_1(\mu) \in \mathcal{F}_+^b(X,Y)$, $F_2(\mu) \in \mathcal{F}_+^b(X,Z)$ and $F_3(\mu) \in \mathcal{F}_+^b(Y,Z)$ then $\sigma_{eap}(S_1(\mu))$ and $\sigma_{eap}(\overline{S}_2(\mu))$ does not depend on the choice of μ.*

(ii) If $F_1(\mu) \in \mathcal{F}_-^b(X,Y)$, $F_2(\mu) \in \mathcal{F}_-^b(X,Z)$ *and* $F_3(\mu) \in \mathcal{F}_-^b(Y,Z)$ *then* $\sigma_{e\delta}(\overline{S}_2(\mu))$ *and* $\sigma_{e\delta}(S_1(\mu))$ *does not depend on the choice of* μ.

Proof. (i) Let $(\lambda,\mu) \in (\rho_b(A_1))^2$, using Eq. (3.1) and Proposition 2.1(i), we will have $\sigma_{eap}(S_1(\mu)) = \sigma_{eap}(S_1(\lambda))$. This implies that $\sigma_{eap}(S_1(\mu))$ does not depend on μ. Now, let $\lambda \in \rho_b(A_1) \cap \rho_b S_1(\lambda)$ then from Proposition (i), we deduce that the difference $\overline{S}_2(\mu) - \overline{S}_2(\lambda) \in \mathcal{F}_+^b(Z)$. Hence by [14, Remark 3.3], we infer that $\sigma_{eap}(\overline{S}_2(\mu))$ does not depend on the choice of μ.
(ii) This assertion can be proved in a similar way as (i). □

We will denote for $\mu \in \rho_b(A) \cap \rho_b(S_1(\mu))$ by $\mathbb{Q}(\mu)$ the operator the following

$$\mathbb{Q}(\mu) := \begin{pmatrix} 0 & \widehat{G}_1(\mu) & \widehat{G}_2(\mu) \\ F_1(\mu) & F_1(\mu)\widehat{G}_1(\mu) & F_1(\mu)\widehat{G}_2(\mu) + \widehat{G}_3(\mu) \\ F_2(\mu) & F_2(\mu)\widehat{G}_1(\mu) + F_3(\mu) & F_2(\mu)\widehat{G}_2(\mu) + F_3(\mu)\widehat{G}_3(\mu) \end{pmatrix}.$$

Theorem 4.1. *Suppose that the assumptions* $(H1)$*-*$(H13)$ *are satisfied.*
(i) *If for some* $\mu \in \rho_b(A_1) \cap \rho_b(S_1(\mu))$, *we have* $F_1(\mu) \in \mathcal{F}_+^b(X,Y)$, $F_2(\mu) \in \mathcal{F}_+^b(X,Z)$,
$F_3(\mu) \in \mathcal{F}_+^b(Y,Z)$ *and* $\mathbb{Q}(\mu) \in \mathcal{F}_+(X,Y,Z)$, *then*

$$\sigma_{eap}(\mathcal{A}) \subseteq \sigma_{eap}(A_1) \cup \sigma_{eap}(S_1(\mu)) \cup \sigma_{eap}(\overline{S}_2(\mu)).$$

If in the addition we suppose that the sets $\Phi_{\mathcal{A}}$, Φ_{A_1}, $\Phi_{S_1(\mu)}$ *and* $\Phi_{\overline{S}_2(\mu)}$ *are connected and the sets* $\rho(\overline{S}_2(\mu))$ *and* $\rho(\mathcal{L})$ *are note empty, then*

$$\sigma_{eap}(\mathcal{A}) = \sigma_{eap}(A_1) \cup \sigma_{eap}(S_1(\mu)) \cup \sigma_{eap}(\overline{S}_2(\mu)).$$

(ii) *If for some* $\mu \in \rho(A_1) \cap \rho(S_1(\mu))$, *we have* $F_1(\mu) \in \mathcal{F}_-^b(X,Y)$, $F_2(\mu) \in \mathcal{F}_-^b(X,Z)$, $F_3(\mu) \in \mathcal{F}_-^b(Y,Z)$ *and* $\mathbb{Q}(\mu) \in \mathcal{F}_-(X,Y,Z)$, *then*

$$\sigma_{e\delta}(\mathcal{A}) \subseteq \sigma_{e\delta}(A_1) \cup \sigma_{e\delta}(S_1(\mu)) \cup \sigma_{e\delta}(\overline{S}_2(\mu)).$$

If in the addition we suppose that the sets $\Phi_{\mathcal{A}}$, Φ_{A_1}, $\Phi_{S_1(\mu)}$ *and* $\Phi_{\overline{S}_2(\mu)}$ *are connected and the sets* $\rho(\overline{S}_2(\mu))$ *and* $\rho(\mathcal{A})$ *are not empty, then*

$$\sigma_{e\delta}(\mathcal{A}) = \sigma_{e\delta}(A_1) \cup \sigma_{e\delta}(S_1(\mu)) \cup \sigma_{e\delta}(\overline{S}_2(\mu)).$$

Proof. Fix $\lambda \in \rho_b(A_1) \cap \rho_b(S_1(\mu))$. Then for $\mu \in \mathbb{C}$ we have

$$\mathcal{A} - \lambda I = \mathcal{G}_1(\mu)\mathbb{V}(\lambda)\mathcal{G}_2(\mu) + (\lambda - \mu)\mathbb{Q}(\mu) + \mathbb{P}(\mu) + \mathbb{N}(\mu),$$

The matrices-operators $\mathbb{V}(\lambda)$ and $\mathbb{P}(\lambda)$ are defined by

$$\mathbb{V}(\lambda) = \begin{pmatrix} A - \lambda & 0 & 0 \\ 0 & S_1(\mu) - \lambda & 0 \\ 0 & 0 & \overline{S_2(\mu)} - \lambda \end{pmatrix},$$

$$\mathbb{P}(\mu) = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix},$$

where :

- $P_{11} = [A_1 - (\mu+1)]P_\mu$,
- $P_{12} = [A_1 - (\mu+1)]P_\mu \widehat{G}_1(\mu)$,
- $P_{13} = [A_1 - (\mu+1)]P_\mu \widehat{G}_2(\mu)$,
- $P_{21} = F_1(\mu)[A_1 - (\mu+1)]P_\mu$,
- $P_{22} = F_1(\mu)[A_1 - (\mu+1)]P_\mu \widehat{G}_1(\mu) + [S_1(\mu) - (\mu+1)]P'_\mu$,
- $F_{23} = F_1(\mu)[A_1 - (\mu+1)]P_\mu \widehat{G}_2(\mu) + [S_1(\mu) - (\mu+1)]P'_\mu F_1(\mu)\widehat{G}_3(\mu)$,
- $P_{31} = F_2(\mu)[A_1 - (\mu+1)]P_\mu$,
- $P_{32} = F_2(\mu)[A_1 - (\mu+1)]P_\mu \widehat{G}_1(\mu) + F_3(\mu)[S_1(\mu) - (\mu+1)]P'_\mu$,
- $P_{33} = F_2(\mu)[A_1 - (\mu+1)]P_\mu \widehat{G}_2(\mu) + F_3(\mu)[S_1(\mu) - (\mu+1)]P'_\mu \widehat{G}_3(\mu)$.

Since $\mathcal{G}_1(\lambda)$ and $\mathcal{G}_2(\lambda)$ are bounded and have bounded inverses, $\mathbb{N}(\lambda)$ and $\mathbb{P}(\lambda)$ are finite rank matrices operators and $\mathbb{Q}(\lambda) \in \mathcal{F}_+(X,Y,Z)$, therefore $(\mathcal{A} - \mu I)$ is an upper semi-Fredholm operator if only if $\mathbb{V}(\mu)$ has this property and

$$i(\mathcal{A} - \mu I) = i(A_1 - \mu I) + i(S_1(\mu) - \mu) + i(\overline{S_2}(\mu) - \mu).$$

This shows that $\sigma_{eap}(\mathcal{A}) = \sigma_{eap}(A_1) \cup \sigma_{eap}(S_1(\mu)) \cup \sigma_{eap}(\overline{S}_2(\mu))$.
Since $\Phi_{\mathcal{A}}$, Φ_{A_1}, $\Phi_{S_1(\mu)}$ and $\Phi_{\overline{S}_2(\mu)}$ are connected and the sets $\rho(\overline{S}_2(\mu))$ and $\rho(\mathcal{A})$ are not empty, then using [4, Proposition 2.3] and [14, Theorem 3.2], we completed the proof of (i).
A same reasoning allows us to reach the result (ii). $\square$

5. Application to transport operators

In this section, we will apply Theorem 4.1 to study the essential spectra of a class of linear operators on L_p-spaces, $1 \le p < \infty$. Let

$$X = Y = Z = X_p = L_p\big([-a,a] \times [-1,1], dx\, d\xi\big), a > 0 \text{ and } p \in [1,\infty).$$

We consider the following three-group transport operator with abstract boundary conditions

$$\mathcal{A} = T_H + K := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where T_H (resp. K) is defined in Eq. (1.2) (resp. Eq. (1.3)). We consider the boundary spaces

$$X_p^o := L_p[\{-a\} \times [-1,0], |\xi| d\xi] \times L_p[\{a\} \times [0,1], |\xi| d\xi] := X_{1,p}^o \times X_{2,p}^o$$

and

$$X_p^i := L_p[\{-a\} \times [0,1], |\xi| d\xi] \times L_p[\{a\} \times [-1,0], |\xi| d\xi] := X_{1,p}^i \times X_{2,p}^i$$

respectively equipped with the norms

$$\|\varphi^o\|_{X_p^o} = \left(\|\varphi_1^o\|^p_{X_{1,p}^o} + \|\varphi_2^o\|^p_{X_{2,p}^o}\right)^{\frac{1}{p}} = \left[\int_{-1}^0 |\varphi(-a,\xi)|^p |\xi|\, d\xi + \int_0^1 |\varphi(a,\xi)|^p |\xi|\, d\xi\right]^{\frac{1}{p}}$$

and

$$\|\varphi^i\|_{X_p^i} = \left(\|\varphi_1^i\|_{X_{1,p}^i}^p + \|\varphi_2^i\|_{X_{2,p}^i}^p\right)^{\frac{1}{p}} = \left[\int_0^1 |\varphi(-a,\xi)|^p|\xi|\,d\xi + \int_{-1}^0 |\varphi(a,\xi)|^p|\xi|\,d\xi\right]^{\frac{1}{p}}.$$

It is well known that any function $\varphi \in \mathcal{W}_p$ has traces on the spacial boundary sets $\{-a\} \times (-1,0)$ and $\{a\} \times (1,0)$ in X_1^o and X_1^i. They are denoted by φ^o and φ^i, and represent the outgoing and the incoming fluxes, respectively (”o” for outgoing and ”i” for incoming). The function $\varphi(x,\xi)$ represents the number density of gas particles with position x and the cosine of direction of propagation ξ; that is, ξ is the cosine of the angle between the velocity vector and the x-direction of the particles. The functions $\sigma_j,\ j = 1,2,3$, which are supposed to be measurable, are called collision frequencies.

In the next, we will define the operator $\mathcal{A}$ on the domain

$$\mathcal{D}(\mathcal{A}) = \left\{\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_2\end{pmatrix} : \ \psi_1 \in \mathcal{W}_1,\ \psi_2 \in \mathcal{D}(T_2),\ \psi_3 \in \mathcal{D}(T_3),\ \psi_1^i = \psi_2^i = \psi_3^i\right\}.$$

and introduce the boundary operators Γ_X, Γ_Y and Γ_Z:

$$\begin{cases}\Gamma_X : X_p \longrightarrow X_p^i\\ \\ \psi_1 \mapsto \psi_1^i\end{cases}, \quad \begin{cases}\Gamma_Y : X_p \longrightarrow X_p^i\\ \\ \psi_2 \longrightarrow \psi_2^i\end{cases} \text{ and } \begin{cases}\Gamma_Z : X_p \longrightarrow X_p^i\\ \\ \psi_3 \mapsto \psi_3^i = \psi_3^i\,.\end{cases}$$

Let $\lambda_j^* \in \mathbb{R}$ be defined by $\lambda_j^* := \liminf_{|\xi|\to 0} \sigma_j(\xi),\ j = 1,2,3$ and let A_1 the operator defined by

$$\begin{cases}A_1 = T_1\\ \\ \mathcal{D}(A_1) = \{\psi_1 \in \mathcal{W}_p \,\text{such that}\, \psi_1^i = 0\}.\end{cases}$$

Let us now consider the resolvent equation for T_1

$$(\lambda - T_1)\psi = \varphi$$

where φ is a given element of X_p and the unknown ψ must be sought in $\mathcal{D}(T_1)$. For $\mathrm{Re}\lambda + \lambda_1^* > 0$, the solution formally given by:

$$\begin{cases}\psi(x,\xi) = \psi(-a,\xi)\, e^{-\frac{(\lambda+\sigma_1(\xi))|a+x|}{|\xi|}}, \quad \xi \in (0,1)\\ \\ \psi(x,\xi) = \psi(a,\xi)\, e^{-\frac{(\lambda+\sigma_1(\xi))|a-x|}{|\xi|}}, \quad \xi \in (-1,0).\end{cases}$$

Thus, the operator K_λ is defined on X_p^i by

$$\begin{cases}K_\lambda : X_p^i \longrightarrow X_p,\ K_\lambda u := \chi_{(0,1)}(\xi) K_\lambda^+ u + \chi_{(-1,0)}(\xi) K_\lambda^- u \quad \text{with}\\ \\ (K_\lambda^+ u)(x,\xi) := u(-a,\xi)\, e^{-\frac{(\lambda+\sigma_1(\xi))|a+x|}{|\xi|}}, \qquad \xi \in (0,1)\\ \\ (K_\lambda^- u)(x,\xi) := u(a,\xi)\, e^{-\frac{(\lambda+\sigma_1(\xi))|a-x|}{|\xi|}}, \qquad \xi \in (-1,0)\end{cases}$$

where $\chi_{(-1,0)}(.)$ and $\chi_{(0,1)}(.)$ denote, respectively, the characteristic functions of the intervals $(-1,0)$ and $(0,1)$. It is easy to see that the operator K_λ is bounded and $\|K_\lambda\| \leq (pRe\lambda + \lambda_1^*)^{-\frac{1}{p}}$. The domain Y_1 is given by

$$Y_1 = \{\psi_2 \in \mathcal{W}_1 \text{ such that } \psi_2^i \in \Gamma_X(\mathcal{W}_1)\}.$$

The operator J_λ is defined on the domain $D(J_\lambda) := \left\{\psi_2^i \text{ such that } \psi_2 \in Y_1\right\}$

Remark 5.1. *To verify that the operator $\mathbb{Q}(\lambda)$ is compact on $X_p \times X_p \times X_p$, $1 < p < \infty$ (resp. weakly compact on $X_1 \times X_1 \times X_1$) we shall prove that the operators $F_i(\lambda)$ and $\widehat{G}_i(\lambda)$, $i = 1, 2, 3$*

Notice that the collision operators K_{ij}, $i, j = 1, 2, 3$, $(i, j) \neq (1, 1)$, $(2, 3)$, defined in Eq. (1.4), act only on the velocity v, so $x \in [-a, a]$ may be seen simply as a parameter. Thus, we consider K_{ij} as a function

$$K_{ij} : x \in [-a, a] \longrightarrow K_{ij}(x) \in \mathcal{L}(L_p([-1, 1]; d\xi).$$

In the following we will make the assumptions:

$$(\mathfrak{H}1) : \begin{cases} -\text{ the function } K_{ij}(.) \text{ is measurable,} \\ -\text{ there exists a compact subset } \mathcal{C} \subset \mathcal{L}(L_p([-1,1]; d\xi)) \text{ such that :} \\ \quad K_{ij}(x) \in \mathcal{C} \text{ a.e. on } [-a, a], \\ -\ K_{ij}(x) \in \mathcal{K}(L_p([-1,1]; d\xi)) \text{ a.e. on } [-a, a] \end{cases}$$

where $\mathcal{K}(L_p([-1, 1]; d\xi))$ is the set of compact operators on $L_p([-1, 1], d\xi)$.

Definition 5.1. *A collision operator K is said to be regular if it satisfies the assumptions $(\mathfrak{H}1)$.*

Lemma 5.1. [20] *Let $\lambda \in \rho(A_1)$.*

(i) If the operators K_{21}, K_{31} are non-negative and their kernels $\dfrac{\kappa_{21}(x, \xi, \xi')}{|\xi'|}$, $\dfrac{\kappa_{31}(x, \xi, \xi')}{|\xi'|}$ define regular operators, then for any $\lambda \in \mathbb{C}$ satisfying $Re\lambda > -\lambda_1^*$, the operators $F_1(\lambda) = K_{21}(A_1 - \lambda)^{-1}$ and $F_2(\lambda) = K_{31}(A_1 - \lambda)^{-1}$ are weakly compact on X_1.

(ii) If K_{31}, K_{21} are non-negative regular operators, then for any $\lambda \in \mathbb{C}$ with $Re\lambda > -\lambda_1^*$ the operators $K_{31}(A_1 - \lambda)^{-1}$, $K_{21}(A_1 - \lambda)^{-1}$, is compact on X_p for $1 < p < \infty$.
(iii) If K_{13}, K_{12} are non-negative regular operators, then for any $\lambda \in \mathbb{C}$ with $Re\lambda > -\lambda_1^*$ the operators $(A_1 - \lambda)^{-1}K_{13}$, $(A_1 - \lambda)^{-1}K_{12}$ are compact on X_p for $1 < p < \infty$ and are weakly compact in X_1.

Theorem 5.1. *If $Q \in \mathcal{S}(X_p)$ positive operator, $K_{12}, K_{21}, K_{13}, K_{31}, K_{33}$ are non-negative regular operators and, in addition, $\dfrac{\kappa_{21}(x, \xi, \xi')}{|\xi'|}$ and $\dfrac{\kappa_{31}(x, \xi, \xi')}{|\xi'|}$ define regular operators on on X_p for $1 < p < \infty$ (resp. on X_1), then*

$$\sigma_{eap}(\mathcal{A}) = \sigma_{e\delta}(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -min(\lambda_1^*, \lambda_2^*, \lambda_3^*)\right\}.$$

Proof. First, it was shown in [4] that

$$\sigma_{eap}(A_1) = \sigma_{e\delta}(A_1) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\lambda_1^*\}. \tag{5.1}$$

Second, if $\lambda \in \rho(T_1)$ using Lemma 5.1, $S_1(\lambda) - T_2 - K_{22}$ is compact on X_p, $1 < p < \infty$ and weakly compact on X_1 this implies that $\sigma_{eap}(S_1(\lambda)) = \sigma_{eap}(A_{22})$. Let $\lambda \in \rho(T_2)$ such that $r_\sigma((\lambda - T_2)^{-1}K_{22}) < 1$, then $\lambda \in \rho(A_{22}) \cap \rho(T_2)$ and we have,

$$(\lambda - T_2 - K_{22})^{-1} - (\lambda - T_2)^{-1} = \sum_{n\geq 1}[(\lambda - T_2)^{-1}K_{22}]^n(\lambda - T_2)^{-1}.$$

Since K_{22} is regular, then it follows from [11, Theorem 2]that the operator

$$(\lambda - T_2 - K_{22})^{-1} - (\lambda - T_2)^{-1}$$

is compact on X_p, $1 < p < \infty$ and weakly compact on X_1. Then the use of Remark 3.3 in [14, Remark 3.3] leads to

$$\sigma_{eap}(S_1(\mu)) = \sigma_{eap}(A_{22}) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\lambda_2^*\}. \tag{5.2}$$

Now, let $\lambda \in \rho(A_1) \cap \rho(S_1(\lambda))$ the operator $S_2(\lambda)$ is given by

$$S_2(\lambda) = (A_{33}) - F_2(\lambda)K_{13} + \Psi(\lambda)\big(J_\lambda\Gamma_Z + (S_1(\lambda) - \lambda)^{-1}F_1(\lambda)K_{13}\big).$$

Since the operator Q is strictly singular on X_p then Γ_Z has also this property. This together with Lemma 5.1 make us conclude that $S_2(\lambda) - A_{33}$ is compact on X_p, $1 < p < \infty$ and weakly compact on X_1, then $\sigma_{eap}(A_{33}) = \sigma_{eap}(S_2(\lambda)$,

$$\sigma_{eap}(S_2(\lambda)) = \sigma_{eap}(A_{33}) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\lambda_3^*\}, \tag{5.3}$$

Applying Theorem 4.1, and using Eqs. (5.1), (5.2) and (5.3) we get

$$\sigma_{eap}(\mathcal{A}) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leq -\min(\lambda_1^*, \lambda_2^*, \lambda_3^*)\}.$$

A same reasoning allows us to show that $\sigma_{e\delta}(\mathcal{A})$. □

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