THE ROOTS AND LINKS IN A CLASS OF M-MATRICES

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This paper is dedicated to Professor Tsuyoshi Ando

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ABSTRACT. In this paper, we discuss exiting roots of sub-kernel transient matrices P associated with a class of $M-$ matrices which are related to generalized ultrametric matrices. Then the results are used to describe completely all links of the class of matrices in terms of structure of the supporting tree.

1. INTRODUCTION

Let I be a finite set and $|I| = n$. $U = (U_{ij}, i, j \in I)$ is *ultrametric matrix* if it is symmetric, nonnegative and satisfies the ultrametric inequality

$$U_{ij} \geq \min\{U_{ik}, U_{kj}\} \quad \text{for all} \quad i, j, k \in I.$$

The ultrametric matrices have an important property that if U is nonsingular ultrametric matrix, then the inverse of U is row and column diagonally dominant $M-$ matrix (see [7] and [13]). A construction also was given in [13] to describe all such ultrametric matrices. Later, nonsymmetric ultrametric matrices were independently defined by McDonald, Neumann, Schneider and Tsatsomeros in [11] and Nabben and Varge in [14], i.e., *nested block form(NBF) matrices* and *generalized ultrametric(GU) matrices*. After a suitable permutation, every GU matrix can be put in NBF. They satisfy ultrametric inequality and are described with dyadic trees in [11], [14] and [4]. On the inequality of $M-$ matrices and inverse $M-$ matrices, Ando in [1] presents many nice and excellent inequalities which play an key role in the nonnegative matrix theory. Zhang [15] characterized equality cases in Fisher, Oppenheim and Ando inequalities. For more detail information on inverse $M-$ and $Z-$ matrices, the reader is referred to [6], [8],

[9], [10], [16] and the references in there. In this paper, we follows closely the global frame work and notation on generalized ultrametric matrices supplied by Dellacherie, Martínez and Martín in [4]. Recently, Nabben was motivated by the result of Fiedler in [5] and defined a new class of matrices: $\mathcal{U}$-matrices (see Section 2) which satisfy ultrametric inequality and are related to GU matrices. There is a common characterization in these matrices that if they are nonsingular, then their inverses are column diagonally dominant $M-$matrices. For each $\eta \geq \eta(U) = \max\{(U^{-1})_{ii}, \ i \in I\}$, define matrix $P = E - \eta^{-1}U^{-1}$, where E is the identity matrix. Then P is sub-Markov kernel: $P_{ij} \geq 0$, for all $i, j \in I$ and $\mathbf{1}^t P \leq \mathbf{1}^t$ (entry-wise), where $\mathbf{1}$ is the column vector of all ones. Therefore

$$\eta U = (E - P)^{-1} = \sum_{m \geq 0} P^m$$

and U is proportional to the potential matrix associated to the transient kernel P. Since $P_{ij} > 0$ if and only if $(U^{-1})_{ij} < 0$ for $i \neq j$, the existence of links between different points does not depend on η, while the condition $P_{ii} > 0$ depends on the value of η. Define the potential vector $\mu = \mu_U$ associated to U by $\mu := U^{-1}\mathbf{1}$ and its total mass $\overline{\mu} := \mathbf{1}^t \mu$. Note that the following equivalence holds

$$\mu_i > 0 \Leftrightarrow (U^{-1}\mathbf{1})_i > 0 \Leftrightarrow (P\mathbf{1})_i < 1.$$

Every i satisfying this property is called an *exiting root* of U(or of P). The set of them is denoted by $\mathcal{R} := \mathcal{R}_U$. The *potential vector* $\nu := \nu_U$ associated to U^t is given by $\nu := (U^t)^{-1}\mathbf{1}$ and $\overline{\nu} := \mathbf{1}^t \nu$. Notice that $\overline{\mu} = \overline{\nu}$, since $\mathbf{1}^t \nu = \mathbf{1}^t (U^t)^{-1}\mathbf{1} = \mathbf{1}^t \mu$.

Our main results in this paper are to characterize the following properties (which do not depend on η) " i is a exiting root of P and P^t"; and " link of P, i.e., for a given couple $i \neq j$, whether $P_{ij} > 0$ for $\mathcal{U}$-matrices". These properties and other related problems were completely investigated for symmetric ultrametric matrices and GU matrices in [3] and [4], respectively. In Section 2, we revisit $\mathcal{U}$- matrices by means of dyadic tree and give some preliminary results which are very useful. In Section 3, we describe exiting root of P and P^t with associated trees. In Section 4, we characterize completely the links of P.

2. $\mathcal{U}$ MATRICES

A *tree* $(T, \mathcal{J})$ is a finite unoriented and connected acyclic graph. For $(t, s) \in T \times T$, $t \neq s$, there is a unique path $\text{geod}(t, s)$ of minimum length, which is called the *geodic* between t and s, while $\text{geod}(t, t) = \{t\}$ which is of length 0. Sometime, we use $\text{geod}(t, s)$ to stand for its edge set. Fixed $r \in T$, we call it the *root* of tree T. If $s \in geod(t, r)$, we denote $s \preceq t$, which is a partial order relation on T. For $s, t \in T$, $s \wedge t = \sup\{v, v \in geod(s, r) \cap geod(t, r)\}$ denotes the closest common *ancestor* of s and t. The set of *successors* of t is $s(t) = \{s \in T, \ s \succeq t, \ (s, t) \in \mathcal{J}\}$. Then $I(\mathcal{J}) = \{i \in T, \ s(i) = \emptyset\}$ is the set of *leaves* of the tree T. A tree is said *dyadic* if $|s(t)| = 2$ for $t \notin I(\mathcal{J})$. The successors of t are denoted by t^- and t^+. For $t \in T$, the set $L(t) := \{i \in I(\mathcal{J}), t \in geod(i, r)\}$ completely characterizes t. Hence we can identify t and $L(t)$. In particular, r is identified with $L(r) = I(\mathcal{J})$ and $i \in I(\mathcal{J})$ with the singleton i. Hence we can assume that each vertex of T

is a subset of the set of leaves $I(\mathcal{J})$. The distinction between the roles of L, as $L \in T$ (mean that L is a vertex of tree T) and $L \subseteq I$ (mean that L is regarded as the subset of $I(\mathcal{J})$ corresponding to the vertex of T), will be clear in the context when we use them. By the above notations and definition of GU matrices in [4], The definition of $\mathcal{U}$ matrices in [12] may be restated in the following way

Definition 2.1. $U = (U_{ij} : i, j \in I)$ is a $\mathcal{U}$ matrix if there exists a dyadic tree $(T, \mathcal{J})$ with fixed a root r and a leaf $n \in I$, and nonnegative real vectors $\overrightarrow{\alpha} = (\alpha_t : t \in T)$, $\overrightarrow{\beta} = (\beta_t : t \in T)$ satisfying

(i). $I = I(\mathcal{J})$, $\overrightarrow{\alpha}|_I = \overrightarrow{\beta}|_I$; and $\alpha_t = \alpha_{t \wedge n}$ for $t \in r^+, t \notin I$;

(ii). $\alpha_t \leq \beta_t$ for $t \in T$;

(iii). $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ are $\preceq$- increasing, i.e., $t \preceq s$ implies $\alpha_t \leq \alpha_s$ and $\beta_t \leq \beta_s$;

(iv). $t^+ \in geod(r, n)$ for $t \in geod(r, n)$ and $t \neq n$; and $\alpha_t = \beta_t$ for $t \in geod(r, n)$.

(v). $U_{ij} = \alpha_t$ if $(i, j) \in (t^-, t^+)$ and $U_{ij} = \beta_s$ if $(i, j) \in (t^+, t^-)$, where $t = i \wedge j$ and $s = \max\{i \wedge j, i \wedge n\}$; $U_{ii} = \alpha_i = \beta_i$ for $i \in I$.

We say that $(T, \mathcal{J})$ support U and U is $\mathcal{U}$ associated with tree $(T, \mathcal{J})$.

It is easy to show that this definition is equivalent to Definition 2.1 in [12]. Observe that for each $L \in T$, the matrix $U|_{L \times L}$ is either GU or $\mathcal{U}$ matrix, where the GU matrix consistent with the definition of GU matrix in [4]. The tree supporting it, denoted by $(T|_L, \mathcal{J}_L)$, is the restriction of $(T, \mathcal{J})$ on L and the associated vectors which are the restrictions of $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ on $T|_L$. In other words, $(T|_L, \mathcal{J}_L)$ is the subtree of $(T, \mathcal{J})$ with the root L and the leaves set L. The potential vectors and the exiting roots of $U|_L, U^t|_L$ are denoted, respectively by μ_L, ν_L, $\mathcal{R}_L$, $\mathcal{R}^t{}_L$. The sub-kernel corresponding to $U|_L, U^t|_L$ is denoted by P^L, $(P^t)^L$. If U is nonsingular $\mathcal{U}$ matrix, it can be shown that $U|_L$ is also nonsingular GU or $\mathcal{U}$ matrix by Schur decomposition and induction argument.

We now introduce the following relation $\leq_{\mathcal{J}}$ in the set of leaves I. For $i \neq j$, we say $i <_{\mathcal{J}} j$ if $i \in t^-, j \in t^+$ with $t = i \wedge j$. Assume that $I = \{1, 2, \cdots, n\}$. By permuting I, we can suppose $\leq_{\mathcal{J}}$ is the usual relation $\leq$ on I. Therefore, we will assume that this is standard presentation of $\mathcal{U}$ matrices in this paper. In the other words, Let $U \in \mathcal{U}$ and $I = I^- \cup I^+$. Denote $J := I^-$ and $K := I^+$. Thus

$$U = \begin{pmatrix} U_J & \alpha_I \mathbf{1}_J \mathbf{1}_K^t \\ b_K \mathbf{1}_J^t & U_K \end{pmatrix},$$

where $\alpha_I = \min\{U_{ij} : i, j \in I\}$ and $b_K = U_K e_K$ with $e_K = (0, \cdots, 0, 1)^t$ unit vector, i.e., b_K is the last column of U_K. Note that in here, U_J is GU matrix and U_K is still $\mathcal{U}$ matrix also, which has a similar 2×2 block structure, and its the first diagonal block is a special GU matrix. We begin with the following theorem in which we re-prove some known result in [12].

Theorem 2.2. [12] *If U is nonsingular $\mathcal{U}$ matrix, then*

(i). $\alpha_I \overline{\mu}_J < 1$ *and*

$$U^{-1} = \begin{pmatrix} C & D \\ E & F \end{pmatrix},$$

where

$$C = U_J^{-1} + \frac{\alpha_I}{1-\alpha_I\overline{\mu}_J}\mu_J\nu_J^t, \qquad D = \frac{-\alpha_I}{1-\alpha_I\overline{\mu}_J}\mu_J\nu_K^t,$$

$$E = \frac{-1}{1-\alpha_I\overline{\mu}_J}e_K\nu_J^t, \qquad F = U_K^{-1} + \frac{\alpha_I\overline{\mu}_J}{1-\alpha_I\overline{\mu}_J}e_K\nu_K^t.$$

(ii).

$$\mu_I = \begin{pmatrix} \frac{1-\alpha_I\overline{\mu}_K}{1-\alpha_I\overline{\mu}_J}\mu_J \\ \mu_K - \frac{\overline{\mu}_J(1-\alpha_I\overline{\mu}_K)}{1-\alpha_I\overline{\mu}_J}e_K \end{pmatrix}; \quad \nu_I = \begin{pmatrix} 0 \\ \nu_K \end{pmatrix}.$$

(iii). $\overline{\mu}_I = \overline{\mu}_K$.
(iv). $(\mu_I)_i \geq 0$ *for* $i = 1, 2, \cdots, n-1$.
(v). $(\nu_I)_i = 0$ *for* $i = 1, 2, \cdots, n-1$; *and* $(\nu_I)_n = \overline{\mu}_I = \frac{1}{U_{nn}}$.

Proof. Since U is nonsingular, U_J is nonsingular GU matrix. By Theorem 3.6(i) in [11], $\alpha_J\overline{\mu}_J \leq 1$, where α_J is smallest entry in U_J. Hence $\alpha_I\overline{\mu}_J \leq \alpha_J\overline{\mu}_J \leq 1$. Suppose that $\alpha_I\overline{\mu}_J = 1$, by theorem 3.6(ii) in [11], U_J has a row whose entries are all equal to α_I. Noting that the last row whose entries are equal to U_{nn}, there are two rows which are proportional, which implies U is singular, a contradiction. Therefore $\alpha_I\overline{\mu}_J < 1$. By Schur decomposition and the inverse of matrix formula, it is not difficult to show that the rest of (i) holds. Since $\overline{\mu}_J = \overline{\nu}_J$ and $\overline{\mu}_K = \overline{\nu}_K$,

$$\begin{aligned} C\mathbf{1}_J + D\mathbf{1}_K &= U_J^{-1}\mathbf{1}_J + \frac{\alpha_I}{1-\alpha_I\overline{\mu}_J}\mu_J\nu_J^t\mathbf{1}_J + \frac{-\alpha_I}{1-\alpha_I\overline{\mu}_J}\mu_J\nu_K^t\mathbf{1}_K \\ &= \frac{1-\alpha_I\overline{\mu}_K}{1-\alpha_I\overline{\mu}_J}\mu_J, \end{aligned}$$

$$\begin{aligned} E\mathbf{1}_J + F\mathbf{1}_K &= \frac{-1}{1-\alpha_I\overline{\mu}_J}e_K\nu_J^t\mathbf{1}_J + U_K^{-1}\mathbf{1}_K + \frac{\alpha_I\overline{\mu}_J}{1-\alpha_I\overline{\mu}_J}e_K\nu_K^t\mathbf{1}_K \\ &= \mu_K - \frac{\overline{\mu}_J(1-\alpha_I\overline{\mu}_K)}{1-\alpha_I\overline{\mu}_J}e_K, \end{aligned}$$

$$\begin{aligned} \mathbf{1}_J^t C + \mathbf{1}_K^t E &= \mathbf{1}_J^t U_J^{-1} + \frac{\alpha_I}{1-\alpha_I\overline{\mu}_J}\mathbf{1}_J^t\mu_J\nu_J^t + \mathbf{1}_K^t\frac{-1}{1-\alpha_I\overline{\mu}_J}e_K\nu_J^t \\ &= 0, \end{aligned}$$

$$\begin{aligned} \mathbf{1}_J^t D + \mathbf{1}_K^t F &= \frac{-\alpha_I}{1-\alpha_I\overline{\mu}_J}\mathbf{1}_J^t\mu_J\nu_K^t + \mathbf{1}_K^t U_K^{-1} + \mathbf{1}_K^t\frac{\alpha_I\overline{\mu}_J}{1-\alpha_I\overline{\mu}_J}e_K\nu_K^t \\ &= \nu_K^t. \end{aligned}$$

So (ii) holds. Furthermore,

$$\overline{\mu}_I = \frac{1-\alpha_I\overline{\mu}_K}{1-\alpha_I\overline{\mu}_J}\overline{\mu}_J + \overline{\mu}_K - \frac{\overline{\mu}_J(1-\alpha_I\overline{\mu}_K)}{1-\alpha_I\overline{\mu}_J} = \overline{\mu}_K.$$

Thus (iii) holds. Since $\frac{1}{U_{nn}}e_I^t U = \mathbf{1}^t$, $\nu_I = \frac{1}{U_{nn}}e_I$ which implies $\overline{\mu}_I = \overline{\nu}_I = \frac{1}{U_{nn}}$. By (iii), we have $1 - \alpha_I\overline{\mu}_K = 1 - \alpha_I\overline{\mu}_I \geq 1 - \frac{\alpha_I}{U_{nn}} \geq 0$. Hence it is easy to show that (iv) and (v) hold by using the induction on the dimension of U. $\qquad\square$

3. Exiting roots of P

In order to characterize the exiting roots of P, we introduce some notations and symbols. Let U be a $\mathcal{U}$ matrix with supporting tree $(T, \mathcal{J})$ and fixed a root r and a leaf n. For $i \in I(\mathcal{J})$, denote by $N_i^+ = \{L \in T : L \preceq i, \ \alpha_L = \alpha_i\}$ and $N_i^- = \{L \in T : L \preceq i, \ \beta_L = \beta_i\}$. Now we can construct the set Γ^t: for $L \notin geod(r, n)$, $(L, L^-) \in \Gamma^t$ if and only if there exists a $i \in L^+$ such that $L \in N_i^-$; $(L, L^+) \in \Gamma^t$ if and only if there exists a $i \in L^-$ such that $L \in N_i^+$. For $L \in geod(r, n)$, $(L, L^-) \in \Gamma^t$ and $(L, L^+) \notin \Gamma^t$.

Theorem 3.1. *Let U be nonsingular $\mathcal{U}$. Then*
(i). $\mathcal{R}_I^t = \{n\}$.
(ii). *For $L \in T$, $i \in \mathcal{R}_L^t$ if and only if $geod(i, L) \cap \Gamma^t = \emptyset$.*

Proof. (i) follows from Theorem 2.2 (v). We prove the assertion (ii) by using an induction on the dimension n of U. It is trivial for $n = 1, 2$. Assume that

$$U = \begin{pmatrix} U_J & \alpha_I \mathbf{1}_J \mathbf{1}_K^t \\ b_K \mathbf{1}_J^t & U_K \end{pmatrix}.$$

If $L \subseteq I^-$, then $U|_L = (U_J)|_L$ is GU matrix. By Theorem 3 in [4] and $(r, r^-) \notin geod(i, L)$, the assertion (ii) holds. If $L \subseteq I^+$, then $U|_L = (U_K)|_L$ is still $\mathcal{U}$ matrix. By the hypothesis and $(r, r^+) \notin geod(i, L)$, $i \in \mathcal{R}_L^t$ if and only if $geod(i, L) \cap \Gamma^t = \emptyset$. If $L = I$, then for $i \neq n$, $(i \wedge n, (i \wedge n)^-) \in \Gamma^t$ and $geod(i, L) \cap \Gamma^t \neq \emptyset$; for $i = n$, $geod(i, L) \cap \Gamma^t = \emptyset$. Hence by (i), $L \in T$, $i \in \mathcal{R}_L^t$ if and only if $geod(i, L) \cap \Gamma^t = \emptyset$. $\square$

In order to describe the exiting of P, we construct the set Γ: For each $L \in T$, $(L, L^-) \in \Gamma$, $(L, L^+) \in \Gamma$, if and only if there exists $i \in L^+$, $i \in L^-$, such that $L \in N_i^+$, $L \in N^-$, respectively.

Theorem 3.2. *Let $U = (U_{ij}, i, j \in I)$ be a nonsingular $\mathcal{U}$ matrix. Then*
(i). $n \neq i \in \mathcal{R}$ *if and only if $geod(i, I) \cap \Gamma = \emptyset$.*
(ii). *If $L \in T$, then $n \neq i \in \mathcal{R}_L$ if and only if $geod(i, I) \cap \Gamma = \emptyset$.*

Proof. (i) We prove the assertion by the induction. It is trivial for $n = 1, 2$. We assume that

$$U = \begin{pmatrix} U_J & \alpha_I \mathbf{1}_J \mathbf{1}_K^t \\ b_K \mathbf{1}_J^t & U_K \end{pmatrix},$$

where $\alpha_I = \min\{U_{ij} : i, j \in I\}$, $b_K = U_K e_K$ with $e_K = (0, \cdots, 0, 1)^t$ unit vector, U_J is GU matrix and U_K is $\mathcal{U}$ matrix. By Theorem 2.2(ii), we have

$$\mu_I = \begin{pmatrix} \frac{1 - \alpha_I \overline{\mu}_K}{1 - \alpha_I \overline{\mu}_J} \mu_J \\ \mu_K - \frac{\overline{\mu}_J (1 - \alpha_I \overline{\mu}_K)}{1 - \alpha_I \overline{\mu}_J} e_K \end{pmatrix}.$$

Now we consider the following two cases.

Case 1: $i \in I^-$. Then $i \in \mathcal{R}$ if and only if $i \in \mathcal{R}_J$ and $1 - \alpha_I \overline{\mu}_K > 0$ by Theorem 2.2 (ii). Note that $\overline{\mu}_I = \overline{\mu}_K$ from Theorem 2.2 (iii). $1 - \alpha_I \overline{\mu}_K = 0$ if and only if $\alpha_I = \alpha_n$ if and only if $(I, I^-) \in \Gamma$ because it follows from the definition of Γ, and if there exists an $n \neq q \in I^+$ such that $I \in N_q^+$ which implies $\alpha_I = \alpha_q$ by definition of 2.1(i). Therefore each entries of the $q-$th row is α_q which implies

that U is singular. Hence $1 - \alpha_I \overline{\mu}_K > 0$ if and only if $(I, I^-) \notin \Gamma$. By the induction hypothesis, $n \neq i \in \mathcal{R}$ if and only if $geod(i, I) \cap \Gamma = \emptyset$.

Case 2: $i \in I^+$. Suppose that $(I, I^+) \in \Gamma$. Then there exists $j_0 \in I^-$ such that $I \in N_{j_0}^-$ which implies $\beta_I = \beta_{j_0}$. Hence $\beta_{j_0} = \alpha_I$ follows from $\alpha_I = \beta_I$. So U is singular, a contradiction. Therefore $(I, I^+) \notin \Gamma$. By Theorem 2.2 (ii), $n \neq i \in \mathcal{R}$ if and only if $n \neq i \in \mathcal{R}_K$. Because U_K is $\mathcal{U}$ matrix, $n \neq i \in \mathcal{R}$ if and only if $geod(i, I^+) \cap \Gamma = \emptyset$ by the induction hypothesis, so if and only $geod(i, I) \cap \Gamma = \emptyset$.

(ii) Since $U|_L$ is GU matrix or $\mathcal{U}$ matrix, the assertion follows from Theorem 3 in [4] or (i). $\square$

Theorem 3.3. *Let U be a $\mathcal{U}$ matrix. Then*

(i). *U^{-1} is row diagonal dominant $M-$ matrix if and only if*

$$\overline{\mu}_n \geq \sum_{L \in geod(r,n), L \neq n} \frac{\overline{\mu}_{L^-}(1 - \alpha_L \overline{\mu}_{L^+})}{1 - \alpha_L \overline{\mu}_{L^-}}. \tag{3.1}$$

(ii). *$n \in \mathcal{R}$ if and only if (3.1) becomes strict inequality.*

Proof. From Theorem 2.2 (ii), the sum of $n-$th row of U^{-1} is

$$(\mu_K)_n - \frac{\overline{\mu}_{I^-}(1 - \alpha_I \overline{\mu}_{I^+})}{1 - \alpha_I \overline{\mu}_{I^-}},$$

where $(\mu_K)_n$ is the last component of μ_K. Hence by the induction hypothesis, the sum of $n-$th row of U^{-1} is

$$\overline{\mu}_n - \sum_{L \in geod(r,n), L \neq n} \frac{\overline{\mu}_{L^-}(1 - \alpha_L \overline{\mu}_{L^+})}{1 - \alpha_L \overline{\mu}_{L^-}}.$$

Therefore (i) follows from Theorem 2.2 (iii). (ii) is just a consequence of (i) and the definition of exiting. $\square$

Lemma 3.4. *Let $U = (U_{ij}, i, j \in I)$ a nonsingular GU matrix. Then $U_{ii}\overline{\mu}_I \geq 1$ for all $i \in I$. Moreover, $\max\{U_{ii}, i \in I\}\overline{\mu}_I = 1$ if and only if there exist a column whose entries are equal to $\max\{U_{ii}, i \in I\}$.*

Proof. Since U is a GU matrix and $\mathbf{1} = UU^{-1}\mathbf{1} = U\mu$, $1 = \sum_{j=1}^{n} U_{ij}(\mu_I)_j \leq \sum_{j=1}^{n} U_{ii}(\mu_I)_j$. Hence we have $U_{ii}\overline{\mu}_I \geq 1$ for $i \in I$. Let $\max\{U_{ii}\} = U_{qq}$ and suppose that $U_{qq}\overline{\mu}_I = 1$. Then $1 = \sum_{j=1}^{n} U_{ij}(\mu_I)_j \leq \sum_{j=1}^{n} U_{ii}(\mu_I)_j \leq U_{qq}\overline{\mu}_I = 1$. Hence $\sum_{j=1}^{n}(U_{ij} - U_{qq})\mu_j = 0$ for $i \in I$, which yields the result. Conversely, let $\max\{U_{ii}\} = U_{qq}$ and $e_q = (0, \cdots, \frac{1}{U_{qq}}, \cdots, 0)^t$. Then $Ue_q = \mathbf{1}$ Hence $\mu = e_q$ and $U_{qq}\overline{\mu}_I = 1$. $\square$

Theorem 3.5. *Let U be a nonsingular $\mathcal{U}$ matrix. If there exists i with $i \neq n$ such that $U_{ii} < U_{nn}$. Then U is not a row diagonally dominant matrix, neither n is a root of P.*

Proof. We prove the assertion by the induction on the dimension of matrix U. It is easy to show that the assertion holds for order $n = 1, 2$. Now we assume that

$$U = \begin{pmatrix} U_J & \alpha_I \mathbf{1}_J \mathbf{1}_K^t \\ b_K \mathbf{1}_J^t & U_K \end{pmatrix}.$$

If there exists a $i \in K$ such that $(U)_{ii} < U_{nn}$, then by the induction hypothesis, the last component of μ_K is less than 0. So $(\mu_I)_n < 0$ by theorem 2.2(ii). Hence we assume that there exists a $i \in J$ such that $U_{ii} < U_{nn}$. Then by Lemma 3.4, $\overline{\mu}_J \geq \frac{1}{(U_J)_{ii}} = \frac{1}{U_{ii}}$. Hence

$$\sum_{i \in K} (\mu_I)_i = \overline{\mu}_K - \frac{\overline{\mu}_J(1 - \alpha_I \overline{\mu}_K)}{1 - \alpha_I \overline{\mu}_J} = \frac{\overline{\mu}_K - \overline{\mu}_J}{1 - \alpha_I \overline{\mu}_J}$$

$$\leq \frac{1}{1 - \alpha_I \overline{\mu}_J}(\frac{1}{U_{nn}} - \frac{1}{U_{ii}}) < 0,$$

since $\overline{\mu}_K = \overline{\mu}_I = \frac{1}{U_{nn}}$ by Theorem 2.2 (iii) and (v). On the other hand, $(\mu_I)_j \geq 0$ for $i \in K \setminus \{n\}$. Therefore, $(\mu_I)_n < 0$. So the assertion holds. $\square$

Corollary 3.6. *Let U be a nonsingular $\mathcal{U}$ matrix. If there exists a i with $i \neq n$ such that $U_{ii} \leq U_{nn}$ then n is not root of P.*

Proof. It follows from Theorem 3.5 and its proof. $\square$

Lemma 3.7. *Let U be a $\mathcal{U}$ matrix. If $\sum_{j \in I} U_{nj} \leq \sum_{j \in I} U_{ij}$ for $i = 1, 2, \cdots, n-1$. Then U is a row and column diagonally dominant M-matrix and n is an exiting of P.*

Proof. From $U^{-1}U = I_n$, $1 = \sum_{j \in I} \sum_{l \in I} (U^{-1})_{nl} U_{lj} = \sum_{l \in I, l \neq n} \sum_{j \in I} U_{lj}(U^{-1})_{nl} + \sum_{j \in I} U_{nj}(U^{-1})_{nn}$. Since $(U^{-1})_{nl} \leq 0$ for $l \neq n$ and $\sum_{j \in I} U_{nj} \leq \sum_{j \in I} U_{lj}$ for $l \neq n$, $1 \leq \sum_{l \in I, l \neq n} \sum_{j \in I} U_{nj}(U^{-1})_{nl} + \sum_{j \in I} U_{nj}(U^{-1})_{nn} = \sum_{j \in I} U_{nj} \sum_{l \in I}(U^{-1})_{nl}$. Hence $\sum_{l \in I}(U^{-1})_{nl} \geq \frac{1}{\sum_{j \in I} U_{nj}} > 0$. Hence the result follows form theorem 2.2. $\square$

4. Links of P

In this section, we describe completely the links of transient kernel P associated with a class of $\mathcal{U}$ matrices. Firstly we give some lemmas.

Lemma 4.1. *Let U be a nonsingular $\mathcal{U}$ matrix. Then for $i, j \in I^- = J$, $i \neq j$, $(U^{-1})_{ij} < 0$ if and only if $(U_J^{-1})_{ij} < 0$ and $U_{ij} > \alpha_I$.*

Proof. Necessity. By Theorem 2.2, we have

$$(U^{-1})_J = U_J^{-1} + \frac{\alpha_I}{1 - \alpha_I \overline{\mu}_J} \mu_J \nu_J^t = (U_J(\alpha_I))^{-1},$$

where $U_J(\alpha_I) = U_J - \alpha_I \mathbf{1}_J \mathbf{1}_J^t$ is a nonsingular GU matrix. Hence for $i, j \in I^- = J$, $(U^{-1})_{ij} < 0$ implies $(U_J^{-1})_{ij} < 0$. Further, by Theorem 3.6 in [14], $(U^{-1})_{ij} = (U_J(\alpha_I))_{ij}^{-1} < 0$ implies that $(U_J(\alpha_I))_{ij} > 0$. So $U_{ij} > \alpha_I$.

Sufficiency. Note that $\alpha_I < \min\{(U_J)_{ii}\}$ (otherwise every entries of some row rather than n is α_I which yields U is singular.) Note that U_J is a GU matrix. By Theorem 6 in [4], $(U_J^{-1})_{ij} < 0$ implies that $(U^{-1})_{ij} = (U_J(\alpha_I)^{-1})_{ij} < 0$. $\square$

Lemma 4.2. *Let U be a nonsingular $\mathcal{U}$ matrix of order n. Then*
  (i). *for $i \in I^- = J$, $j \in I^+ = K$, $(U^{-1})_{ij} < 0$ if and only if $i \in \mathcal{R}_J$ and $j = n$.*
  (ii). *for $i \in I^+$, $j \in I^-$, $(U^{-1})_{ij} < 0$ if and only if $i = n$ and $j \in \mathcal{R}_J$.*

Proof. (i). For $i \in I^- = J$, $j \in I^+ = K$, by Theorem 2.2 (i), $(U^{-1})_{ij} = (\frac{-\alpha_I}{1-\alpha_I \bar{\mu}_J} \mu_J \nu_K^t)_{ij} < 0$ if and only if $(\mu_J)_i > 0$ and $(\nu_K)_j > 0$ if and only if $i \in \mathcal{R}_J$ and $j = n$.

(ii). For $i \in I^- = J$, $j \in I^+ = K$, by Theorem 2.2, $(U^{-1})_{ij} < 0$ if and only if $(\frac{-1}{1-\alpha_I \bar{\mu}_J} e_K \nu_J^t)_{ij} < 0$ if and only if $i = n$ and $j \in \mathcal{R}_J$. $\qquad \square$

Lemma 4.3. *Let U be a nonsingular $\mathcal{U}$ matrix. Then for $i \in I^+ = K$, $j \in I^+$ and $i \neq j$, $(U^{-1})_{ij} < 0$ if and only if $(U_K^{-1})_{ij} < 0$.*

Proof. The assertion follows from Theorem 2.2 (V). $\qquad \square$

Now we can state the main result in this section.

Theorem 4.4. *Let U be a nonsingular $\mathcal{U}$ matrix associated supporting tree $(T, \mathcal{T})$ with fixed the leaf n. Suppose that $i \neq j$ and $i \wedge j = L$.*

(i). *If $L \in geod(I, n)$, then $P_{ij} > 0$ if and only if $(P^L)_{ij} > 0$; i.e., if and only if*

(i.a). *for $i \in L^-$, $j \in L^+$, $i \in \mathcal{R}_{L^-}$ and $j = n$.*

(i.b). *for $i \in I^+$, $j \in I^-$, $i = n$ and $j \in \mathcal{R}_{L^-}$.*

(ii) *If $L \notin geod(I, n)$ and $L_1 = (i \wedge j) \wedge n$, then*

(ii.a). *$(P^L)_{ij} > 0$ if and only if $i \in \mathcal{R}_{L^-}$ and $j \in \mathcal{R}_{L^+}$ for $i \in L^-$, $j \in L^+$; and $i \in \mathcal{R}_{L^+}$ and $j \in \mathcal{R}_{L^-}$ for $i \in L^+$, $j \in L^-$.*

(ii.b). *$(P^{L_1^-})_{ij} > 0$ if and only if $(P^L)_{ij} > 0$; and either $U_{ij} > \alpha_L$, or $U_{ij} = \alpha_{ij}$ and for every $M \prec L$ such that $\alpha_M = \alpha_L$ implies that $(M, M^-) \notin \Gamma^t$ for $\{i, j\} \subseteq M^-$, $(M, M^+) \notin \Gamma$ for $\{i, j\} \subseteq M^+$ hold.*

(ii.c). *$P_{ij} > 0$ if and only if $(P^{L_1^-})_{ij} > 0$ and $U_{ij} > \alpha_{L_1}$.*

Proof. We prove the assertion by the dimension of the matrix U. It is trivial for $n = 1, 2$. Now assume that $i \neq j$ and $i \wedge j = L$.

Case 1: $L \in geod(I, n)$. If $L = I$, then $i \in I^-$ (or I^+) and $j \in I^+$ (or I^-). Hence by Lemma 4.2 and (2), the assertion (i) holds. If $L \succ I$, then $L \in I^+$ and $i, j \in I^+$. So by Lemma 4.3, we have $P_{ij} > 0$ if and only if $(U^{-1})_{ij} < 0$ if and only if $(U_{I^+}^{-1})_{ij} < 0$. Since U_{I^+} is a $\mathcal{U}$ matrix, by the induction hypothesis, we have $(U^{-1})_{ij} < 0$ if and only if $(P^L)_{ij} > 0$. Moreover, the rest of (i) follows from Lemma 4.2.

Case 2: $L \notin geod(I, n)$. If $L_1 = I$, then $i, j \in L \subseteq I^-$ and U_J is a GU matrix. Hence (ii.a) and (ii.b) follow from Theorem 4 in [4]. At the same time, (ii.c) follows from Lemma 4.1 and (2) that $P_{ij} > 0$ if and only if $(U_{I^-}^{-1})_{ij} < 0$ and $U_{ij} > \alpha_{L_1}$. If $L_1 \succ I$, then $P_{ij} > 0$ if and only if $(U^{-1})_{ij} < 0$ if and only if $(U_{I^+}^{-1})_{ij} < 0$ if and only if $(P^{I^+})_{ij} > 0$. By the induction hypothesis and Theorem 4 in [4], the assertions of (i) and (ii) hold. $\qquad \square$

Corollary 4.5. *Let U be a nonsingular $\mathcal{U}$ matrix. If $A, B \in geod(I, n)$ and $A \neq B \neq n$, then $(U^{-1})_{ij} = 0$ for $i \in A$ and $j \in B$.*

Proof. Since $A \wedge B = A$ or B, The result follows from Theorem 4.4 (i). $\qquad \square$

Corollary 4.6. *Let U be nonsingular $\mathcal{U}$ matrix with supporting tree. If $\beta_i = \beta_t$ for all $t \prec i$ and all $i \in I$, then the inverse of U has the following structure:*

$$U^{-1} = \begin{pmatrix} W_{11} & 0 & \cdots & 0 & W_{1s} \\ 0 & W_{22} & \cdots & 0 & W_{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ W_{s1} & W_{s2} & \cdots & W_{s,s-1} & W_{ss} \end{pmatrix},$$

where W_{ii} is a lower triangular matrix for $i = 2, 3 \cdots, s - 1$; W_{ss} is a 1×1 matrix. Moreover, if $\beta_i > \beta_t$ for all $t \prec i$ and all $i \in I$; and both $\alpha_{I^-} > \alpha_I$ and $\beta_{A^-} > \beta_A$ for $A \in geod(I, n)$; then each entry of W_{11} is nonzero, each entry of W_{is} and W_{si} is nonzero for $i = 1, \cdots, s$; and each each entry of the lower part of lower triangular matrix W_{ii} is zero for $i = 2, \cdots, s - 1$.

Proof. We partition the blocks of $U^{-1} = (W_{ij})$ corresponding to the leaves sets of vertices of $geod(I, n)$. In particular, W_{ss} is corresponding to fixed vertex n. By Corollary 4.5, $W_{ij} = 0$ for $i \neq j \neq s$. Further, it follows from Theorem 4.4(ii.c) that W_{ii} is a lower triangular matrix for $i = 2, \cdots, s - 1$, since for $A \in I^+$, $\alpha_A = \alpha_{A \wedge n}$. Let $n \neq A \in geod(A, n)$. Since $\beta_i > \beta_t$ for all $t \prec i$ and all $i \in I$, $U|_A^-$ is a strictly generalized ultrametric matrix. Hence by Theorem 4.4(ii.a) and (ii.b) or Theorem 3.5 in [14], each entry of W_{11} is nonzero, since $\alpha_{I^-} > \alpha_I$; and each entry of the lower part of the lower triangular matrix W_{ii} is nonzero for $i = 2, \cdots, s - 1$, since $\beta_{A^-} > \beta_A = \alpha_A$. Moreover, since $\mathcal{R}_{A^-} = A^-$, each entry of W_{is} and W_{si} is nonzero by Theorem 4.4 (i). The proof is finished. $\square$

Remark 4.7. From Theorem 4.4, it is easy to see that the links of $U \in \mathcal{U}$ are not involved in whether $n \in \mathcal{R}$ or not. Hence we may directly determine whether each entries of U^{-1} is zero or not from the structure of support tree. Let us give an example to illustrate Theorems 3.2 3.1, 4.4

Example 4.8. Let U be a $\mathcal{U}$ matrix of order 7 with support tree $(T, \mathcal{J})$ as in the Figure 1, where I is root and 6 is fixed leaf.
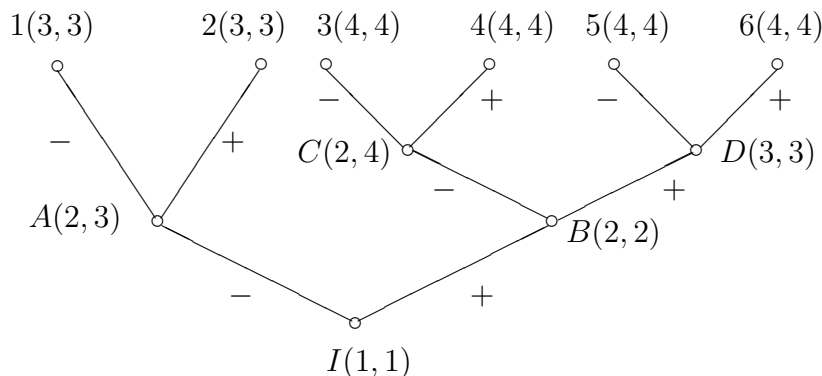

Fig. 1
Then the matrix U and inverse of U are

$$U = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 2 & 4 & 4 & 2 & 2 \\ 3 & 3 & 3 & 3 & 4 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix} \quad U^{-1} = \frac{1}{8} \begin{pmatrix} 8 & -4 & 0 & 0 & 0 & -1 \\ -8 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & -2 \\ 0 & 0 & -4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -6 \\ 0 & -4 & 0 & -4 & -8 & 11 \end{pmatrix}.$$

It is easy to see that $\Gamma = \{(A,2),(C,4)\}$ and

$$\Gamma^t = \{(A,1),(C,3),(I,A),(B,C),(D,5)\}.$$

By Theorem 3.2 and Corollary 3.3, we have $\mathcal{R} = \{1,3,5\}$. Further, we determine all links of P by Theorem 4.4. For instance, in order to determine P_{43}, we consider $3 \wedge 4 = C \notin geod(I,6)$ and $(3 \wedge 4) \wedge 6 = B$. By Theorems 3.2 and 3.1, $4 \in \mathcal{R}_C$, $3 \in \mathcal{R}_{C^+}^t$. Hence by Theorem 4.4 (ii.a), $(P^C)_{43} > 0$. Further, by Theorem 4.4(ii.b) and $U_{43} = 4 > \alpha_C = 2$, $(P^{B^-})_{43} > 0$. Therefore $P_{43} > 0$ follows from Theorem 4.4(ii.c) and $U_{43} = 4 > \alpha_B = 2$.

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