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Perp-systems and partial geometries

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Abstract. A perp-system $\mathscr{R}(r)$ is a maximal set of r-dimensional subspaces of $\mathrm{PG}(N, q)$ equipped with a polarity ρ, such that the tangent space of an element of $\mathscr{R}(r)$ does not intersect any element of $\mathscr{R}(r)$. We prove that a perp-system yields partial geometries, strongly regular graphs, two-weight codes, maximal arcs and k-ovoids. We also give some examples, one of them yielding a new $\mathrm{pg}(8, 20, 2)$.

1 Introduction

1.1 Strongly regular graphs and partial geometries. A *strongly regular graph* denoted by $\mathrm{srg}(v, k, \lambda, \mu)$ is a graph Γ with v vertices, which is regular of degree k and such that any two adjacent vertices have exactly λ common neighbours while any two different non-adjacent vertices have exactly μ common neighbours. If Γ or its complement is a complete graph, then Γ is called a *trivial* strongly regular graph.

Let $\mathscr{S}$ be a connected partial linear space of order (s, t), i.e. every two different points are incident with at most one line, every point is incident with $t + 1$ lines, while every line is incident with $s + 1$ points. The *incidence number* $\alpha(x, L)$ of an antiflag (x, L) (i.e. x is a point which is not incident with the line L) is the number of points on L collinear with x, or equivalently the number of lines through x concurrent with L.

A *partial geometry* with parameters s, t, α, which we denote by $\mathrm{pg}(s, t, \alpha)$, is a partial linear space of order (s, t) such that for all antiflags (x, L) the incidence number $\alpha(x, L)$ is a constant α $(\neq 0)$. A *semipartial geometry* with parameters s, t, α, μ, which we denote by $\mathrm{spg}(s, t, \alpha, \mu)$, is a partial linear space of order (s, t) such that for all antiflags (x, L) the incidence number $\alpha(x, L)$ equals 0 or a constant α $(\neq 0)$ and such that for any two points which are not collinear, there are μ (> 0) points collinear with both points. Partial geometries were introduced by Bose [2] and semipartial geometries by Debroey and Thas [7].

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Partial geometries can be divided into four (non-disjoint) classes:

1. the partial geometries with $\alpha = 1$, the generalized quadrangles [13];
2. the partial geometries with $\alpha = s + 1$ or dually $\alpha = t + 1$; that is the 2-$(v, s + 1, 1)$ designs and their duals;
3. the partial geometries with $\alpha = s$ or dually $\alpha = t$; the partial geometries with $\alpha = t$ are the *Bruck nets* of order $s + 1$ and degree $t + 1$;
4. *proper* partial geometries with $1 < \alpha < \min\{s, t\}$.

For the description of some examples and for further references see [6]. In this article we will only consider the proper partial geometries.

The *point graph* of a partial geometry is the graph whose vertices are the points of the geometry, two distinct vertices being adjacent whenever they are collinear.

The point graph of a partial geometry $\mathrm{pg}(s, t, \alpha)$ is an

$$\mathrm{srg}\Big((s+1)\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\Big).$$

A strongly regular graph Γ with these parameters (and $t \geqslant 1$, $s \geqslant 1$, $1 \leqslant \alpha \leqslant \min\{s+1, t+1\}$) is called a *pseudo-geometric* (s, t, α)-graph. If the graph Γ is indeed the point graph of at least one partial geometry then Γ is called *geometric*.

1.2 Linear representations and SPG reguli. Let $\mathscr{K}$ be a set of points in $\mathrm{PG}(N, q)$ and embed this $\mathrm{PG}(N, q)$ as a hyperplane into a $\mathrm{PG}(N+1, q)$. Define a graph $\Gamma_N^*(\mathscr{K})$ with vertices the points of $\mathrm{PG}(N+1, q) \backslash \mathrm{PG}(N, q)$. Two vertices are adjacent whenever the line joining them intersects $\mathrm{PG}(N, q)$ in an element of $\mathscr{K}$. This graph is a regular graph with $v = q^{N+1}$ and valency $k = (q-1)|\mathscr{K}|$. Delsarte [8] proved that this graph is strongly regular if and only if there are two integers w_1 and w_2 such that the complement of any hyperplane of $\mathrm{PG}(N, q)$ meets $\mathscr{K}$ in w_1 or w_2 points and then the other parameters of the graph are $\lambda = k - 1 + (k - qw_1 + 1)(k - qw_2 + 1)$ and $\mu = k + (k - qw_1)(k - qw_2) = q^{1-N} w_1 w_2$. By regarding the coordinates of the elements of $\mathscr{K}$ as columns of the generator matrix of a code, the property that the complement of any hyperplane meets $\mathscr{K}$ in w_1 or w_2 points is equivalent to the property that the code has two weights w_1 and w_2. For an extensive discussion see [3].

In [19] a new construction method for semipartial geometries is introduced. An *SPG regulus* is a set $\mathscr{R}$ of r-dimensional subspaces $\pi_1, \dots, \pi_k$, $k > 1$ of $\mathrm{PG}(N, q)$ satisfying the following conditions.

(SPG-R1) $\pi_i \cap \pi_j = \varnothing$ for all $i \neq j$.

(SPG-R2) If $\mathrm{PG}(r+1, q)$ contains π_i then it has a point in common with either 0 or α (> 0) spaces in $\mathscr{R} \backslash \{\pi_i\}$; if this $\mathrm{PG}(r+1, q)$ has no point in common with π_j for all $j \neq i$, then it is called a *tangent space* of $\mathscr{R}$ at π_i.

(SPG-R3) If a point x of $\mathrm{PG}(N, q)$ is not contained in an element of $\mathscr{R}$, then it is contained in a constant number θ $(\geqslant 0)$ of tangent $(r+1)$-spaces of $\mathscr{R}$.

Embed $PG(N,q)$ as a hyperplane in $PG(N+1,q)$, and define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines as follows. Points of $\mathcal{S}$ are the points of $PG(N+1,q) \backslash PG(N,q)$. Lines of $\mathcal{S}$ are the $(r+1)$-dimensional subspaces of $PG(N+1,q)$ which contain an element of $\mathcal{R}$, but are not contained in $PG(N,q)$. Incidence is that of $PG(N+1,q)$. Thas [19] proved that $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(q^{r+1}-1, k-1, \alpha, (k-\theta)\alpha)$. If $\theta = 0$, then $\mu = k\alpha$ and hence $\mathcal{S}$ is a partial geometry $\mathrm{pg}(q^{r+1}-1, k-1, \alpha)$.

Recently, Thas [21] proved that if $N = 2r+2$ and a set $\mathcal{R} = \{\pi_1, \ldots, \pi_k\}$ of r-dimensional spaces in $PG(2r+2,q)$ satisfies (SPG-R1) and (SPG-R2) then

$$\alpha(k(q^{r+2}-1) - (q^{2r+3}-1)) \leqslant k^2(q^{r+1}-1) - k(q^{2r+2}+q^{r+1}-2) + q^{2r+3} - 1. \quad (1)$$

If equality holds then $\mathcal{R}$ is an SPG regulus, and conversely.

A *linear representation* of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t) in $AG(N+1, s+1)$ is an embedding of $\mathcal{S}$ in $AG(N+1, s+1)$ such that the line set $\mathcal{L}$ of $\mathcal{S}$ is a union of parallel classes of lines of $AG(N+1, s+1)$ and hence the point set $\mathcal{P}$ of $\mathcal{S}$ is the point set of $AG(N+1, s+1)$. The lines of $\mathcal{S}$ define in the hyperplane at infinity Π_∞ a set $\mathcal{K}$ of points of size $t+1$. A common notation for a linear representation of a partial linear space is $T_N^*(\mathcal{K})$. Note that the point graph of $T_N^*(\mathcal{K})$ is the graph $\Gamma_N^*(\mathcal{K})$. For an extensive discussion see [5]. If $T_N^*(\mathcal{K})$ is a partial geometry then every line of Π_∞ intersects $\mathcal{K}$ in either 0 or $\alpha+1$ points and either $N=1$ and $\mathcal{K}$ is any subset of order $\alpha+1$ of the line at infinity, or $N=2$ and $\mathcal{K}$ is a maximal arc, or $\mathcal{K}$ is the complement of a hyperplane of $PG(N,q)$ and hence $\alpha = s = q-1$. See [18] for more details.

2 Perp-systems

Let ρ be a polarity in $PG(N,q)$ $(N \geqslant 2)$. Let n $(n \geqslant 2)$ be the rank of the related polar space P. A *partial m-system* M of P, with $0 \leqslant m \leqslant n-1$, is a set $\{\pi_1, \ldots, \pi_k\}$ $(k > 1)$ of totally singular m-dimensional spaces of P such that no maximal totally singular space containing π_i has a point in common with an element of $M \backslash \{\pi_i\}, i = 1, 2, \ldots, k$. If the set M is maximal then M is called an *m-system*. For the maximal size of M for each of the polar spaces P and for more information we refer to [14, 16].

We introduce an object which has very strong connections with m-systems and SPG reguli, not only because of the geometrical construction but also because of other similarities of their properties such as bound (1) for SPG reguli.

Again, let ρ be a polarity of $PG(N,q)$. Define a partial *perp-system* $\mathcal{R}(r)$ to be any set $\{\pi_1, \ldots, \pi_k\}$ of k $(k > 1)$ mutual disjoint r-dimensional subspaces of $PG(N,q)$ such that no π_i^ρ meets an element of $\mathcal{R}(r)$. Hence each π_i is non-singular with respect to ρ. Note that $N \geqslant 2r+1$.

Theorem 2.1. *Let* $\mathcal{R}(r)$ *be a partial perp-system of* $PG(N,q)$ *equipped with a polarity* ρ. *Then*

$$|\mathcal{R}(r)| \leqslant \frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)}{q^{(N-2r-1)/2}+1}. \quad (2)$$

Proof. Consider a partial perp-system $\mathcal{R}(r) = \{\pi_1, \dots, \pi_k\}$ $(k > 1)$ of r-dimensional subspaces π_i of $\mathrm{PG}(N, q)$. We count in two different ways the number of ordered pairs (p_i, π^ρ), $\pi \in \mathcal{R}(r)$ and p_i a point of π^ρ. If t_i is the number of $(N - r - 1)$-dimensional spaces π^ρ $(\pi \in \mathcal{R}(r))$ containing p_i then

$$\sum t_i = |\mathcal{R}(r)| \frac{q^{N-r} - 1}{q - 1}.$$

Next we count in two different ways the number of ordered triples $(p_i, \pi^\rho, \pi'^\rho)$, with $\pi, \pi' \in \mathcal{R}(r)$ $(\pi \neq \pi')$ and p_i a point of $\pi^\rho \cap \pi'^\rho$. Then we obtain

$$\sum t_i(t_i - 1) = |\mathcal{R}(r)|(|\mathcal{R}(r)| - 1) \frac{q^{N-2r-1} - 1}{q - 1}.$$

The number of points p_i equals

$$|I| = \frac{q^{N+1} - 1}{q - 1} - |\mathcal{R}(r)| \frac{q^{r+1} - 1}{q - 1}.$$

Then the inequality $|I| \sum t_i^2 - (\sum t_i)^2 \geqslant 0$ yields after some calculation the bound in the statement of the theorem. □

Corollaries. *If equality holds in* (2) *then* $\mathcal{R}(r)$ *is called a* perp-system. *This is equivalent to the fact that every point* p_i *of* $\mathrm{PG}(N, q)$ *not contained in an element of* $\mathcal{R}(r)$ *is incident with a constant number* $\bar{t}$ *of* $(N - r - 1)$*-dimensional spaces* π^ρ *with*

$$\bar{t} = \frac{\sum t_i}{|I|} = q^{(N-2r-1)/2}.$$

Assume that $N = 2r + 1$, *then a perp-system contains* $\frac{q^{r+1}+1}{2}$ *elements. In this case* q *has to be odd and every point not contained in an element of the perp-system is incident with exactly one space* π^ρ $(\pi \in \mathcal{R}(r))$, *which is an* r*-dimensional space.*

Assume $N > 2r + 1$, *then the right hand side of* (2) *is an integer if and only if* $\frac{N+1}{N-2r-1}$ *is an odd integer, say* $2l + 1$. *This is equivalent to*

$$N = 2r + 1 + \frac{r + 1}{l}.$$

Hence l *divides* $r + 1$ *and*

$$2r + 1 \leqslant N \leqslant 3r + 2. \qquad (3)$$

If N *is even then equality in* (2) *implies that* q *is a square.*

Assume that $N = 3r + 2$ *then a perp-system contains* $q^{(r+1)/2}(q^{r+1} - q^{(r+1)/2} + 1)$ *elements. Hence if* r *is even then* q *has to be a square.*

Theorem 2.2. *Let $\mathcal{R}(r)$ be a perp-system of* $\mathrm{PG}(N,q)$ *equipped with a polarity ρ and let $\overline{\mathcal{R}(r)}$ denote the union of the point sets of the elements of $\mathcal{R}(r)$. Then $\overline{\mathcal{R}(r)}$ has two intersection sizes with respect to hyperplanes.*

Proof. Suppose that p is a point of $\mathrm{PG}(N,q)$ which is not contained in an element of $\mathcal{R}(r)$. Then p is incident with $q^{(N-2r-1)/2}$ $(N-r-1)$-dimensional spaces π^{ρ} $(\pi \in \mathcal{R}(r))$. Therefore the hyperplane p^{ρ} contains

$$h_1 = \frac{q^{r+1}-1}{q-1}q^{(N-2r-1)/2} + \frac{q^r-1}{q-1}\left(\frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)}{q^{(N-2r-1)/2}+1} - q^{(N-2r-1)/2}\right)$$

points of $\overline{\mathcal{R}(r)}$.

Suppose that p is contained in an element of $\mathcal{R}(r)$. Since all elements of $\mathcal{R}(r)$ are non-singular, we obtain that p^{ρ} contains

$$h_2 = \frac{q^r-1}{q-1}\,\frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)}{q^{(N-2r-1)/2}+1}$$

points of $\overline{\mathcal{R}(r)}$. □

Remark. Theorem 2.2 implies that $\mathcal{R}(r)$ yields a two-weight code and a strongly regular graph $\Gamma^*(\overline{\mathcal{R}(r)})$ [3]. One easily checks that this graph is a pseudo-geometric

$$\left(q^{r+1}-1, \frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)}{q^{(N-2r-1)/2}+1} - 1, \frac{q^{r+1}-1}{q^{(N-2r-1)/2}+1}\right)\text{-graph.}$$

Recall that the existence of the perp-system $\mathcal{R}(r)$ implies that $\frac{N+1}{N-2r-1}$ is odd, say $2l+1$, which implies that $\frac{2(r+1)}{N-2r-1} = 2l$, hence is even; so $\frac{q^{r+1}-1}{q^{(N-2r-1)/2}+1}$ is a positive integer.

Theorem 2.3. *Let $\mathcal{R}(r)$ be a perp-system of* $\mathrm{PG}(N,q)$ *equipped with a polarity ρ. Then the graph $\Gamma^*(\overline{\mathcal{R}(r)})$ is geometric.*

Proof. The vertices of $\Gamma^*(\overline{\mathcal{R}(r)})$ are the points of $\mathrm{PG}(N+1,q)\backslash\mathrm{PG}(N,q)$. The incidence structure $\mathcal{S}$ with this set of points as point set and with lines the $(r+1)$-dimensional subspaces of $\mathrm{PG}(N+1,q)$ which contain an element of $\mathcal{R}(r)$ but that are not contained in $\mathrm{PG}(N,q)$, is a partial linear space with point graph $\Gamma^*(\overline{\mathcal{R}(r)})$. It is well-known that the point graph $\Gamma^*(\overline{\mathcal{R}(r)})$ of $\mathcal{S}$ being pseudo-geometric implies that $\mathcal{S}$ is a partial geometry. For a proof of this result we refer to [10]. □

Remark. Actually $\mathcal{R}(r)$ is an SPG regulus with $\theta = 0$. In Section 5 we will come back to these partial geometries for the extremal cases of N. However we will first discuss a few properties that are similar to those for m-systems.

3 Perp-systems and intersections with generators

Assume that the polarity ρ is a non-singular symplectic polarity in $\mathrm{PG}(N,q)$, hence N is odd. Let $|\Sigma(W(N,q))|$ denote the number of generators of the symplectic polar space $W(N,q)$ (see for example [12] for more information on classical polar spaces and their notation). Then, as for m-systems, we can calculate the intersection of a perp-system with a generator of $W(N,q)$.

Theorem 3.1. *Let $\mathscr{R}(r)$ be a perp-system of the finite classical polar space $W(N,q)$ and let $\overline{\mathscr{R}(r)}$ denote the union of the point sets of the elements of $\mathscr{R}(r)$. Then for any maximal isotropic subspace (also called generator) G of $W(N,q)$*

$$|G\cap\overline{\mathscr{R}(r)}| = \frac{q^{(N-2r-1)/2}(q^{r+1}-1)}{(q^{(N-2r-1)/2}+1)(q-1)}.$$

Proof. Recall that $|\Sigma(W(N,q))| = (q^{(N+1)/2}+1)|\Sigma(W(N-2,q))|$. We count in two ways the number of ordered pairs (p,G_i) with $p\in\overline{\mathscr{R}(r)}$ and G_i a generator of the polar space $W(N,q)$ such that $p\in G_i$. If $t_i = |G_i\cap\overline{\mathscr{R}(r)}|$ then

$$\begin{aligned}\sum t_i &= |\mathscr{R}(r)|\frac{q^{r+1}-1}{q-1}|\Sigma(W(N-2,q))|\\ &= \frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)(q^{r+1}-1)}{(q^{(N-2r-1)/2}+1)(q-1)}(q^{(N-1)/2}+1)|\Sigma(W(N-4,q))|.\end{aligned}$$

Next we count in two ways the number of ordered triples (p,p',G_i), with p and p' different points of $\overline{\mathscr{R}(r)}$ contained in the generator G_i. Then we obtain

$$\begin{aligned}\sum t_i(t_i-1) &= |\mathscr{R}(r)|\frac{q^{r+1}-1}{q-1}\left(\frac{q^r-q}{q-1}+(|\mathscr{R}(r)|-1)\frac{q^r-1}{q-1}\right)|\Sigma(W(N-4,q))|\\ &= |\mathscr{R}(r)|\frac{q^{r+1}-1}{q-1}\left((|\mathscr{R}(r)|)\frac{q^r-1}{q-1}-1\right)|\Sigma(W(N-4,q))|.\end{aligned}$$

And so

$$\begin{aligned}\sum t_i^2 = {}& \frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)(q^{r+1}-1)}{(q^{(N-2r-1)/2}+1)(q-1)}\\ &\cdot\left(q^{(N-1)/2}+\frac{q^{(N-2r-1)/2}(q^{(N+1)/2}+1)(q^r-1)}{(q^{(N-2r-1)/2}+1)(q-1)}\right)|\Sigma(W(N-4,q))|.\end{aligned}$$

Finally we obtain for the cardinality of the index set

$$|I| = |\Sigma(W(N,q))| = (q^{(N+1)/2}+1)(q^{(N-1)/2}+1)|\Sigma(W(N-4,q))|.$$

An easy calculation now shows that $|I|\sum t_i^2 - (\sum t_i)^2 = 0$. Therefore

$$t_i = \bar{t} = \frac{\sum t_i}{|I|} = \frac{q^{(N-2r-1)/2}(q^{r+1}-1)}{(q^{(N-2r-1)/2}+1)(q-1)}. \qquad \square$$

Remark. The counting arguments of the proof of Theorem 3.1 do not work for the other classical polar spaces.

Let P be a finite classical polar space of rank $n \geqslant 2$. In [20] Thas introduced the concept of a *k-ovoid* of P, that is a point set $\mathscr{K}$ of P such that each generator of P contains exactly k points of $\mathscr{K}$. Note that a k-ovoid with $k = 1$ is an ovoid. By Theorem 3.1 a perp-system $\mathscr{R}(r)$ of $W(N,q)$ yields a k-ovoid with $k = \frac{q^{(N-2r-1)/2}(q^{r+1}-1)}{(q^{(N-2r-1)/2}+1)(q-1)}$. In Section 5 we will give an example of a perp-system $\mathscr{R}(1)$ in $W_5(3)$ yielding a new 3-ovoid.

4 Perp-systems arising from a given one

The next lemma is commonly known but we give a proof for the sake of completeness.

Lemma 4.1. *Let B be a non-degenerate reflexive sesquilinear form on the vector space $V(N+1,q^n)$ of dimension $N+1$ over the field* GF(q^n), *and let T be the trace map from* GF(q^n) *to* GF(q). *Then the map $B' = T \circ B$ is a non-degenerate reflexive sesquilinear form on the vector space $V((N+1)n,q)$.*

Proof. The fact that B' is sesquilinear on $V((N+1)n,q)$ follows immediately from B being sesquilinear and T being additive.

Assume that x is some non-zero element of the vector space $V(N+1,q^n)$. Then the map $y \mapsto B(x,y)$ maps the vector space *onto* GF(q^n). Since there exist elements of GF(q^n) that have non-zero trace, there must be some y such that $T \circ B(x,y) \neq 0$. Hence B' is non-degenerate.

It remains to be shown that B' is reflexive, that is $B'(x,y) = 0$ implies $B'(y,x) = 0$. By the classification of the non-degenerate reflexive sesquilinear forms, B is either symmetric $(B(x,y) = B(y,x))$, anti-symmetric $(B(x,y) = -B(y,x))$ or Hermitean $(B(x,y) = B(y,x)^\sigma$ for some $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$. In the first and second case it is obvious that $B' = T \circ B$ is reflexive. When B is Hermitean, then $B'(x,y) = T \circ B(x,y) = T \circ (B(y,x)^\sigma) = (T \circ B(y,x))^\sigma = B'(y,x)^\sigma$, and so is reflexive. $\square$

Remark. It is well known that a non-degenerate reflexive sesquilinear form on a vector space $V(N+1,q^n)$ gives rise to a polarity of the related projective space PG(N,q^n) and conversely.

Note however that a polarity of PG$(Nn+n-1,q)$ obtained from a polarity of PG(N,q^n) by composition with the trace map is not neccesarily of the same type as the original polarity. For instance a Hermitean polarity may under certain conditions give rise to an orthogonal polarity. See [15, Section 9] for examples.

Theorem 4.2. *Let* $\mathscr{R}(r)$ *be a perp-system with respect to some polarity of* $\mathrm{PG}(N, q^n)$ *then there exists a perp-system* $\mathscr{R}'((r+1)n-1)$ *with respect to some polarity of* $\mathrm{PG}((N+1)n-1, q)$.

Proof. Let ρ be a polarity such $\mathscr{R}(r)$ is a perp-system with respect to ρ. Let B be the non-degenerate reflexive sesquilinear form on $V(N+1, q^n)$ associated with ρ. Then $B' = T \circ B$ induces as in the previous lemma a polarity of $\mathrm{PG}((N+1)n-1, q)$. The elements of $\mathscr{R}(r)$ can be considered as $((r+1)n-1)$-dimensional subspaces of $\mathrm{PG}((N+1)n-1, q)$. Denote this set of subspaces by $\mathscr{R}'((r+1)n-1)$. We show that $\mathscr{R}'((r+1)n-1)$ is a perp-system of $\mathrm{PG}((N+1)n-1, q)$ with respect to the polarity ρ' induced by B'.

First note that the size of $\mathscr{R}'((r+1)n-1)$ is the correct size to be a perp-system of $\mathrm{PG}((N+1)n-1, q)$. Then consider an element M of $\mathscr{R}(r)$ and let M' be the corresponding element of $\mathscr{R}'((r+1)n-1)$. The tangent space $M^{\perp_B}$ of M is defined to be

$$M^{\perp_B} = \{x \in \mathrm{PG}(N, q^n) \mid B(x, y) = 0 \text{ for all } y \in M\}.$$

It has projective dimension $N-r-1$ over $\mathrm{GF}(q^n)$, and considered as a subspace of $\mathrm{PG}((N+1)n-1, q)$ has projective dimension $(N-r)n-1$. Also

$$M'^{\perp_{B'}} = \{x \in \mathrm{PG}((N+1)n-1, q) \mid B'(x, y) = 0 \text{ for all } y \in M'\}$$

has projective dimension $(N-r)n-1$ over $\mathrm{GF}(q)$. Now if x is such that $B(x, y) = 0$ then $B'(x, y) = 0$, so it follows that the tangent space of M' with respect to B' is exactly the tangent space of M with respect to B considered as a subspace of $\mathrm{PG}((N+1)n-1, q)$. Hence since $M^{\perp_B}$ is disjoint from the set of points of elements of $\mathscr{R}(r)$, also $M'^{\perp_{B'}}$ is disjoint from the set of points of elements of $\mathscr{R}'((r+1)n-1)$. □

Remark. It is possible to calculate the type of the polar space obtained by taking the trace of a reflexive sesquilinear form (cf. [15]). But in some sense perp-systems do not care about the type of the underlying polar space since the size of a perp-system is only dependent on the dimension of the projective space it is embedded in. Actually the perp-system $\mathscr{R}(1)$ in $\mathrm{PG}(5, 3)$ that we describe in the next section is related to a symplectic polarity as well as to an elliptic one.

5 Examples

We recall, see (3), that if $\mathscr{R}(r)$ is perp-system in $\mathrm{PG}(N, q)$ then $2r+1 \leqslant N \leqslant 3r+2$. The authors have no knowledge of examples for N not equal to one of the bounds.

5.1 Perp-systems in PG(2*r* + 1, *q*). Assume that $N = 2r+1$, then a perp-system $\mathscr{R}(r)$ in $\mathrm{PG}(2r+1, q)$ yields a

$$\mathrm{pg}\left(q^{r+1}-1,\frac{q^{r+1}-1}{2},\frac{q^{r+1}-1}{2}\right),$$

which is a Bruck net of order q^{r+1} and degree $\frac{q^{r+1}+1}{2}$.

Note that q is odd and that a Bruck net of order q^{r+1} and degree $\frac{q^{r+1}+1}{2}$ coming from a perp system $\mathscr{R}(r)$ in $\mathrm{PG}(2r+1,q)$ is in fact a net that is embeddable in an affine plane of order q^{r+1}. Actually, assume that Φ is an r-spread of $\mathrm{PG}(2r+1,q)$, then $|\Phi|=q^{r+1}+1$ and taking half of the elements of Φ yields a net with requested parameters. However this does not immediately imply that there exists a polarity ρ such that these $\frac{q^{r+1}+1}{2}$ elements form a perp system with respect to ρ. However examples do exist. Take an arbitrary involution without fix points on the line at infinity $\mathrm{PG}(1,q^{r+1})$ of $\mathrm{AG}(2,q^{r+1})$. Using Theorem 4.2 this yields a perp-system $\mathscr{R}'(r)$ in $\mathrm{PG}(2r+1,q)$.

5.2 Perp-systems in PG($3r+2,q$). We will now describe perp-systems $\mathscr{R}(r)$ in $\mathrm{PG}(3r+2,q)$. Note that the partial geometry related to such a perp-system is a

$$\mathrm{pg}(q^{r+1}-1,(q^{r+1}+1)(q^{(r+1)/2}-1),q^{(r+1)/2}-1).$$

Such a partial geometry has the parameters of a partial geometry of type $T_2^*(\mathscr{K})$ with $\mathscr{K}$ a maximal arc of degree $q^{(r+1)/2}$ in a projective plane $\mathrm{PG}(2,q^{r+1})$. As we will see in what follows there do exist partial geometries related to perp-systems and isomorphic to a $T_2^*(\mathscr{K})$ while there exist partial geometries coming from perp-systems $\mathscr{R}(r)$ in $\mathrm{PG}(3r+2,q)$ that are not isomorphic to a $T_2^*(\mathscr{K})$.

Example 1. A perp-system $\mathscr{R}(0)$ in $\mathrm{PG}(2,q^2)$ equipped with a polarity ρ is equivalent to a self-polar maximal arc $\mathscr{K}$ of degree q in the projective plane $\mathrm{PG}(2,q^2)$; i.e. for every point $p\in\mathscr{K}$, the line p^ρ is an exterior line with respect to $\mathscr{K}$. The partial geometry is a $\mathrm{pg}(q^2-1,(q^2+1)(q-1),q-1)T_2^*(\mathscr{K})$. Note that a necessary condition for the existence of a maximal arc of degree d in $\mathrm{PG}(2,q)$ is $d\,|\,q$; this condition is sufficient for q even [9, 17], while non-trivial maximal arcs (i.e. $d<q$) do not exist for q odd [1].

Self-polar maximal arcs do exist as is proven in the next lemma.

Lemma 5.1. *In* $\mathrm{PG}(2,q^2)$ *there exists a self-polar maximal arc of degree q for all even q.*

Proof. We show that certain maximal arcs constructed by Denniston admit a polarity. In what follows the Desarguesian plane $\mathrm{PG}(2,2^e)$ is represented via homogeneous coordinates over the Galois field $\mathrm{GF}(2^e)$. Let $\xi^2+\alpha\xi+1$ be an irreducible polynomial over $\mathrm{GF}(2^e)$, and let $\mathscr{F}$ be the set of conics given by the pencil

$$F_\lambda : x^2 + \alpha xy + y^2 + \lambda z^2 = 0, \quad \lambda \in \mathrm{GF}(2^e) \cup \{\infty\}.$$

Then F_0 is the point $(0, 0, 1)$, F_∞ is the line $z^2 = 0$ (which we shall call the *line at infinity*). Every other conic in the pencil is non-degenerate and has nucleus F_0. Further, the pencil is a partition of the points of the plane. For convenience, this pencil of conics will be referred to as the *standard pencil.*

In 1969, Denniston showed that if A is an additive subgroup of $\mathrm{GF}(2^e)$ of order n, then the set of points of all F_λ for $\lambda \in A$ forms a $\{2^e(n-1)+n; n\}$-arc $\mathcal{K}$, i.e. a maximal arc of degree n in $\mathrm{PG}(2, 2^e)$ [9].

In [11, Theorem 2.2.4], Hamilton showed that if $\mathcal{F}$ is the standard pencil of conics, A an additive subgroup of $\mathrm{GF}(2^e)$, and $\mathcal{K}$ the Denniston maximal arc in $\mathrm{PG}(2, 2^e)$ determined by A and $\mathcal{F}$, then the dual maximal arc $\mathcal{K}'$ of $\mathcal{K}$ has points determined by the standard pencil and additive subgroup

$$A' = \{\alpha^2 s \mid s \in \mathrm{GF}(2^e)^* \text{ and } T(\lambda s) = 0 \text{ for each } \lambda \in A\} \cup \{0\},$$

where T denotes the trace map from $\mathrm{GF}(2^e)$ to $\mathrm{GF}(2)$.

In the case when e is even and $\mathrm{GF}(q^2) = \mathrm{GF}(2^e)$, it follows that if A is the additive group of $\mathrm{GF}(q)$ then $A' = \alpha^2 A$. Simple calculations then show that the homology matrix

$$H = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a collineation that maps the Denniston maximal arc determined by A to that determined by A'. Furthermore, $HH^{-t} = H^{-t}H = I$, where H^{-t} is the inverse transpose of H. It follows that the function, mapping the point (x, y, z) to the line with coordinate $(x, y, z)H$, is a polarity that maps the Denniston maximal arc of degree q determined by the additive group A of $\mathrm{GF}(q)$ to its set of external lines. □

By expanding over a subfield we can obtain an SPG regulus (with $\theta = 0$) from a maximal arc $\mathcal{K}$, but the corresponding partial geometry is isomorphic to $T_2^*(\mathcal{K})$.

A self-polar maximal arc of degree q^n in $\mathrm{PG}(2, q^{2n})$ is a perp-system $\mathcal{R}(0)$. Applying Theorem 4.2 gives a perp-system with $r = n-1$ in $\mathrm{PG}(3n-1, q^2)$ and a perp-system with $r = 2n-1$ in $\mathrm{PG}(6n-1, q)$.

Example 2. A perp-system $\mathcal{R}(1)$ in $\mathrm{PG}(5, q)$ equipped with a polarity ρ will yield a $\mathrm{pg}(q^2-1, (q^2+1)(q-1), q-1)$. The fourth author found by computer search such a system M in $\mathrm{PG}(5, 3)$ yielding a $\mathrm{pg}(8, 20, 2)$.

We represent the set M as follows. A point of $\mathrm{PG}(5, 3)$ is given as a triple *abc* where a, b and c are in the range 0 to 8. Taking the base 3 representation of each digit then gives a vector of length 6 over $\mathrm{GF}(3)$. Each of the following columns of 4 points corresponds to a line of the set in $\mathrm{PG}(5, 3)$.

300	330	630	310	610	440	540	470	570	713	813	343	843
100	103	203	201	101	707	137	134	404	831	531	741	351
700	763	563	821	421	387	827	684	254	157	657	407	717
400	433	833	511	711	247	377	514	674	344	144	184	264

373	773	723	823	353	453	383	583
451	641	381	671	881	761	571	461
177	267	867	187	537	347	227	217
704	424	174	564	214	224	834	654

This set M is the unique perp-system with respect to a symplectic polarity in $PG(5,3)$ but also with respect to an elliptic orthogonal polarity. The set has many interesting properties.

(i) The stabilizer of M in PG(5,3) has order 120 and has two orbits on M of size 24 and 60 containing 6 and 15 lines, respectively. The group G of the $pg(8,20,2)$ has order $120 \cdot 729$, acts transitively on the points and has two orbits on the lines. Since each line of M generates a spread of lines in $pg(8,20,2)$ it contains a parallelism. The subgroup of G fixing this parallelism is isomorphic to S_5.

(ii) There are 7 solids S_i in PG(5,3) which contain 3 lines of M each. The S_i meet in a common line L (disjoint from the lines of M).

(iii) Every point of $PG(5,3) \backslash M$ is incident with a unique line with 3 points in M. These 280 lines meet each of the 21 lines of M 40 times and each pair of lines 4 times, hence forming a 2-(21,3,4) design.

(iv) The set M contains exactly 21 lines of $PG(5,3)$, these lines form a partial spread. $PG(5,3) \backslash M$ contains exactly 21 solids of $PG(5,3)$, these solids intersect mutually in a line, and there are exactly 3 solids through any point of $PG(5,3) \backslash M$. An exhaustive computer search established that any set of 21 solids in $PG(5,3)$ satisfying the above properties is isomorphic to the complement of our set M.

Note that the related partial geometry $pg(8,20,2)$ has the same parameters as one of type $T_2^*(\mathscr{K})$, with $\mathscr{K}$ a maximal arc of degree 3 in $PG(2,9)$ which can not exist by [1], but that was already proved for this small case by Cossu [4]. The graph $\Gamma_5^*(\overline{M})$ which is a $srg(729,168,27,42)$ seems to be new although graphs with the same parameters are known.

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