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Near hexagons with four points on a line

Bart De Bruyn*

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Abstract. We classify, up to four open cases, all near hexagons with four points on a line and with quads through every two points at distance 2.

1 The examples

A *near polygon* is a connected partial linear space satisfying the property that for every point x and every line L, there is a unique point on L nearest to x (distances are measured in the collinearity graph Γ). If d is the diameter of Γ, then the near polygon is called a near $2d$-gon. A near polygon is said to have order (s,t) if every line is incident with $s+1$ points and if every point is incident with $t+1$ lines. Near polygons were introduced in [12]. The near quadrangles are just the *generalized quadrangles*, see [10] and [13] for a detailed survey of these geometries. A generalized quadrangle (GQ for short) is called *nondegenerate* if every point is incident with at least two lines. A *near hexagon* is called *regular with parameters* s, t, t_2 if it has order (s,t) and if every two points at distance 2 have exactly t_2+1 common neighbours. In this paper we classify, up to four open cases, all near hexagons satisfying the following two properties: (i) every line is incident with 4 points ($s=3$); (ii) every two points at distance 2 have at least two common neighbours. The corresponding classification with $s=2$ was done in [2]. By Yanushka's lemma (Proposition 2.5 of [12]), the near hexagons satisfying (i) and (ii) above have the *quad-property*, i.e. every two points at distance two are contained in a (necessarily unique) quad. Such a *quad* is defined as a set Q of points satisfying: (i) every point which is collinear with two points of Q belongs as well to Q; (ii) the partial linear space, with points the elements of Q and with lines those lines of the near polygon which have all their points in Q, is a nondegenerate generalized quadrangle. When we talk about a quad in the sequel, this can be as well in the sense of the set Q as in the sense of the related GQ; from the context it is always clear what is meant. A near polygon is called *classical* if it satisfies the quad-property and if for every point x and every quad Q, there exists a unique point

* Postdoctoral Fellow of the Fund for Scientific Research—Flanders (Belgium)

in Q, nearest to x. Classical near polygons are also called *dual polar spaces*, because they can be constructed from polar spaces, see [8] and [12].

We now give several examples of near hexagons which have four points on a line and which satisfy the quad-property. We divide them into the following two classes.

(A) The classical near hexagons.

Each generalized quadrangle can be used to construct classical near hexagons. Let $\mathcal{Q} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ denote a GQ and let X denote a set of order $s+1 \geqslant 2$. Let $\mathcal{S}$ be the incidence system with $\mathcal{P} \times X$ as point set, with $\mathcal{P} \cup (\mathcal{L} \times X)$ as line set and with the following incidence relation:

- $(p,i)\,\mathrm{I}(L,i')$ if and only if $p\,\mathrm{I}\,L$ and $i = i'$;
- $(p,i)\,\mathrm{I}\,q$ if and only if $p = q$.

Then $\mathcal{S}$ is a classical near hexagon and it is called *the direct product* of $\mathcal{Q}$ with a line of length $s+1$. Applying the above construction with s equal to 3 and with $\mathcal{Q}$ one of the five GQ's of order $(3,t)$ (see Section 2) yields five classical near hexagons. The remaining classical examples are associated with the following polar spaces:

(1) $W(5,3)$: the polar space associated with a symplectic polarity of $PG(5,3)$;

(2) $Q(6,3)$: the polar space associated with a nonsingular quadric in $PG(6,3)$;

(3) $H(5,9)$: the polar space associated with a nonsingular Hermitian variety in $PG(5,9)$.

For each of these polar spaces, one can construct a classical near hexagon in the following way: the points of the near hexagon are the totally isotropic planes, the lines are the totally isotropic lines and incidence is the natural one. All these near hexagons are regular with parameters given below.

Type	s	t_2	t
$W(5,3)$	3	3	12
$Q(6,3)$	3	3	12
$H(5,9)$	3	9	90

(B) The nonclassical near hexagons.

The following two near hexagons are examples of glued near hexagons (see Section 3.2 for the definition).

Consider in $PG(5,4)$ a hyperplane Π and two planes α_1 and α_2 meeting in a point p. Let O_i ($i \in \{1,2\}$) be a hyperoval in α_i containing p. We use these objects to define the so-called *linear representation* of the geometry $T_4^*(O_1 \cup O_2)$: the points of $T_4^*(O_1 \cup O_2)$ are the points of $PG(5,4)$ not in Π, the lines are the lines of $PG(5,4)$

intersecting Π in a point of $O_1 \cup O_2$, and incidence is the natural one. The geometry $T_4^*(O_1 \cup O_2)$ is then a near hexagon, see [7]. Since there is a unique hyperoval in $PG(2,4)$ (up to projective equivalence) and since the stabilizer of such a hyperoval in the group of all automorphisms of $PG(2,4)$ acts transitively on the set of its points, there is up to isomorphism only one such near hexagon.

Put $K = \{x \in GF(9) \mid x^4 = 1\}$. Consider in the vector space $V(3,9)$ a nonsingular Hermitian form $(\cdot,\cdot)$ and let U be the corresponding unital of $PG(2,9)$. Let $\alpha = \langle \bar{a} \rangle$ be a fixed point of U. For two points $\beta = \langle \bar{b} \rangle$ and $\gamma = \langle \bar{c} \rangle$ of U, we define

$$\begin{aligned}\Delta(\beta,\gamma) &= -(\bar{a},\bar{b})^2(\bar{b},\bar{c})^2(\bar{c},\bar{a})^2 \quad \text{if } \alpha \neq \beta \neq \gamma \neq \alpha \\ &= 1 \quad \text{otherwise.}\end{aligned}$$

This is well defined. For, if we replace $\bar{b}$ by $\mu\bar{b}$ and $\bar{c}$ by $\lambda\bar{c}$ with $\mu, \lambda \in GF(9) \backslash \{0\}$, then the above value for $\Delta(\beta,\gamma)$ is unaltered. We now define a graph Γ with vertex set $K \times U \times U$. Two different vertices (k_1, α_1, β_1) and (k_2, α_2, β_2) are adjacent if and only if one of the following conditions is satisfied:

(a) $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$,

(b) $\alpha_1 = \alpha_2$, $\beta_1 \neq \beta_2$ and $k_2 = k_1 \Delta(\beta_1, \beta_2)$;

(c) $\alpha_1 \neq \alpha_2$, $\beta_1 = \beta_2$ and $k_2 = k_1 \Delta(\alpha_1, \alpha_2)$.

Every two adjacent vertices of Γ are contained in a unique maximal clique. The geometry, with points the vertices of Γ, with lines the maximal cliques of Γ, and with natural incidence, is then a near hexagon.

There are also four cases which have not yet been settled. These cases are described in Theorem 4.17 which is the main result of this paper.

2 The GQ's with $s = 3$ and their ovoids

The near hexagons which we try to classify have a lot of GQ's as subgeometries (the quads). It is therefore important to know all GQ's of order $(3,t)$. Also the knowledge of some properties of these GQ's will help during the classification. As we will motivate in the following subsection, the study of (configurations of) ovoids of GQ's can indeed yield important information about the structure of a near hexagon. Such an *ovoid* is defined as a set of points having the property that every line meets it in exacly one point. If the GQ has order (s,t), then such an ovoid contains exactly $st+1$ points. A *fan of ovoids* is a set of $s+1$ ovoids partitioning the point set; a *rosette of ovoids* is a set of s ovoids through a certain point p, partitioning the set of points at distance 2 from p.

2.1 Motivation. The following theorem gives the possible relations between a point and a quad of an arbitrary near polygon.

Theorem 2.1 ([11] and [12]). *If p is a point and Q a quad of a near polygon $\mathcal{S}$, then exactly one of the following cases occurs.*

(1) *There is a unique point* $q \in Q$ *such that* $d(p,r) = d(p,q) + d(q,r)$ *for all points* $r \in Q$. *In this case* (p,Q) *is called classical.*

(2) *The points of* Q *which are nearest to* p *form an ovoid of* Q. *In this case* (p,Q) *is called ovoidal.*

(3) *Every line of* Q *is incident with exactly two points. If* A, B, C, *respectively, denote the set of points of* Q *at distance* k *from* p, $k+1$ *from* p, $k+2$ *from* p, *respectively, then*

- $|A| \geqslant 2$ *and* $|C| \geqslant 1$,
- B *and* $A \cup C$ *are the two maximal cocliques of the point graph of* Q.

In this case (p,Q) *is called thin ovoidal.*

Consider now the case where $\mathscr{S}$ is a near hexagon. If p is a point and Q a quad, then there are three possibilities:

(1) $\mathrm{d}(p,Q) = 0$ or $p \in Q$;

(2) $\mathrm{d}(p,Q) = 1$, then (p,Q) is classical;

(3) $\mathrm{d}(p,Q) = 2$, then (p,Q) is ovoidal.

If (p,Q) is ovoidal and if L is a line through p, then from the results in [2], it follows that one of the following cases occurs.

(1) Every point of L has distance 2 from Q. The four ovoids determined by the points of L form a fan of ovoids.

(2) L contains a unique point at distance 1 from a point q of Q, the other three points lie at distance 2 from Q and determine a rosette of ovoids with q as common point.

If Q is a quad having no ovoid, then every point has distance at most 1 from Q. Similar remarks can be made if Q has no fan or no rosette of ovoids (see Section 2.3).

2.2 GQ's with $s = 3$. There are five GQ's with $s = 3$ (see [10]), namely the following ones.

(1) The (4×4)-grid. This is the GQ with points x_{ij} $(1 \leqslant i,j \leqslant 4)$; the lines are L_k and M_k with $1 \leqslant k \leqslant 4$; the point x_{ij} is incident with L_k, respectively M_k, if and only if $i = k$, respectively $j = k$.

(2) $W(3)$. The points of $W(3)$ are the points of $\mathrm{PG}(3,3)$, the lines of $W(3)$ are the totally isotropic lines of $\mathrm{PG}(3,3)$ with respect to a symplectic polarity and the incidence is the natural one. Its order is $(3,3)$.

(3) $Q(4,3)$. $Q(4,3)$ is the GQ of the points and the lines of a nonsingular quadric in $\mathrm{PG}(4,3)$. Its order is $(3,3)$. One can prove that it is the point-line dual of the symplectic quadrangle $W(3)$.

(4) $T_2^*(H)$ with H a hyperoval in PG(2, 4). Let the plane PG(2, 4) be embedded as a hyperplane π in $\Pi = PG(3,4)$. The points of $T_2^*(H)$ are the points of $\Pi \backslash \pi$, the lines of $T_2^*(H)$ are the lines of Π which intersect H in a unique point and incidence is the natural one. Notice that all hyperovals in PG(2, 4) are projectively equivalent (a conic union its nucleus). $T_2^*(H)$ has order $(3, 5)$.

(5) $Q(5,3)$. $Q(5,3)$ is the GQ of the points and the lines of a nonsingular elliptic quadric in PG(5, 3). Its order is $(3, 9)$.

The following theorem settles the existence (of certain configurations) of ovoids in these GQ's.

Lemma 2.2. (a) *$W(3)$ and $Q(5,3)$ have no ovoids.*

(b) *$T_2^*(H)$ contains ovoids but no rosettes of ovoids.*

(c) *$Q(4,3)$ contains ovoids but no fans of ovoids.*

Proof. (a) The following statements were proved in [10].

- $W(q)$, q odd, has no ovoids.
- Every GQ of order (s, s^2), $s > 1$ has no ovoids.

(b) Let the plane PG(2, 4) be embedded as a hyperplane π in $\Pi = PG(3,4)$, and let H be a hyperoval in π. Consider a plane α in Π intersecting π in a line exterior to H. It follows that the points of α not in π form an ovoid of $T_2^*(H)$. Conversely, every ovoid of $T_2^*(H)$ arises this way (see [9]). Hence, two different ovoids are disjoint or intersect in 4 points and as a consequence no rosette of ovoids occurs.

(c) Let Q be a nonsingular quadric in PG(4, 3). If π is a hyperplane of PG(4, 3) intersecting Q in an elliptic quadric, then $\pi \cap Q$ is an ovoid of $Q(4,3)$. Conversely, every ovoid is obtained this way (see e.g. [3]). Let O_1 and O_2 be two ovoids and let α_1 and α_2 be the hyperplanes such that $O_1 = \alpha_1 \cap Q$ and $O_2 = \alpha_2 \cap Q$. Now, $O_1 \cap O_2 = (\alpha_1 \cap \alpha_2) \cap Q \neq \emptyset$; hence, $Q(4,3)$ has no fans of ovoids.

2.3 Applications. From now on, we will always assume that $\mathcal{S}$ is a near hexagon satisfying the following properties:

(1) all lines of $\mathcal{S}$ have 4 points;

(2) every two points at distance two have at least two common neighbours.

Definition. A quad Q of $\mathcal{S}$ is called *big* if every point of $\mathcal{S}$ has distance at most 1 from Q.

Lemma 2.2 now has the following corollaries.

Corollary 2.3. *If Q is a quad of $\mathcal{S}$ which is not big, then Q is isomorphic to the (4×4)-grid or to $Q(4,3)$.*

Proof. Let x be a point at distance 2 from Q, then (x, Q) is ovoidal. Since $W(3)$ and $Q(5,3)$ have no ovoids, Q is not isomorphic to one of these GQ's. Now, let L be a line through x having a point at distance 1 from Q, then L determines a rosette of ovoids. Hence Q is not isomorphic to $T_2^*(H)$.

Corollary 2.4. *If a quad* Q *of* $\mathcal{S}$ *is isomorphic to* $Q(4,3)$, *then no line of* $\mathcal{S}$ *is contained in* $\Gamma_2(Q)$.

Proof. If the line L would be contained in $\Gamma_2(Q)$, then this line determines a fan of ovoids, a contradiction.

3 The classification for some special cases

As mentioned before, every two points of $\mathcal{S}$ at distance 2 are contained in a unique quad. The existence of these quads can then be used to prove that every point of $\mathcal{S}$ is incident with exactly $t+1$ lines (see Lemma 19 of [3]). One of the important tools which we will use in the classification is that of the local space. With every point x of $\mathcal{S}$, there is associated the following incidence system $\mathcal{S}_x$, called the *local space at* x:

(a) the points of $\mathcal{S}_x$ are the lines of $\mathcal{S}$ through x;

(b) the lines of $\mathcal{S}_x$ are all the sets L_Q, where Q is a quad through x and L_Q denotes the set of lines of Q through x;

(c) incidence is the symmetrized containment.

Every local space is a linear space. If $\mathcal{G}$ is a local space of one of the ten near hexagons given in Section 1, then

(a) $\mathcal{G}$ is a (possible degenerate) projective plane, or

(b) $\mathcal{G}$ has a thin point, i.e. a point incident with exactly two lines.

The linear spaces with a thin point are exactly the linear spaces $S_{u,v}$ $(u, v \geqslant 1)$ defined as follows:

(1) the point set of $S_{u,v}$ is equal to $\{\alpha, \beta_1, \ldots, \beta_u, \gamma_1, \ldots, \gamma_v\}$;

(2) the line set of $S_{u,v}$ is equal to $\{\{\alpha, \beta_1, \ldots, \beta_u\}, \{\alpha, \gamma_1, \ldots, \gamma_v\}\} \cup \{\{\beta_i, \gamma_j\} \mid 1 \leqslant i \leqslant u \text{ and } 1 \leqslant j \leqslant v\}$;

(3) incidence is the symmetrized containment.

We now carry out the classification for some special cases.

3.1 The case of classical near hexagons. If $\mathcal{S}$ is a classical near hexagon, then $\mathcal{S}$ is one of the 8 examples described in Section 1 (see [8], the classification of polar spaces in [14], and Theorem 7.1 of [5]). It is known (or easy to verify) that the near hexagon $\mathcal{S}$ is classical if and only if every local space is a (possibly degenerate) projective plane.

3.2 The case of glued near hexagons. In [5], the author presents the following construction for near hexagons. Let $\mathcal{Q}_i$, $i \in \{1,2\}$, be a GQ of order (s, t_i), let $S_i = \{L_1^{(i)}, L_2^{(i)}, \dots, L_{1+st_i}^{(i)}\}$ be a spread of $\mathcal{Q}_i$ (i.e. a set of lines partitioning the point set) and let θ be a bijection from $L_1^{(1)}$ to $L_1^{(2)}$ (the lines of the GQ's are regarded as sets of points). We use the following notation: if $x \in L_j^{(i)}$, then $x^{[L_j^{(i)}, L_k^{(i)}]}$ denotes the unique point of $L_k^{(i)}$ nearest to x. Consider now the following graph Γ on the vertex set $L_1^{(1)} \times S_1 \times S_2$. Two different vertices $(x, L_i^{(1)}, L_j^{(2)})$ and $(y, L_k^{(1)}, L_l^{(2)})$ are adjacent whenever at least one of the following conditions is satisfied:

(1) $i = k$ and $x^{\theta[L_1^{(2)}, L_j^{(2)}]} \sim y^{\theta[L_1^{(2)}, L_l^{(2)}]}$ (in $\mathcal{Q}_2$),

(2) $j = l$ and $x^{[L_1^{(1)}, L_i^{(1)}]} \sim y^{[L_1^{(1)}, L_k^{(1)}]}$ (in $\mathcal{Q}_1$).

It is proved in [5] that any two adjacent vertices are contained in a unique maximal clique. We can regard these maximal cliques as the lines of a geometry $\mathcal{A}$ which has the vertex set of Γ as point set. In [5], a necessary and sufficient condition is determined for $\mathcal{A}$ to be a near hexagon; if this condition is satisfied, then $\mathcal{A}$ is called a *glued near hexagon*.

Suppose now that $\mathcal{S}$ is a glued near hexagon. From the classification results obtained in [4] and [6] we have that $\mathcal{S}$ is one of the following examples.

(1) $\mathcal{Q}_1$ is a (4×4)-grid and $\mathcal{Q}_2$ is a GQ of order $(3, t)$ not isomorphic to $Q(4, 3)$ (this GQ has no spread). In this case the above mentioned condition is always satisfied and $\mathcal{A}$ is the direct product of $\mathcal{Q}_2$ with a line of size 4.

(2) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are isomorphic to $T_2^*(H)$. Up to an automorphism of $T_2^*(H)$, there is a unique choice for S_1 and S_2: the spread consists of the lines of $\mathrm{PG}(3, 4)$ through a fixed point of H. From the results in [6] it follows that there is a unique example (up to isomorphism). This example was first given in [7] using the same description as we did in the first section of the present paper.

(3) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are isomorphic to $Q(5, 3)$. From the results in [6], it follows that there is a unique glued near hexagon. With the same notation as in Section 1, we can define a graph Γ' with vertex set $K \times U$. Two different vertices (k_1, α_1) and (k_2, α_2) are adjacent if and only if one of the following conditions is satisfied:

(a) $\alpha_1 = \alpha_2$;

(b) $\alpha_1 \neq \alpha_2$ and $k_2 = k_1 \Delta(\alpha_1, \alpha_2)$.

Every two adjacent vertices of Γ' are contained in a unique maximal clique. The geometry, with points the vertices of Γ', with lines the maximal cliques of Γ', and with natural incidence, is then a generalized quadrangle $\mathcal{Q}$ isomorphic to $Q(5, 3)$, see [4] for a proof. If we put $L_\alpha = \{(k, \alpha) \mid k \in K\}$ for every $\alpha \in U$, then $S = \{L_\alpha \mid \alpha \in U\}$ is a spread of $\mathcal{Q}$. The unique glued near hexagon is then obtained by putting

- $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{Q}$;
- $S_1 = S_2 = S$;

- $L_1^{(1)} = L_1^{(2)}$ an arbitrary line of S;
- θ the identical map of $L_1^{(1)}$.

The model given in Section 1 now follows easily.

3.3 The case of regular near hexagons. In [11], the following classification result appeared.

Theorem 3.1 ([11]). *If $\mathcal{S}$ is a regular near hexagon with parameters $s = 3$, $t_2 > 0$ and t, then one of the following possibilities occurs*:

(1) $t_2 = 1$ *and* $t = 2$,

(2) $t_2 = 3$ *and* $t = 12$,

(3) $t_2 = 9$ *and* $t = 90$,

(4) $t_2 = 1$ *and* $t = 9$,

(5) $t_2 = 1$ *and* $t = 34$,

(6) $t_2 = 3$ *and* $t = 27$,

(7) $t_2 = 3$ *and* $t = 48$.

The parameters given in (1), (2) and (3) satisfy $t = t_2^2 + t_2$ and hence correspond to classical near hexagons (see [3]); (4) corresponds to no near hexagon (see [1]); it is known that (6) and (7) correspond to no near hexagon, but this will also follow from the treatment given here. Whether there exists a regular near hexagon with parameters $(s, t, t_2) = (3, 34, 1)$ is still an open problem.

4 The classification

We distinguish two cases.

4.1 Case I: all quads of type $Q(4,3)$ are big. In this section we suppose that all quads of type $Q(4,3)$ are big. We first prove the following lemma.

Lemma 4.1. *Suppose $\mathcal{G}$ is a linear space satisfying the following properties*:

(a) *$\mathcal{G}$ has at least two lines,*

(b) *every line has size* 2, 4, 6 *or* 10,

(c) *if two lines are disjoint*, *then they have both size* 2.

Then $\mathcal{G}$ is one of the following examples:

(1) *K_n with $n \geqslant 3$,*

(2) *a projective plane of order* 3, 5 *or* 9,

(3) *S_{t_1,t_2} with $t_1, t_2 \in \{1, 3, 5, 9\}$.*

Proof. If all lines are thin, then we have example (1). If all lines have the same size $\alpha + 1 \neq 2$, then (a), (b) and (c) imply that $\mathcal{G}$ is a projective plane of order 3, 5 or 9.

Suppose that the line sizes $\alpha + 1$ and $\beta + 1$ occur with $2 < \alpha + 1 < \beta + 1$. Let L_1 and L_2 be two lines of sizes $\alpha + 1$ and $\beta + 1$ respectively. If a is a point not on L_1 (respectively L_2), then there are exactly $\alpha + 1$ (respectively $\beta + 1$) lines through a. Since $\alpha + 1 \neq \beta + 1$, all points are contained in $L_1 \cup L_2$ and hence $\mathcal{G} \simeq S_{\alpha,\beta}$.

Suppose now that 2 and $\alpha + 1 \in \{4, 6, 10\}$ are the only line sizes that occur. Consider a line L of size $\alpha + 1$ and let N be the set of points not on L. If $|N| = 1$, then $\mathcal{G} \simeq S_{1,\alpha}$. If $|N| > 1$, then $|N| \geqslant \alpha$. If $|N| = \alpha$, then $\mathcal{G} \simeq S_{\alpha,\alpha}$. Suppose therefore that $|N| > \alpha$. Let $L' \neq L$ be any line of size $\alpha + 1$ and let $p \notin L \cup L'$. Through p there are α or $\alpha + 1$ lines of size $\alpha + 1$. This implies that $|N| = \alpha^2 - \alpha + 1$ or $|N| = \alpha^2$. If $|N| = \alpha^2$, then every line of $\mathcal{G}$ has size $\alpha + 1$, a contradiction. If $|N| = \alpha^2 - \alpha + 1$, then $\mathcal{G}$ induces a linear space $\mathcal{A}$ on the set N which is a Steiner system $S(2, \alpha, \alpha^2 - \alpha + 1)$ and hence a projective plane of order $\alpha - 1$. The linear space $\mathcal{A}$ contains $\alpha^2 - \alpha + 1$ lines and every such line determines a point on L. If L_1 and L_2 are two different lines of $\mathcal{A}$, then they determine different points on L, hence $\alpha^2 - \alpha + 1 < \alpha + 1$, a contradiction.

Lemma 4.2. *Every local space of* $\mathcal{S}$ *satisfies* (a), (b) *and* (c) *from the previous lemma.*

Proof. Clearly (a) and (b) are satisfied. Let Q_1 and Q_2 be two quads intersecting only in a point x. Every point of Q_2 at distance 2 from x has distance 2 from Q_1. Our assumption and Corollary 2.3 then imply that Q_1 is the (4×4)-grid. A similar reasoning proves that Q_2 is the (4×4)-grid.

Corollary 4.3. *If* $\mathcal{S}$ *contains a local space not isomorphic to a* K_n *with* $n \geqslant 3$, *then* $t + 1 \in \{5, 7, 9, 11, 13, 15, 19, 31, 91\}$.

Lemma 4.4. *All local spaces of* $\mathcal{S}$ *are isomorphic.*

Proof. If a local space is isomorphic to $S_{1,\alpha}$ with $\alpha \in \{1, 3, 5, 9\}$, then, by Theorem 7.1 of [5], $\mathcal{S}$ is the direct product of a GQ with a line and the lemma is true in this case. We therefore suppose that no local space is isomorphic to a $S_{1,\alpha}$ for some $\alpha \in \{1, 3, 5, 9\}$. We may also suppose that there exists a point x for which $\mathcal{S}_x$ is not isomorphic to K_{t+1}. These assumptions imply that $t + 1 \neq 5$. If $t + 1$ is equal to 7, 9, 11, 15, 19, 31, respectively, then $\mathcal{S}_x \simeq S_{3,3}$, $S_{3,5}$, $S_{5,5}$, $S_{5,9}$, $S_{9,9}$, $\mathrm{PG}(2, 5)$, respectively. If y is a point collinear with x, then $\mathcal{S}_y$ is not isomorphic to K_{t+1}. Hence $\mathcal{S}_y \simeq \mathcal{S}_x$ and the result follows by connectivity of $\mathcal{S}$. If $t + 1 = 91$, then a similar reasoning yields that all local spaces are projective planes of order 9. Then $\mathcal{S}$ is classical and hence all local spaces are isomorphic to $\mathrm{PG}(2, 9)$ (see Section 3.1). If $t + 1 = 13$, then $\mathcal{S}_x \simeq S_{3,9}$ or $\mathcal{S}_x \simeq \mathrm{PG}(2, 3)$. If y is a point collinear with x, then $\mathcal{S}_y \simeq \mathcal{S}_x$ and the result follows once again by connectivity of $\mathcal{S}$.

The following theorem completes the classification (for Case I), up to the open case appearing in the regular case.

Theorem 4.5. $\mathcal{S}$ *is a regular, a classical or a glued near hexagon.*

Proof. If all local spaces are isomorphic to K_{t+1}, then $\mathcal{S}$ is regular with $t_2 = 1$. If all local spaces are (possible degenerate) projective planes, then $\mathcal{S}$ is a classical near hexagon. If all local spaces are isomorphic to $S_{u,v}$ with $u, v \geqslant 2$, then $\mathcal{S}$ is glued by Theorem 7.2 of [5].

4.2 Case II: there is a quad of type $Q(4,3)$ which is not big. We suppose that there is a quad of type $Q(4,3)$ which is not big. If v is the number of vertices of $\mathcal{S}$, then $v > 40 + 120(t-3)$, hence every quad of type $W(3)$ or $Q(4,3)$ is not big. Since $W(3)$ has no ovoids, quads of type $W(3)$ do not occur. Some of the following lemmas are just adapted versions of results mentioned in [2].

Lemma 4.6. *Let x be a point at distance 2 from a quad Q of type $Q(4,3)$, then x is contained in $\alpha = \frac{39-t}{2}$ grids meeting Q and $\beta = \frac{t-19}{2}$ quads of type $Q(4,3)$ meeting Q. Hence $19 \leqslant t \leqslant 39$ and t is odd.*

Proof. The point x determines an ovoid O of Q. Let o and o' be two different points of O. Let Q_1 be the quad through x and o. The point o' is ovoidal with respect to Q_1. Hence Q_1 is either a (4×4)-grid or a GQ isomorphic to $Q(4,3)$, see Corollary 2.3. As a consequence, every quad through x meeting Q determines 2 or 4 lines through x. It is impossible that a line through x is contained in at least two such quads. If a line through x is not contained in one such quad, then this line determines a fan of ovoids in Q, contradicting Corollary 2.4. The lemma now follows from the following equalities:

$$\alpha + \beta = 10,$$

$$2\alpha + 4\beta = t + 1.$$

Lemma 4.7. *There are no quads of type $W(3)$, $T_2^*(H)$ and $Q(5,3)$.*

Proof. As we already mentioned, quads of type $W(3)$ cannot occur. Suppose that a quad of type $Q(5,3)$ occurs. Consider a local space $\mathcal{S}_x$ with a line L of size 10 and let p be a point of $\mathcal{S}_x$ not on L. Every line through p meets L and contains an even number of points. Since there are 10 such lines the number t must be even, contradicting Lemma 4.6. A totally similar argument yields that there are no quads of type $T_2^*(H)$.

Lemma 4.8. *There are constants a and b such that each point is contained in a grids and b quads of type $Q(4,3)$. One has that*

$$a = \frac{t(t+1)}{2} - 6b \quad \textit{and} \quad v = 18t^2 - 6t + 4 - 108b.$$

Proof. Let x be a any point of $\mathcal{S}$ and suppose that x is contained in a grids and b quads of type $Q(4,3)$. There are $9a + 27b$ points at distance 2 from x and $v - 1 - 3(t+1) - 9a - 27b$ points at distance 3 from x. Counting triples (L_1, L_2, Q), where L_1 and L_2 are two different lines through x and where Q is a quad through

L_1 and L_2, yields $t(t+1) = 2a + 12b$ or $a = \frac{t(t+1)}{2} - 6b$. Counting pairs (y, z) with $d(y, z) = 1$, $d(x, y) = 2$ and $d(x, z) = 3$ yields

$$((v-1) - 3(t+1) - 9a - 27b)(t+1) = 27a(t-1) + 81b(t-3).$$

From $a = \frac{t(t+1)}{2} - 6b$, it follows that $v = 18t^2 - 6t + 4 - 108b$.

Lemma 4.9. $t \neq 39$.

Proof. Suppose $t = 39$ and let x be any point of $\mathcal{S}$. The local space $\mathcal{S}_x$ has lines of cardinality two or four. We will call them 2-lines, respectively 4-lines. By Lemma 4.6, a 2-line and a 4-line of $\mathcal{S}_x$ always intersect. Now, take a 4-line L and a point u not on L. Through u, there is a 4-line M disjoint with L. Let v be a point not on $L \cup M$ and let N be a 4-line through v not meeting $L \cup M$. Any 2-line of $\mathcal{S}_x$ will meet L, M and N; hence there are only 4-lines. Then $\mathcal{S}$ is a regular near hexagon with parameters $s = 3$, $t_2 = 3$ and $t = 39$, contradicting Theorem 3.1.

Lemma 4.10. *If* $t = 19$, *then* $b \leqslant 13$.

Proof. Let x be any point of $\mathcal{S}$. From Lemma 4.6, it follows that every two different 4-lines of the local space $\mathcal{S}_x$ always meet. Consider now two different 4-lines L_1 and L_2 and let p be their common point. If all 4-lines go through p, then $b \leqslant 6$. If there exists a line L not through p, then there are at most four 4-lines through p (since every such line meets L). Since every 4-line of $\mathcal{S}_x$ meets L_1 and L_2, we have that $b \leqslant 4 + 3 \times 3 = 13$.

Let N_2 be the number of points which are ovoidal with respect to a quad of type $Q(4, 3)$; one easily checks that

$$N_2 = 18(t^2 - 7t + 18 - 6b).$$

Lemma 4.11. *Let* L *be a line of size* 4 *in the local space* $\mathcal{S}_x$. *If* $t \neq 21$, *then there are* $\frac{1}{2160}(t-19)N_2$ *lines of size* 4 *and* $\frac{1}{720}(39-t)N_2$ *lines of size* 2 *disjoint with* L; *hence these numbers are integers.*

Proof. (a) Let Q be the quad corresponding with L. For a point z of Q, let A_z be the set of quads of type $Q(4, 3)$ intersecting Q only in the point z. Let y be a point of Q at distance 2 from x. Counting pairs (R, S) with $R \in A_x$, $S \in A_y$ and $|R \cap S| = 1$ yields $|A_x|\frac{t-21}{2} = |A_y|\frac{t-21}{2}$. Hence $|A_x| = |A_y|$. By connectivity of the noncollinearity graph of $Q(4, 3)$ it follows that $|A_z| = |A_x|$ for all points z of Q. Since there are $\frac{1}{27}\binom{t-19}{2}N_2$ quads of type $Q(4, 3)$ intersecting Q in only one point, it follows that $|A_x| = \frac{1}{2160}(t-19)N_2$.

(b) The numbers $|\Gamma_2(z)|$, $|\Gamma_2(z) \cap Q|$ and $|\Gamma_2(z) \cap \Gamma(Q)|$ are independent of $z \in Q$. Hence also $|\Gamma_2(z) \cap \Gamma_2(Q)|$ is independent of $z \in Q$. Hence $|\Gamma_2(x) \cap \Gamma_2(Q)| = \frac{1}{4}N_2$. Counting $|\Gamma_2(x) \cap \Gamma_2(Q)|$ in another way yields $27\frac{1}{2160}(t-19)N_2 + 9\delta = \frac{1}{4}N_2$, where δ is the number of 2-lines disjoint from L. Hence $\delta = \frac{1}{720}(39-t)N_2$.

Corollary 4.12. *$t \neq 23$, $t \neq 29$ and $t \neq 35$.*

Proof. If $t \neq 21$, then $\frac{1}{2160}(t-19)N_2 = \frac{1}{60}(t-19)\left[\frac{t(t-7)}{2} + 9 - 3b\right] \in \mathbb{N}$; hence $3 \mid t(t-19)(t-7)$, from which the corollary follows.

Lemma 4.13.

$$\frac{20t(t+1) - (39-t)(t^2-7t+18)}{6(t+1)} \geqslant b \geqslant \frac{(t-19)(t^2-7t+18)+120}{6(t+1)}.$$

Proof. Let Q be a quad of type $Q(4,3)$. There are $\frac{1}{54}(t-19)N_2$ quads of type $Q(4,3)$ which intersect Q in exactly one point; this number is at most $40(b-1)$, from which the lower bound for b follows. There are $\frac{1}{18}(39-t)N_2$ grids which intersect Q in exactly one point; hence this number is at most $40a$, from which the upper bound for b follows.

We also have the following lower bound for b.

Lemma 4.14.

$$b \geqslant \frac{(t+1)(t^3-26t^2+151t-702)}{6(t^2+2t-319)}.$$

Proof. Let x be any point of $\mathcal{S}$. For every point p of the local space $\mathcal{S}_x$, let α_p denote the number of 4-lines through the point p. Elementary counting yields

$$\sum 1 = t+1,$$

$$\sum \alpha_p = 4b,$$

$$\sum \alpha_p(\alpha_p - 1) = b\left[(b-1) - \frac{1}{2160}(t-19)N_2\right],$$

where the summation ranges over all points p of $\mathcal{S}_x$. The inequality $\sum\left(\alpha_p - \frac{4b}{t+1}\right)^2 \geqslant 0$ yields the bound for b.

Remark. If $t \neq 19, 21$, then the lower bound of b given in Lemma 4.14 is stronger then the one given in Lemma 4.13. Collecting the results of the above lemmas, we find the following bounds for b:

- $t = 19 : 1 \leqslant b \leqslant 13$;
- $t = 21 : 6 \leqslant b \leqslant 27$;
- $t = 25 : 30 \leqslant b \leqslant 41$;
- $t = 27 : 42 \leqslant b \leqslant 50$;
- $t = 31 : 67 \leqslant b \leqslant 71$;
- $t = 33 : 81 \leqslant b \leqslant 84$;
- $t = 37 : b = 113$.

Note that in each case the number of grids is equal to $\frac{va}{16}$ while the number of quads of type $Q(4,3)$ is equal to $\frac{vb}{40}$. Hence these numbers are integers.

Lemma 4.15. *The following congruences hold.*

- *If* $t \neq 19, 21$, *then* $b \equiv t^2 - 2t + 3 \pmod 5$.
- *If* $t = 19$ *or* $t = 21$, *then* $b \equiv 0 \pmod 5$ *or* $b \equiv t^2 - 2t + 3 \pmod 5$.
- *If* $t \equiv 1 \pmod 8$, *then* $b \equiv 0 \pmod 4$.
- *If* $t \equiv 3 \pmod 8$, *then* b *is odd.*
- *If* $t \equiv 5 \pmod 8$, *then* $b \equiv 2 \pmod 4$.

Proof. From $\frac{vb}{40} \in \mathbb{N}$, it follows that $b(3t^2 - t - 1 - 3b) \equiv 0 \pmod 5$. Hence $b \equiv 0 \pmod 5$ or $6t^2 - 2t - 2 - 6b \equiv t^2 - 2t + 3 - b \equiv 0 \pmod 5$. If $t \neq 19, 21$, then $(t - 19)N_2 \equiv 0 \pmod 5$ and, since $t \neq 29, 39$, we even can say $N_2 \equiv 0 \pmod 5$, from which it follows that $b \equiv t^2 - 2t + 3 \pmod 5$. From $\frac{va}{16} \in \mathbb{N}$, it follows that $\left(\frac{9t^2 - 3t}{2} + 1 - 27b\right)\left(\frac{t(t+1)}{2} - 6b\right) \equiv 0 \pmod 4$ from which the remaining congruences readily follow.

Corollary. *From the above lemmas there remain the following possibilities*:

(1) $(t, b) = (19, 1)$;

(2) $(t, b) = (19, 5)$;

(3) $(t, b) = (19, 11)$;

(4) $(t, b) = (21, 10)$;

(5) $(t, b) = (21, 22)$;

(6) $(t, b) = (27, 43)$;

(7) $(t, b) = (31, 67)$.

The quad-quad relation. For a quad Q of $\mathcal{S}$, let $\Gamma_{\leqslant 1}(Q) := Q \cup \Gamma_1(Q)$. Now, let Q and Q' be two quads. We will summarize the possibilities for $\Gamma_{\leqslant 1}(Q) \cap Q'$ regarded as a substructure of Q'. If a line of Q' meets two points of $\Gamma_{\leqslant 1}(Q)$, then it is completely contained in $\Gamma_{\leqslant 1}(Q)$.

(1) Suppose that Q and Q' are quads of type $Q(4,3)$. Since $Q(4,3)$ has no fan of ovoids, every line of $\mathcal{S}$ meets $\Gamma_{\leqslant 1}(Q)$. Hence the following possibilities may occur (see also Theorem 2.3.1 of [10]):

(A) $\Gamma_{\leqslant 1}(Q) \cap Q'$ is an ovoid of Q';

(B) $\Gamma_{\leqslant 1}(Q) \cap Q'$ consists of the four lines through a fixed point of Q';

(C) $\Gamma_{\leqslant 1}(Q) \cap Q'$ is a grid of Q';

(D) $\Gamma_{\leqslant 1}(Q) \cap Q' = Q'$.

(2) Suppose that Q is a quad of type $Q(4,3)$ and that Q' is a grid. With a similar reasoning as above, we have the following possibilities for $\Gamma_{\leqslant 1}(Q) \cap Q'$:

(A) $\Gamma_{\leqslant 1}(Q) \cap Q'$ is an ovoid of Q';

(B) $\Gamma_{\leqslant 1}(Q) \cap Q'$ consists of the two lines through a fixed point of Q';

(C) $\Gamma_{\leqslant 1}(Q) \cap Q' = Q'$.

(3) Suppose that Q and Q' are two grids. We find the following possibilities for $\Gamma_{\leqslant 1}(Q) \cap Q'$:

(A) $\Gamma_{\leqslant 1}(Q) \cap Q'$ is a set of i noncollinear points and $i \in \{0, 1, 2, 4\}$;

(B) $\Gamma_{\leqslant 1}(Q) \cap Q'$ is a line of Q';

(C) $\Gamma_{\leqslant 1}(Q) \cap Q'$ consists of two intersecting lines;

(D) $\Gamma_{\leqslant 1}(Q) \cap Q' = Q'$.

The possibility that $\Gamma_{\leqslant 1}(Q) \cap Q'$ is a set of three noncollinear points is ruled out by a reasoning which one can find in [1].

Theorem 4.16. *The following cases cannot occur*:

(A) $(t, b) = (19, 1)$,

(B) $(t, b) = (19, 11)$,

(C) $(t, b) = (21, 22)$,

(D) $(t, b) = (31, 67)$.

Proof. (A) *The case* $(t, b) = (19, 1)$. Let Q be any quad of type $Q(4,3)$. If G is a grid of $\mathcal{S}$, then $|G \cap \Gamma_2(Q)| \in \{0, 9, 12\}$. Let M_i, $i \in \{0, 9, 12\}$, be the number of grids G for which $|G \cap \Gamma_2(Q)| = i$. Through every point of $\Gamma_1(Q)$ there is one line meeting Q, four lines contained in $\Gamma_1(Q)$ and 15 other lines. Hence $M_{12} \leqslant \frac{1}{4}|\Gamma_1(Q)|\frac{15 \cdot 14}{2} =$

50400. There are $10a - \frac{(39-t)N_2}{72} = 640$ grids intersecting Q in a line and $|Q|\frac{(t-3)(t-4)}{2} = 4800$ grids intersecting Q in only one point. Counting triples (G, L_1, L_2), where $L_1, L_2 \subseteq \Gamma_1(Q)$ are two intersecting lines of the grid G, yields $16(M_0 - 640 - 1) + (M_9 - 4800) \leqslant |\Gamma_1(Q)|\frac{4 \cdot 3}{2} = 11520$. Hence $M_9 \leqslant 16320$ and $M_0 \leqslant 1361$. Hence there are at most $(50400 + 16320 + 1361) = 68081$ grids, a contradiction, since there are exactly $\frac{va}{16} = 72220$ grids.

(B) *The case* $(t, b) = (19, 11)$. Let x be a point of $\mathcal{S}$ and consider the local space $\mathcal{S}_x$. By Lemma 4.11 every two 4-lines meet. We will suppose now that there is no point which is incident with exactly three 4-lines and derive a contradiction. There are at most four 4-lines through every point p of $\mathcal{S}_x$. Indeed, there exists a 4-line not containing p and every 4-line through p meets this line. Now, let $M = \{m_1, m_2, m_3, m_4\}$ be a fixed 4-line. Since there are exactly eleven 4-lines, we may suppose that m_i, $i \in \{1, 2, 3\}$, is incident with four 4-lines and that m_4 is incident with two 4-lines, say M and T. Let L be a 4-line through m_1, different from M and let r be a point of L not on M or T. Every 4-line through r meets M. The line through r and m_i, $i \in \{1, 2, 3\}$, is a 4-line and the line through r and m_4 is a 2-line, a contradiction. Hence there exists a line K through x, which is contained in exactly three quads of type $Q(4, 3)$, say Q_1, Q_2 and Q_3. Let Q_4 be another quad of type $Q(4, 3)$ through x and let y be a point of K different from x. There are 10 lines through y contained in $\Gamma_1(Q_4)$; one of these lines, say S, is not contained in $Q_1 \cup Q_2 \cup Q_3$. Every quad of type $Q(4, 3)$ through y, but not through K, meets Q_i, $i \in \{1, 2, 3\}$, in a line. Hence at least three of its lines through y are contained in $\Gamma_1(Q_4)$. From the quad-quad relation, it then follows that also the fourth line through y is contained in $\Gamma_1(Q_4)$ and must coincide with S. Hence, there are at most $3 + 3$ quads of type $Q(4, 3)$ through y, a contradiction.

(C) *The case* $(t, b) = (21, 22)$. For a quad Q of type $Q(4, 3)$ and a point $x \in Q$, let $N_{x,Q}$ denote the number of quads of type $Q(4, 3)$ intersecting Q only in the point x. We have

$$\sum_Q \sum_x N_{x,Q} = \sum_Q \frac{1}{54}(t - 19)N_2 = \frac{vb(t - 19)N_2}{2160} = 3vb. \qquad (1)$$

On the other hand $\sum_Q \sum_x N_{x,Q} = \sum_x \sum_Q N_{x,Q}$. Now, let x be fixed and consider the local space $\mathcal{S}_x$. For a 4-line $L = \{l_1, l_2, l_3, l_4\}$ of $\mathcal{S}_x$, we define $\alpha_L = \alpha_{l_1} + \alpha_{l_2} + \alpha_{l_3} + \alpha_{l_4}$, with α_{l_i} the number of 4-lines through l_i in $\mathcal{S}_x$.

Suppose that there exists a 4-line $L = \{l_1, l_2, l_3, l_4\}$ of $\mathcal{S}_x$ for which $\alpha_L \geqslant 23$. Let $M \neq L$ be one of the $(\alpha_L - 4)$ 4-lines meeting L, say in the point l_1. There are at least

$$(\alpha_{l_2} - 4) + (\alpha_{l_3} - 4) + (\alpha_{l_4} - 4) = \alpha_L - \alpha_{l_1} - 12 \geqslant 23 - 7 - 12 = 4$$

4-lines disjoint with M. Hence

$$\sum_Q N_{x,Q} \geqslant (\alpha_L - 4)4 \geqslant 76 > 3b. \tag{2}$$

Suppose that $\alpha_L \leqslant 22$ for all 4-lines L of $\mathscr{S}_x$. For every line L of $\mathscr{S}_x$, there are $(25 - \alpha_L)$ 4-lines disjoint with L. Hence

$$\sum_Q N_{x,Q} = \sum_L (25 - \alpha_L) \geqslant 3b. \tag{3}$$

From equations (1), (2) and (3) it follows that $\alpha_L = 22$ for all 4-lines L of $\mathscr{S}_x$. Once again let $L = \{l_1, l_2, l_3, l_4\}$ be a 4-line of $\mathscr{S}_x$ and let $M \neq L$ be one of the 4-lines meeting L, say in l_1. There are at least

$$(\alpha_{l_2} - 4) + (\alpha_{l_3} - 4) + (\alpha_{l_4} - 4) = 10 - \alpha_{l_1}$$

4-lines disjoint with M. Hence $\alpha_{l_1} = 7$. For similar reasons, we may suppose that $\alpha_{l_2} = \alpha_{l_3} = 7$ and $\alpha_{l_4} = 1$. Hence, every 4-line contains a unique point p for which $\alpha_p = 1$. Since there are 22 4-lines and 22 points, all points p of $\mathscr{S}_x$ satisfy $\alpha_p = 1$, a contradiction.

(D) *The case* $(t, b) = (31, 67)$. Consider a local space $\mathscr{S}_x$. With the same notation as above we have

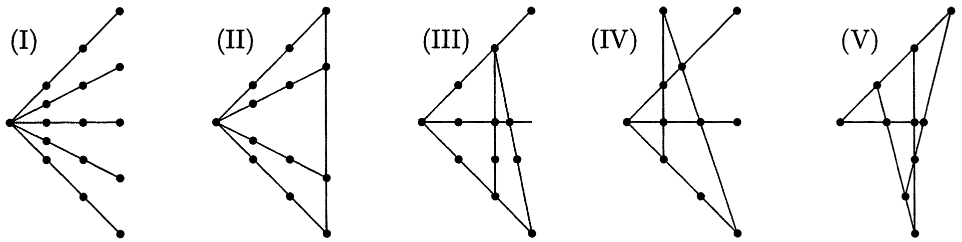
$$\sum 1 = 32,$$
$$\sum \alpha_p = 268,$$
$$\sum \alpha_p(\alpha_p - 1) = 2010,$$

where the summation ranges over all points p of $\mathscr{S}_x$. For every point p of $\mathscr{S}_x$, we have $\alpha_p \leqslant 10$. Suppose $\alpha_p = 10$ for some point p of $\mathscr{S}_x$; then there exists a unique point q such that p and q are not contained in a 4-line. For every line $\{p, p_1, p_2, p_3\}$ through p, $\alpha_{p_1} + \alpha_{p_2} + \alpha_{p_3} = 24$. As a consequence $\alpha_q = 268 - 10 \times 24 - 10 = 18$, contradicting $\alpha_q \leqslant 10$. Hence $\alpha_p \leqslant 9$ for every point p of $\mathscr{S}_x$. Since the average value of the α's is $\frac{67}{8}$, there exists a point p such that $\alpha_p = 9$. Let q_i, $i \in \{1, 2, 3, 4\}$, be those points of $\mathscr{S}_x$ such that p and q_i are not contained in a 4-line. As before, one calculates that $\alpha_{q_1} + \alpha_{q_2} + \alpha_{q_3} + \alpha_{q_4} = 34$; hence $\alpha_{q_i} \in \{7, 8, 9\}$ for all $i \in \{1, 2, 3, 4\}$. For a line $\{p, p_1, p_2, p_3\}$ through p, $\alpha_{p_1} + \alpha_{p_2} + \alpha_{p_3} = 25$; hence $\alpha_{p_i} \in \{7, 8, 9\}$ for all $i \in \{1, 2, 3\}$. With N_i being the number of points r of $\mathscr{S}_x$ for which $\alpha_r = i$, $i \in \{7, 8, 9\}$, we get

$$N_7 + N_8 + N_9 = 32,$$
$$7N_7 + 8N_8 + 9N_9 = 268,$$
$$42N_7 + 56N_8 + 72N_9 = 2010.$$

Hence, $N_7 = 13$, $N_8 = -6$ and $N_9 = 25$, a contradiction.

Remark on the case $(t, b) = (19, 5)$ We want to make some remarks about the local spaces for this case. If x is a point of $\mathcal{S}$, then any two 4-lines of the local space $\mathcal{S}_x$ always meet. These 4-lines induce one of the following configurations.



Suppose that Q is a quad of type $Q(4,3)$. It is impossible that all local spaces $\mathcal{S}_y$, $y \in Q$, are of type (I). Otherwise, the lines of Q which are contained in five quads of type $Q(4,3)$ determine a spread of Q, but $Q(4,3)$ has no spread. We will prove now that no local space is of type (III). Let L be a line of $\mathcal{S}$ which is contained in exactly three quads of type $Q(4,3)$, say Q_1, Q_2 and Q_3. We prove that $\mathcal{S}_x$ is of type (IV) for every point x of L. Let y be a second point of L and let Q_4 be a quad of type $Q(4,3)$ through y, different from Q_1, Q_2 and Q_3. There are 10 lines through x contained in $\Gamma_1(Q_4)$; one of these lines, say K, is not contained in Q_1, Q_2 or Q_3. Let Q_5 be one of the two quads through x different from Q_1, Q_2 and Q_3. At least three of the lines of Q_5 through x are contained in $\Gamma_1(Q_4)$. From the quad-quad relation, it follows that all four lines through x are contained in $\Gamma_1(Q_4)$. Hence the line K is contained in two quads of type $Q(4,3)$, which proves that $\mathcal{S}_x$ is of type (IV).

4.3 The main theorem. By collecting the previous results we can now state the main theorem of this paper.

Theorem 4.17. *Let $\mathcal{S}$ be a near hexagon satisfying the following properties*:

(1) *all lines of $\mathcal{S}$ have* 4 *points*;

(2) *every two points at distance* 2 *have at least two common neighbours.*

We distinguish two cases.

(A) *If $\mathcal{S}$ is classical or glued, then it is isomorphic to one of the ten examples described in Section* 1.

(B) *If $\mathcal{S}$ is not classical and not glued, then only quads isomorphic to the* (4×4)*-grid or to* $Q(4,3)$ *occur. Moreover, there are numbers a and b such that every point of $\mathcal{S}$ is contained in a grids and b quads isomorphic to* $Q(4,3)$. *Every point is contained in the same number of lines, say $t+1$ lines. We then have the following possibilities for t, a, b and v (= the number of points)*:

- $v = 5848$, $t = 19$, $a = 160$, $b = 5$;
- $v = 6736$, $t = 21$, $a = 171$, $b = 10$;
- $v = 8320$, $t = 27$, $a = 120$, $b = 43$;
- $v = 20608$, $t = 34$, $a = 595$, $b = 0$.

It is still an open problem whether there exist near hexagons with parameters as in (B) of the previous theorem.

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B. De Bruyn, Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Gent, Belgium
E-mail: bdb@cage.rug.ac.be