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On the incidence structures of polar spaces and quadrics

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Abstract. In 1969 Buekenhout characterized non-singular quadrics of finite-dimensional projective spaces as polar spaces spanning the whole space and containing every line of the projective space which they intersect in at least three points (see [2]). Using suitable synthetic properties of the pairs of non-collinear disjoint lines, in this paper I present a new characterization of polar spaces and a combinatorial characterization of non-singular quadrics of a projective space of arbitrary dimension. Moreover, the extension of the theorem of Buekenhout even to the infinite-dimensional case is given.

Key words. Incidence geometry, polar spaces, quadrics.

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1 Introduction and comments

A *semilinear space* is a point-line geometry $(\mathscr{P}, \mathscr{L})$ satisfying the following axioms: *any two distinct points lie on at most one line, every line contains at least two points and every point lies on at least one line.* If every line contains at least three points, then the semilinear space is said to be *irreducible*. Two distinct points p and q are *collinear*, if there exists a line containing p and q. The symbol $p \sim q$ means that the two points p and q are collinear and $p \vee q$ denotes the line of $\mathscr{L}$ joining p and q. For convenience, we also say that every point p is collinear to itself. More generally, two subsets X and Y are *collinear* $(X \sim Y)$, if each point of one of them is collinear with every point of the other. If X is a subset of $\mathscr{P}$, then $X^{\perp}$ denotes the set of points of $\mathscr{P}$ collinear with every point of X. A *singular point* of $(\mathscr{P}, \mathscr{L})$ is a point collinear with all points; the set of all singular points of $(\mathscr{P}, \mathscr{L})$ will be denoted by $\mathrm{Rad}(\mathscr{P}, \mathscr{L})$ and $(\mathscr{P}, \mathscr{L})$ is said to be *non-singular* if $\mathrm{Rad}(\mathscr{P}, \mathscr{L})$ is empty. The *incidence graph* of $(\mathscr{P}, \mathscr{L})$ is the graph $G(\mathscr{P}, \mathscr{L})$ whose vertices are the points, two vertices being adjacent if they are collinear. $(\mathscr{P}, \mathscr{L})$ is *connected* if the graph $G(\mathscr{P}, \mathscr{L})$ is connected, i.e. for every pair p, q

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of points of $\mathscr{P}$ there exists a finite chain of points $p_1 = p, p_2, \dots, p_t = q$, such that $p_i \sim p_{i+1}$ for $i = 1, \dots, t-1$. The *distance* $d(p,q)$ between the points p and q of $\mathscr{P}$ is their distance in $G(\mathscr{P}, \mathscr{L})$, hence $d(p,q) = h$ if $h+1$ is the minimum number of points $p_1 = p, p_2, \dots, p_t = q$, such that $p_i \sim p_{i+1}$. Finally, the distance between two subsets X and Y of points of $\mathscr{P}$ is the positive integer $d(X, Y) := \inf\{d(x,y) \mid x \in X, y \in Y\}$. A *subspace* of $(\mathscr{P}, \mathscr{L})$ is a subset W of $\mathscr{P}$ such that for every two collinear points of W the line joining them is contained in W. Clearly, a non-empty intersection of subspaces is a subspace, thus it is possible to define the *closure* $[X]$ of a subset X of $\mathscr{P}$ as the intersection of all subspaces containing X. Moreover, a *singular subspace* of $(\mathscr{P}, \mathscr{L})$ is a subspace W such that any two points of W are collinear. We say that the *rank* of a singular subspace W of $(\mathscr{P}, \mathscr{L})$ is k if $k+1$ is the maximum length of all saturated chains of singular subspaces $W_0 \subset W_1 \subset \cdots \subset W_t$, such that W_0 is a point and $W_t = W$. It follows that points and lines of $(\mathscr{P}, \mathscr{L})$ are singular subspaces of rank 0 and 1, respectively. Moreover, $(\mathscr{P}, \mathscr{L})$ has *rank* d if $d-1$ is the maximum rank of its proper singular subspaces. Finally, a semilinear space $(\mathscr{P}, \mathscr{L})$ is *embedded* in a semilinear space $(\mathscr{P}', \mathscr{L}')$ if there exists an injection $\varphi : \mathscr{P} \to \mathscr{P}'$ between points transforming lines onto lines and such that $[\mathscr{P}^\varphi] = \mathscr{P}'$, and an *isomorphism* of semilinear spaces is a bijection f between points such that f and f^{-1} are embeddings. Note that for every embedding $\varphi : (\mathscr{P}, \mathscr{L}) \to (\mathscr{P}', \mathscr{L}')$ we can identify $\mathscr{P}$ and $\mathscr{P}^\varphi$ hence, for every line $L \in \mathscr{L}$, we identify L and L^φ. Thus, it is not an essential restriction in the study of embedded semilinear spaces to suppose that every semilinear space $(\mathscr{P}, \mathscr{L})$ embedded in a semilinear space $(\mathscr{P}', \mathscr{L}')$ is contained in $(\mathscr{P}', \mathscr{L}')$.

A *polar space* (see [5]) is a semilinear space $(\mathscr{P}, \mathscr{L})$ satisfying the so-called *one-all axiom*: for every point p and for every line L, $p^\perp$ contains either all points of L or exactly one point of L. Clearly, if $(\mathscr{P}, \mathscr{L})$ is a polar space, then the subset $\mathrm{Rad}(\mathscr{P}, \mathscr{L})$ is a singular subspace.

We briefly recall some basic facts about Hermitian forms, pseudo-quadratic forms, polarities and associated polar spaces. For further details, see for instance [7], [15], or [6]. Let K be a skew-field, V a right K-vector space (not necessarily finite-dimensional), $\sigma : K \to K$ an antiautomorphism of K and ε a non-zero element of K. A function $f : V \times V \to K$ is called a (σ, ε)-*Hermitian form* (or *reflexive* σ-*sesquilinear form*) if it is biadditive and it satisfies the following conditions: $f(\boldsymbol{u}a, \boldsymbol{v}b) = a^\sigma f(\boldsymbol{u}, \boldsymbol{v})b$ and $f(\boldsymbol{v}, \boldsymbol{u}) = f(\boldsymbol{u}, \boldsymbol{v})^\sigma \varepsilon$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $a, b \in K$. If, in addition, $f \neq 0$, then $\varepsilon^\sigma = \varepsilon^{-1}$ and $t^{\sigma^2} = \varepsilon t \varepsilon^{-1}$ for all $t \in K$. Moreover, a (σ, ε)-Hermitian form f is *trace-valued* if $f(\boldsymbol{u}, \boldsymbol{u}) \in \{t + t^\sigma \varepsilon \mid t \in K\}$, for all $\boldsymbol{u} \in V$. Now, let $\eta \in K$ be such that $\eta^\sigma = \eta^{-1}$ and $t^{\sigma^2} = \eta t \eta^{-1}$, for all $t \in K$; assume further that $\eta \neq -1$ if $\sigma = \mathrm{id}_K$ and $\mathrm{char}\, K \neq 2$. The set $K_{\sigma,\eta} = \{t - t^\sigma \eta \mid t \in K\}$ is a subgroup of the additive group of K and we denote by $K^{(\sigma,\eta)}$ the quotient $K/K_{\sigma,\eta}$. A function $q : V \to K^{(\sigma,\eta)}$ is called a (σ, η)-*quadratic form* (or *pseudo-quadratic form*) if for $a \in K$ and $\boldsymbol{u} \in V$ we have $q(\boldsymbol{u}a) = a^\sigma q(\boldsymbol{u})a$ and there exists a trace-valued (σ, η)-Hermitian form $f : V \times V \to K$ such that $q(\boldsymbol{u} + \boldsymbol{v}) = q(\boldsymbol{u}) + q(\boldsymbol{v}) + (f(\boldsymbol{u}, \boldsymbol{v}) + K_{\sigma,\eta})$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$. The form f is uniquely determined and it is called the *sesquilinearized form* of q. Note that if $\mathrm{char}\, K \neq 2$, the (σ, ε)-quadratic forms are in canonical 1-1 correspondence with the (σ, ε)-Hermitian forms. If f is a (σ, ε)-Hermitian form (if q is a (σ, ε)-quadratic form) and $c \in K - \{0\}$, the map $cf : V \times V \to K$ defined by $(cf)(\boldsymbol{u}, \boldsymbol{v}) := c(f(\boldsymbol{u}, \boldsymbol{v}))$

($cq : V \to K$ defined by $(cq)(\boldsymbol{u}) := c(q(\boldsymbol{u}))$) is a (σ', ε')-Hermitian form (a (σ', ε')-quadratic form), where $t^{\sigma'} = ct^{\sigma}c^{-1}$, for all $t \in K$, and $\varepsilon' = c(c^{\sigma})^{-1}\varepsilon$. The two forms f and cf (q and cq) are said to be *proportional*.

Let $\mathbb{P}$ be a projective space and $\mathscr{H}(\mathbb{P})$ the set of all the hyperplanes of $\mathbb{P}$. A *polarity* of $\mathbb{P}$ is a correspondence $\pi : \mathbb{P} \to \mathscr{H}(\mathbb{P})$ satisfying the *reciprocity law*: for all points x, y of $\mathbb{P}$, $x \in \pi(y)$ if, and only if, $y \in \pi(x)$. An *absolute point* of π is a point x of $\mathbb{P}$ such that $x \in \pi(x)$ and a *totally isotropic subspace* of π is a subspace X of $\mathbb{P}$ such that $X \subseteq \pi(X)$. It is easy to see that the incidence geometry whose points are the absolute points and whose lines are the totally isotropic lines of a polarity π of $\mathbb{P}$ is a polar space. Moreover, if $\mathbb{P}$ is a projective space over a right vector space $V(K)$, K a skew-field, and f is a (σ, ε)-Hermitian form of $V(K)$ (possibly f is the sesquilinearized form of a (σ, ε)-quadratic form q), then the orthogonality relation $\boldsymbol{u} \perp \boldsymbol{v} \Leftrightarrow f(\boldsymbol{u}, \boldsymbol{v}) = 0$ induces a polarity π in $\mathbb{P}$. It can be shown that every polarity of $\mathbb{P}$ of rank at least 2 is represented either by a (σ, ε)-Hermitian form or by a (σ, ε)-quadratic form which is unique up to proportionality. The polar space of absolute points and totally isotropic lines of a polarity induced either by a (σ, ε)-Hermitian form or by a (σ, ε)-quadratic form (more precisely, in this case we consider the totally singular lines, on which the (σ, ε)-quadratic form vanishes identically) will be called a *classical polar space*.

A complete classification of polar spaces of rank at least 3 has been obtained by many authors, see Buekenhout–Shult [5], Tits [15], Veldkamp [16], and, for an extension to infinite rank, Buekenhout [3] and Johnson [9]. As a consequence of the classification theorem of polar spaces (see [6], Theorem 3.34), we have that every polar space of rank at least three embedded in a projective space is isomorphic to a classical polar space.

Quadrics of a projective space $\mathrm{PG}(n, K)$ of finite dimension n over a field K have been intensively studied. Some results for elliptic quadrics in $\mathrm{PG}(3, q)$, q odd, have been obtained by Barlotti [1] and Panella [12] and, for general quadrics, by Tallini ([13] and [14]). In this context, the notion of Tallini set (name suggested by Lefévre [10]) naturally arose and there are several characterizations of quadrics as Tallini sets of desarguesian projective spaces satisfying suitable arithmetic or incidence conditions (see [4], [8], [11], [13], [14]). Let now $\mathbb{P}(K)$ be a projective space over a skew-field K. A *Tallini set* of $\mathbb{P}(K)$ is a proper subset $\mathscr{T}$ of points of $\mathbb{P}(K)$ spanning $\mathbb{P}(K)$ (i.e. $[\mathscr{T}] = \mathbb{P}(K)$) and such that every line of $\mathbb{P}(K)$ intersecting $\mathscr{T}$ in at least three points is contained in $\mathscr{T}$. From Theorem 1 of [4] and Theorem 1 of [2] we can obtain the following theorem.

Theorem 1.1. *Let $(\mathscr{P}, \mathscr{L})$ be a non-singular polar space. If $(\mathscr{P}, \mathscr{L})$ is a Tallini set of a projective space* $\mathrm{PG}(n, K)$ *of finite dimension n over a skew-field K, then K is a field and $(\mathscr{P}, \mathscr{L})$ is a quadric.*

In 1996, Lo Re, Melone and I characterized the incidence geometry of the Klein quadric $\mathscr{Q}^{+}(5, K)$ using the notion of Tallini set and a suitable property of the set $L^{\perp} \cap M^{\perp}$, for L and M non-collinear disjoint lines (see [8]). Hence it can be reasonably supposed that Tallini sets and $L^{\perp} \cap M^{\perp}$ are useful tools to characterize the incidence geometry of every quadric. In particular, first of all I obtain a characterization of polar spaces of rank at least three in terms of $L^{\perp} \cap M^{\perp}$, as the following theorem shows.

Theorem 1.2. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a non-singular connected semilinear space satisfying the following conditions*:

(A) *Every line contains at least three points and is not a maximal singular subspace of $\mathscr{S}$*;

(B) *For every pair L and M of non-collinear disjoint lines*, *the following hold*:
(B_1) $L \cap M^{\perp} \neq \emptyset$ *if*, *and only if*, $M \cap L^{\perp} \neq \emptyset$;
(B_2) *The subset* $\mathscr{P}_{L,M} := L^{\perp} \cap M^{\perp}$ *of* $\mathscr{P}$ *is a subspace of* $\mathscr{S}$.

Then $\mathscr{S}$ is a non-singular polar space of rank at least three.

In Section 2, after the proof of the previous theorem, I examine the relations between axioms (A) and (B) and the one-all axiom of Buekenhout and Shult, characterizing polar spaces. Precisely, reformulating the axioms, I want to highlight analogies and differences between axioms (A) and (B) and the one-all axiom. Moreover, in the new formulation, axiom (B) seems to be weaker than the one-all axiom.

In Section 3, I deal with the characterization of quadrics. In particular, I prove that, if (B) holds, the hypothesis that the whole space is a Tallini set is superfluous. Indeed, it is essentially sufficient to suppose that $L^{\perp} \cap M^{\perp}$ is "nearly" a Tallini set, for every pair L and M of non-collinear disjoint lines. Precisely, the following theorem is proved.

Theorem 1.3. *Let $(\mathscr{P}, \mathscr{L})$ be a non-singular connected semilinear space of rank at least three embedded in a projective space $\mathbb{P}(K)$ over a skew-field K. Suppose that $(\mathscr{P}, \mathscr{L})$ satisfies condition* (B) *and the following condition*:

(C) *For every pair L and M of non-collinear disjoint lines of $\mathscr{L}$, every line of $\mathbb{P}(K)$ containing at least three points of $\mathscr{P}_{L,M} = L^{\perp} \cap M^{\perp}$ is a line of $\mathscr{L}$.*

Then K is a field and $(\mathscr{P}, \mathscr{L})$ is a non-singular quadric of $\mathbb{P}(K)$.

The techniques of the proof of Theorem 1.3 allow me to prove the following extension of Theorem 1.1 of Buekenhout to the infinite-dimensional case. Precisely, the following result is proved.

Theorem 1.4. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a non-singular polar space. If $\mathscr{S}$ is a Tallini set of a projective space $\mathbb{P}(K)$ over a skew-field K, then K is a field and $\mathscr{S}$ is a quadric.*

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2 On Axioms (A) and (B) and the proof of Theorem 1.2

In this section, I want to discuss the two axioms (A) and (B), examining analogies and differences between them and the one-all axiom of Buekenhout and Shult characterizing polar spaces. Moreover, the proof of Theorem 1.2 is given.

First of all, we observe that every polar space of rank at least three satisfies axioms (A) and (B). Since the paper [3] contains a reduction of the case in which lines of lenght two occur to polar spaces all of whose lines have at least three points, it is not an essential restriction in the study of polar spaces to restrict to irreducible polar spaces.

Obviously, if $\mathscr{S}$ is an irreducible polar space of rank at least three, then its lines are not maximal singular subspaces, hence property (A) holds. Moreover, by the one-all axiom, for every subset X of points of $\mathscr{S}$, $X^{\perp}$ is a subspace of $\mathscr{S}$, hence property (B_2) holds, since every intersection of subspaces is still a subspace. The following proposition shows that property (B_1) holds for every pair of non-collinear lines of the polar space.

Proposition 2.1. *Let L and M be two non-collinear lines of a polar space. Then $L \cap M^{\perp} \neq \emptyset$ if, and only if, $M \cap L^{\perp} \neq \emptyset$.*

Proof. If L and M intersect at a point p, then $p = L \cap M^{\perp} = M \cap L^{\perp}$. Let L and M be two non-collinear disjoint lines and let p be a point of the line L collinear with M. By the one-all axiom, a point x of L different from p is collinear with exactly one point q of M, hence q is collinear with L, since it is collinear with x and p on L.

In the sequel, $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ will be a non-singular connected semilinear space satisfying properties (A) and (B).

Proposition 2.2. *For every pair L, M of non-collinear disjoint lines of $\mathscr{L}$, $\mathscr{P}_{L,M} = L^{\perp} \cap M^{\perp}$ is non-empty. Consequently, $\mathscr{P}_{L,M}$ contains at least one point external to $L \cup M$.*

Proof. By condition (A), the line L is not a maximal singular subspace of $\mathscr{S}$, thus there exists a maximal singular subspace S of $\mathscr{S}$ properly containing L. Since every point of S is collinear with L, if $M^{\perp} \cap S \neq \emptyset$, then $\mathscr{P}_{L,M}$ is non-empty. We prove that $M^{\perp} \cap S \neq \emptyset$. We proceed by induction on the distance $d := d(M,S)$. If $d = 0$, then M intersects S and we are done. Suppose that $d \geqslant 1$ and the induction hypothesis holds. Let $p_0, p_1, \ldots, p_d$ be points of $\mathscr{P}$ such that $p_i \sim p_{i+1}$, for $i = 1, \ldots, d$, $p_0 \in S$ and $p_d \in M$, and let M' be the line $p_{d-1} \vee p_d$ of $\mathscr{L}$. Since $d(M',S) = d - 1$, by induction hypothesis there exists a point $p \in M'^{\perp} \cap S$. It follows that $p \sim p_d$ and $d = 1$. Without loss of generality, we can assume $q = p_1 \in M$ and $M' = p \vee q$. Let L' be a line of S not through p. $M \cap L' = \emptyset$, since $d(M,S) = 1$. If $M \sim L'$, then $M^{\perp} \cap S \neq \emptyset$ and we are done. Let M and L' be non-collinear lines and consider the two disjoint lines L' and M'. If $M' \sim L'$, then the point q of M is collinear with L' and, from (B_1), there exists a point x of L' collinear with M, hence $x \in M^{\perp} \cap S$. Suppose that M' and L' are non-collinear lines. Since $p \in M' \cap L'^{\perp}$, by condition (B_1) there exists a point $p' \in L' \cap M'^{\perp}$. By condition ($B_2$), the line $L'' = p \vee p'$ is contained in $\mathscr{P}_{L',M'}$, hence it is collinear with the point q of M. Finally, if M and L'' are collinear, then $M^{\perp} \cap S \neq \emptyset$, otherwise, by condition (B_1), since q is a point of M collinear with L'', there exists a point q' on L'' collinear with M, i.e. $q' \in M^{\perp} \cap S$.

Now we can prove that $\mathscr{P}_{L,M}$ contains at least one point external to $L \cup M$. Since $\mathscr{P}_{L,M}$ is a non-empty subspace, it contains at least one point x. If $x \notin L \cup M$, we are done. Let us suppose that $x \in L$. By condition (B_1), there is a point $y \in M$ collinear with L, so x and y are two collinear points of the subspace $\mathscr{P}_{L,M}$, hence the line $x \vee y$ passing through them is contained in $\mathscr{P}_{L,M}$ and, by condition (A), $x \vee y$ contains at least one point different from x and y.

Using the previous result, now we are able to prove that the axioms (A) and (B) are equivalent to the one-all axiom. Precisely, the following result holds.

Theorem 2.3. *If* $(\mathscr{P}, \mathscr{L})$ *is a non-singular connected semilinear space satisfying axioms* (A) *and* (B), *then* $(\mathscr{P}, \mathscr{L})$ *is a polar space of rank at least three.*

Proof. In order to prove the one-all axiom, first we prove that if p is a point of $\mathscr{P}$ collinear with two distinct points a_1 and a_2 of a line L of $\mathscr{L}$, then p is collinear with L. Assume, on the contrary, that p is not collinear with L. By property (A), there exists a maximal singular subspace S of $(\mathscr{P}, \mathscr{L})$ properly containing L. Let q be a point of L not collinear with p, let M be a line of S passing through q and different from L and denote by R_1 and R_2 the lines $p \vee a_1$ and $p \vee a_2$, respectively. The lines M and R_1 are disjoint and non-collinear (since p and q are non-collinear points) and a_1 is a point of R_1 collinear with M. From (B), it follows that there exists a point $b_1 \in M \cap R_1^{\perp}$ and the line $T = a_1 \vee b_1$ is entirely contained in $M^{\perp} \cap R_1^{\perp}$. Since p and a_2 are two collinear points of $R_2^{\perp} \cap T^{\perp}$, from (B_2) it follows that the line R_2 is collinear with T and, in particular, a_1 is collinear with R_2. Let us consider now the two disjoint lines M and R_2. They are non-collinear, since p and q are non-collinear points, and a_1 and a_2 are two collinear points of $R_2^{\perp} \cap M^{\perp}$. From (B_2), it follows that the line $L = a_1 \vee a_2$ is contained in $R_2^{\perp} \cap M^{\perp}$, a contradiction since the point $p \in R_2$ is non-collinear with L.

Now, we prove that if (p, L) is a point-line pair with $L \not\subset p^{\perp}$, then $p^{\perp} \cap L \neq \emptyset$. Since $(\mathscr{P}, \mathscr{L})$ is connected, there exists a line M through p. If $M \cap L \neq \emptyset$, we are done. Otherwise L and M are non-collinear disjoint lines and, by Proposition 2.2, $L^{\perp} \cap M^{\perp}$ contains at least a point $q \notin L \cup M$. The lines L and $N = p \vee q$ are non-collinear, since p is non-collinear with L. If $L \cap N \neq \emptyset$, we are done. Otherwise, L and N are non-collinear disjoint lines and the point q of N is collinear with L, hence, from (B_1), there is a point x of L which is collinear with N. It follows that p is collinear with the point x of L.

By Theorem 2.3, the Theorem 1.2 is completely proved.

Now, we want to reformulate the given axioms in order to highlight the relations between the one-all axiom of Buekenhout and Shult and the axioms (A) and (B). It is easy to see that the first part of axiom (B) holds for either collinear or intersecting lines, too. It follows that the axiom (B_1) is well-reformulated as follows:

(F2) For every pair L, M of lines, $L^{\perp}$ intersects M if, and only if, $M^{\perp}$ intersects L.

In order to reformulate (B_2), we consider the following proposition.

Proposition 2.4. *Let* $(\mathscr{P}, \mathscr{L})$ *be a non-singular connected semilinear space satisfying axiom* (A). *Then* (B_2) *is equivalent to the following axiom*:

(F1) *For every line* L, $L^{\perp}$ *is a subspace of* $(\mathscr{P}, \mathscr{L})$.

Proof. Let us suppose that (B_2) holds and consider a line L of $\mathscr{L}$ and two distinct collinear points x and y of $L^{\perp}$. Let M be the line of $\mathscr{L}$ containing x and y. Since $L^{\perp}$ is a subspace if, and only if, M is collinear with L, by contradiction, let us suppose that M is not collinear with L. If the two non-collinear lines L and M were disjoint, since x and y are two collinear points of $\mathscr{P}_{L,M}$, from (B_2) it follows that the line M is collinear with L, a contradiction. Thus L and M intersect at a point. Without loss of generality, we can suppose that $x = L \cap M$. By property (A), there exists a maximal singular subspace S of $(\mathscr{P}, \mathscr{L})$ containing M, and L intersects S at the point x, since L is not collinear with M. For every point p of $S \backslash M$, if the two disjoint lines L and $N = p \vee y$ were non-collinear, since x and y are two collinear points of $\mathscr{P}_{L,N}$, from (B_2) it follows that M is collinear with L, a contradiction. Thus L and N are collinear, i.e. L is collinear with every point p of $S \backslash M$. Let q be a point of M not collinear with L and let R be a line of S passing through q and different from M. The two lines L and R are disjoint and non-collinear and, by property (A), the line R contains at least two points different from q, thus L is collinear with at least two points of R. From (B_2) it follows that L is collinear with R, a contradiction.

The converse is trivial, since every intersection of subspaces is still a subspace.

We have the following result.

Corollary 2.5. *Let* $(\mathscr{P}, \mathscr{L})$ *be a non-singular connected semilinear space satisfying axioms* (A) *and* (B). *Then* (B_1) *and* (B_2) *hold for every pair of lines of* $\mathscr{L}$. *Moreover, for every pair* L, M *of lines of* $\mathscr{L}$, $\mathscr{P}_{L,M} = L^{\perp} \cap M^{\perp}$ *is non-empty and contains at least one point external to* $L \cup M$.

Proof. If L and M are two collinear lines of $\mathscr{L}$, the statement is trivial. Moreover, if L and M are non-collinear disjoint lines, the statement follows from axiom (B) and Proposition 2.2. Let L and M be two non-collinear lines of $\mathscr{L}$ intersecting at a point p. Since $p \in \mathscr{P}_{L,M}$, (B_1) easily holds and $\mathscr{P}_{L,M}$ is non-empty. Moreover, $\mathscr{P}_{L,M}$ is a subspace, since, from the previous proposition, $L^{\perp}$ and $M^{\perp}$ are subspaces. In order to prove that $\mathscr{P}_{L,M}$ contains at least one point external to L and M, let us consider a maximal singular subspace S of $(\mathscr{P}, \mathscr{L})$ containing M. The line L intersects S at the point p, since L is not collinear with M. Let q be a point of M not collinear with L and let N be a line of S passing through q and different from M (the line N exists, since $(\mathscr{P}, \mathscr{L})$ has rank at least three). By the one-all axiom, a fixed point x of $L \backslash \{p\}$ is collinear with exactly one point y of $N \backslash \{q\}$, hence $y \in \mathscr{P}_{L,M} \backslash \{L, M\}$ and the proof is complete.

It is easy to see that the one-all axiom of Buekenhout and Shult is well-reformulated as follows: $\mathscr{S}$ is a polar space if, and only if, for every point p of $\mathscr{S}$, the following properties hold:

(BS1) $p^{\perp}$ is a subspace of $\mathscr{S}$;

(BS2) $p^{\perp}$ intersects every line of $\mathscr{S}$.

By Proposition 2.4 and Theorem 2.3, in the case of rank at least three the axioms are the following for every line L of $\mathscr{S}$:

(F1) $L^{\perp}$ is a subspace of $\mathscr{S}$;

(F2) $L^{\perp}$ intersects a line M if, and only if, $M^{\perp}$ intersects L.

In this way, it is clear that the axioms are expressed in terms of points in (BS1) and (BS2) and in terms of lines in (F1) and (F2), but axioms (F1) and (F2) seem to be weaker than axioms (BS1) and (BS2).

3 The proofs of Theorem 1.3 and Theorem 1.4

In this section, according to the hypotheses of Theorem 1.3, $(\mathscr{P}, \mathscr{L})$ will be a non-singular connected semilinear space of rank at least three embedded in a projective space $\mathbb{P}(K)$ over a skew-field K. By identifying the set $\mathscr{P}$ with its image under the embedding, without loss of generality, we can suppose that the points and the lines of $(\mathscr{P}, \mathscr{L})$ are points and lines of the projective space $\mathbb{P}(K)$, which is spanned by $\mathscr{P}$ (i.e. $[\mathscr{P}] = \mathbb{P}(K)$). Moreover, suppose that $(\mathscr{P}, \mathscr{L})$ satisfies conditions (B) and (C). Since the rank of $(\mathscr{P}, \mathscr{L})$ is at least three, the lines are not maximal singular subspaces. Finally, since every line of $\mathscr{L}$ is a line of $\mathbb{P}(K)$, every line contains at least three points. It follows that axiom (A) holds and, by Theorem 1.2, $(\mathscr{P}, \mathscr{L})$ is a non-singular polar space of rank at least three. Moreover, if L and M are two non-collinear disjoint lines of $\mathscr{L}$ and ℓ is a line of $\mathbb{P}(K)$ containing at least three points of $\mathscr{P}_{L,M}$, then, by condition (C), ℓ is a line of $\mathscr{L}$ and, since $(\mathscr{P}, \mathscr{L})$ is a polar space, the line ℓ is contained in $\mathscr{P}_{L,M}$.

The following proposition extends condition (C) even to the pairs of non-collinear intersecting lines of $(\mathscr{P}, \mathscr{L})$.

Proposition 3.1. *Let $(\mathscr{P}, \mathscr{L})$ be a non-singular connected semilinear space of rank at least three embedded in a projective space $\mathbb{P}(K)$ over a skew-field K and satisfying conditions* (B) *and* (C). *Then for every pair of non-collinear lines L and M of $\mathscr{L}$, every line of $\mathbb{P}(K)$ containing at least three points of $\mathscr{P}_{L,M} = L^{\perp} \cap M^{\perp}$ is a line of $\mathscr{L}$.*

Proof. Let L and M be two non-collinear lines intersecting at a point p, and let ℓ be a line of $\mathbb{P}(K)$ containing at least three pairwise distinct points a, b and c of $\mathscr{P}_{L,M}$. If $p \in \ell$, then $\ell \in \mathscr{L}$, since p is collinear with a. Hence, we can suppose that $p \notin \ell$. Let α be the plane of $\mathbb{P}(K)$ containing p and ℓ. By Theorem 1.2, $(\mathscr{P}, \mathscr{L})$ is a polar space, thus there exists a polarity π of $\mathbb{P}(K)$ whose absolute points are the points of $\mathscr{P}$ and whose totally isotropic lines are the lines of $\mathscr{L}$. The plane α is contained in the hyperplane $\pi(p)$ of $\mathbb{P}(K)$, since p, a and b lies on $\pi(p)$. Let M' be a line of $\mathscr{L}$ intersecting M at a point $q \neq p$, and non-collinear with M. The two lines L and M' are non-collinear, otherwise every point of L is collinear with the points p and q of M and, by the one-all axiom, L would be collinear with M, a contradiction. Moreover,

$L \cap M' = \emptyset$, otherwise q is collinear with L and hence L and M are collinear. The hyperplane $\pi(q)$ of $\mathbb{P}(K)$ contains the plane α, since the points p, a and b of α are collinear with q, and $\pi(q)$ contains $\pi(M')$, since $q \in M'$. It follows that $\pi(M') \cap \alpha$ is a line ℓ' and $p \notin \ell'$, otherwise $p \in \pi(M')$, i.e. p is collinear with M', a contradiction. The lines $p \vee a$, $p \vee b$ and $p \vee c$ of $\mathscr{L}$ are contained in α, thus they intersect the line ℓ' at three pairwise distinct points a', b' and c' of $\mathscr{P}$. The points a', b' and c' are collinear with L, since α is contained in $\pi(L)$. Moreover, a', b' and c' lie on the line ℓ' of $\pi(M')$, hence they are collinear with M'. By axiom (C), the line ℓ' is a line of $\mathscr{L}$ contained in $\mathscr{P}_{L,M'}$, hence the plane α of $\mathbb{P}(K)$ is a singular subspace of the polar space $(\mathscr{P}, \mathscr{L})$, since it contains the point p of $\mathscr{P}$ and the line ℓ' of $\mathscr{L}$ which is collinear with p. In particular, $\ell \in \mathscr{L}$.

In order to prove Theorems 1.3 and 1.4, we need two lemmas concerning non-singular polar spaces embedded in projective spaces. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a non-singular polar space of rank at least $t \geqslant 2$ whose points and lines are points and lines of a projective space $\mathbb{P}(K)$ over a skew-field K; without loss of generality, we can suppose that $[\mathscr{P}] = \mathbb{P}(K)$. It is easy to see that for every subspace X of $\mathbb{P}(K)$ which is not a singular subspace of $\mathscr{S}$, the incidence geometry $\mathscr{S}_X = (\mathscr{P}_X, \mathscr{L}_X)$ whose points (lines) are the points of $\mathscr{P}$ (the lines of $\mathscr{L}$) contained in X is a polar space embedded in X, and we say that $\mathscr{S}_X$ is the *polar space induced* by $\mathscr{S}$ on X. We have the following lemma.

Lemma 3.2. *Let X_0 be a finite-dimensional subspace of $\mathbb{P}(K)$ which is not a singular subspace of $\mathscr{S}$ and suppose that $X_0 \cap \mathscr{P}$ spans X_0. Then there exists a subspace X of $\mathbb{P}(K)$ containing X_0 and satisfying the following properties*:

(i) *X is a finite-dimensional subspace of* $\mathbb{P}(K)$;

(ii) *$X \cap \mathscr{P}$ spans X*;

(iii) *The polar space $\mathscr{S}_X$ induced by $\mathscr{S}$ on X is non-singular.*

Proof. Let $\mathscr{S}_0$ be the polar space induced by $\mathscr{S}$ on X_0. If the polar space $\mathscr{S}_0$ is non-singular, we are done, hence we can suppose that the singular subspace $R_0 := \mathrm{Rad}(\mathscr{S}_0)$ of $\mathscr{S}_0$ contains at least a point p_0. Since the polar space $\mathscr{S}$ is non-singular, $\mathscr{P}$ contains a point p_1 which is not collinear with p_0. Let X_1 be the subspace $p_1 \vee X_0$ of $\mathbb{P}(K)$. Clearly, X_1 satisfies (i) and (ii), since X_0 is a hyperplane of X_1 and $[X_1 \cap \mathscr{P}] = p_1 \vee [X_0 \cap \mathscr{P}] = p_1 \vee X_0 = X_1$. If the polar space $\mathscr{S}_1$ induced by $\mathscr{S}$ on X_1 is non-singular we are done, hence we can suppose that the singular subspace $R_1 := \mathrm{Rad}(\mathscr{S}_1)$ is not empty. R_1 does not contain p_0, since p_0 is not collinear with the point p_1 of $\mathscr{S}_1$. We prove that $R_1 \subset X_0$. By contradiction, let x be a point of $R_1 \backslash X_0$. The point x is different from p_1, otherwise it is $p_1 \sim p_0$. Moreover, the line $p_1 \vee x$ of X_1 intersects the hyperplane X_0 of X_1 at a point y. Since $p_1 \in \mathscr{S}_1$ and $x \in R_1$, the line $p_1 \vee x$ is a line of $\mathscr{S}_1$ and p_0 is collinear with $y \in \mathscr{S}_0$ (since $p_0 \in R_0$) and it is collinear with x (since $x \in R_1$). By the one-all axiom, p_0 is collinear with $p_1 \vee x$, hence with p_1, a contradiction. It follows that R_1 is contained in X_0, i.e. R_0 properly contains R_1, since $p_0 \in R_0 \backslash R_1$, thus the rank of the singular subspace R_1 is less than the rank of R_0. We can repeat the above construction by replacing $\mathscr{S}_0$ with $\mathscr{S}_1$ and, after at most $t := \mathrm{rank}(R_0) + 1$ times, the statement is proved.

As a consequence of the previous lemma, we have the following result.

Lemma 3.3. *Let p be a point of $\mathbb{P}(K)$ not on $\mathscr{P}$, q be a point of $\mathbb{P}(K)$ different from p and $\mathscr{S}_q$ be the family of all finite-dimensional subspaces X of $\mathbb{P}(K)$ satisfying the following conditions*:

(i) $p, q \in X$;

(ii) *The polar space $\mathscr{S}_X$ induced by $\mathscr{S}$ on X is non-singular and has finite rank at least t, $(2 \leqslant t \leqslant \mathrm{rank}(\mathscr{S}))$;*

(iii) $X \cap \mathscr{P}$ *spans* X.

Then $\mathscr{S}_q$ is not empty. Moreover, for every two subspaces $X_1, X_2 \in \mathscr{S}_q$ there exists a subspace $X \in \mathscr{S}_q$ such that $X_1 \cup X_2 \subseteq X$.

Proof. Since $[\mathscr{P}] = \mathbb{P}(K)$, p and q lie on a subspace of $\mathbb{P}(K)$ spanned by a finite set $\{p_1, \dots, p_k\}$ of points of $\mathscr{P}$. Moreover, since the rank of $\mathscr{S}$ is at least t, for every $i = 1, \dots, k$, we can consider a $(t-1)$-dimensional singular subspace T_i of $\mathscr{S}$ passing through p_i. Let X_0 be the subspace of $\mathbb{P}(K)$ spanned by $\bigcup_{i=1}^{k} T_i$. The finite-dimensional subspace X_0 of $\mathbb{P}(K)$ is not a singular subspace of $\mathscr{S}$, since it contains the point $p \notin \mathscr{P}$, and $[X_0 \cap \mathscr{P}] = X_0$. Note that the rank of the polar space $\mathscr{S}_0$ induced by $\mathscr{S}$ on X_0 is at least t, since $\mathscr{S}_0$ contains every $(t-1)$-dimensional singular subspace T_i of $\mathscr{S}$, for $i = 1, \dots, k$. By Lemma 3.2, $\mathscr{S}_q$ is not empty. Moreover, denote by X_1 and X_2 two subspaces of $\mathscr{S}_q$, and let X_0 be the subspace $X_1 \vee X_2$ of $\mathbb{P}(K)$. Then, by Lemma 3.2 again, there exists a subspace $X \in \mathscr{S}_q$ such that $X_1 \cup X_2 \subseteq X_0 \subseteq X$.

As an easy consequence of the one-all axiom and of some elementary properties of polar spaces, we have the following remark, in which a useful property of a polar space embedded in a finite-dimensional projective space is given.

Remark 3.4. Let $(\mathscr{P}, \mathscr{L})$ be a polar space embedded in a finite-dimensional projective space $\mathrm{PG}(n, K)$ over a skew-field K. Let ℓ be a line of $\mathrm{PG}(n, K)$ intersecting the set $\mathscr{P}$ in at least three distinct points x, y and z and not contained in $\mathscr{L}$ and let S be a maximal singular subspace of $(\mathscr{P}, \mathscr{L})$ passing through x. Then the hyperplanes $y^{\perp} \cap S$ and $z^{\perp} \cap S$ coincide.

Now, we are able to prove that every section of a non-singular polar space $\mathscr{S}$ embedded in a projective space $\mathbb{P}(K)$ and satisfying condition (C) with a suitable subspace X of $\mathbb{P}(K)$ which is not a singular subspace of $\mathscr{S}$ is a quadric of X. Precisely, the following proposition holds.

Proposition 3.5. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a non-singular connected semilinear space of rank at least three embedded in a projective space $\mathbb{P}(K)$ over a skew-field K. Suppose that*

$(\mathscr{P}, \mathscr{L})$ *satisfies conditions* (B) *and* (C). *Then there exists a finite-dimensional subspace* X *of* $\mathbb{P}(K)$ *such that the polar space* $\mathscr{S}_X = (\mathscr{P}_X, \mathscr{L}_X)$ *induced by* $\mathscr{S}$ *on* X *is a Tallini set of* X.

Proof. By Theorem 1.2, $\mathscr{S}$ is a polar space of rank at least three embedded in $\mathbb{P}(K)$. By Lemma 3.3, there exists a finite-dimensional subspace X of $\mathbb{P}(K)$ which is not a singular subspace of the polar space $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ and such that the polar space $\mathscr{S}_X$ induced by $\mathscr{S}$ on X is a non-singular polar space of finite rank $d \geqslant 3$ whose set of points $\mathscr{P}_X = \mathscr{P} \cap X$ is a proper subset of X spanning X. $\mathscr{S}_X$ is a Tallini set of X if, and only if, $\mathscr{P}_X$ contains every line of X with at least three points in common with it. Let ℓ be a line of X containing at least three distinct points x, y and z of $\mathscr{P}_X$, and suppose that ℓ is not contained in $\mathscr{P}_X$. Let S be a maximal singular subspace of $\mathscr{S}_X$ through x and, by Remark 3.4, let H be the hyperplane of S collinear with y and z and not containing x. Let T be a hyperplane of S passing through x. Hence $H \cap T$ is a $(d-3)$-dimensional subspace D of S; moreover, since $\mathscr{S}$ is a polar space, there exists a maximal singular subspace S' of $\mathscr{S}_X$ such that $S \cap S' = T$. By Remark 3.4 again, $y^{\perp} \cap S' = z^{\perp} \cap S'$ is a hyperplane H' of S' and $H \cap H' = D$. Since $d \geqslant 3$, D is not empty and we can consider two lines L of H and L' of H', not entirely contained in D. These two lines are not collinear, since every point of L is collinear with the hyperplane T of S' and L' is not contained in T, but $L^{\perp} \cap L'^{\perp}$ contains the three pairwise non-collinear points x, y and z, contradicting Proposition 3.1.

By Theorem 1.1 and Proposition 3.5, we have the following result.

Corollary 3.6. *K is a field and every non-singular polar space of rank at least three induced by $\mathscr{S}$ on a finite-dimensional subspace X of $\mathbb{P}(K)$ is a non-singular quadric of X.*

Using Lemmas 3.2 and 3.3 and Corollary 3.6, we can prove the following theorem.

Theorem 3.7 (Theorem 1.3). *$\mathscr{S} = (\mathscr{P}, \mathscr{L})$ is a non-singular quadric of the projective space $\mathbb{P}(K)$ over the field K.*

Proof. By Corollary 3.6, K is a field. Let $V(K)$ be a vector space over the field K and let $\rho : V(K)\backslash\{\mathbf{0}\} \to \mathbb{P}(K)$ the projective structure of $\mathbb{P}(K)$, i.e. the surjective map such that, for every $\boldsymbol{u}, \boldsymbol{v} \in V(K)\backslash\{\mathbf{0}\}$, $\rho(\boldsymbol{u}) = \rho(\boldsymbol{v})$ if, and only if, there exists $k \in K\backslash\{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$. Let $p = \rho(\boldsymbol{u})$ be a fixed point of $\mathbb{P}(K)\backslash\mathscr{S}$ and q be a point of $\mathbb{P}(K)$ different from p. By Lemma 3.3 and Corollary 3.6, there exists a finite-dimensional subspace $X \in \mathscr{S}_q$ of $\mathbb{P}(K)$ containing p and q and such that the polar space $\mathscr{S}_X$ induced by $\mathscr{S}$ on X is a non-singular quadric of X. Denote by U the subspace of $V(K)$ such that $X = \rho(U\backslash\{\mathbf{0}\})$. There exists an $(\mathrm{id}_K, 1)$-quadratic form $\varphi_X : U \to K$ inducing a polarity π_X of X whose absolute points and totally isotropic lines (totally singular lines, in the case of a pseudo-quadratic form over a field of characteristic 2) define the quadric $\mathscr{S}_X$. Since φ_X is unique up to proportionality, we can suppose that $\varphi_X(\boldsymbol{u}) = 1$. Let $\varphi : V(K) \to K$ be the map defined by $\varphi(\boldsymbol{w}) = \varphi_X(\boldsymbol{w})$ for every $\boldsymbol{w} = \rho^{-1}(q)$ (and $\varphi(\mathbf{0}) = 0$). By Lemma 3.3 the map φ is well defined. Indeed, let

$X_1 = \rho(U_1 \setminus \{\mathbf{0}\})$ and $X_2 = \rho(U_2 \setminus \{\mathbf{0}\})$ be two subspaces of $\mathscr{S}_q$ and let $\varphi_1 : U_1 \to K$ and $\varphi_2 : U_2 \to K$ be the $(\mathrm{id}_K, 1)$-quadratic forms defining the quadrics $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively. Moreover, suppose that $\varphi_1(\boldsymbol{u}) = \varphi_2(\boldsymbol{u}) = 1$. By Lemma 3.3, there exists a finite-dimensional subspace $Y = \rho(W \setminus \{\mathbf{0}\})$ such that $X_1 \cup X_2 \subseteq Y$, hence the $(\mathrm{id}_K, 1)$-quadratic form $\varphi_Y : W \to K$ induces φ_1 on X_1 and φ_2 on X_2, since φ_1, φ_2 and φ_Y coincide on $\boldsymbol{u}$. It follows that $\varphi_1(\boldsymbol{w}) = \varphi_2(\boldsymbol{w}) = \varphi_Y(\boldsymbol{w})$ for $\boldsymbol{w} \in V(K) \setminus \{\mathbf{0}\}$ such that $q = \rho(\boldsymbol{w})$. Thus, the map $\varphi : V(K) \to K$ is an $(\mathrm{id}_K, 1)$-quadratic form on $V(K)$ inducing a polarity π of $\mathbb{P}(K)$ whose absolute points and totally isotropic lines (totally singular lines, in the case of characteristic 2) are points and lines of $\mathscr{S} = (\mathscr{P}, \mathscr{L})$, i.e. $\mathscr{S}$ is a quadric of $\mathbb{P}(K)$.

Finally, we are able to prove Theorem 1.4, extending the Theorem 1.1 of Buekenhout to the infinite-dimensional case.

Theorem 3.8 (Theorem 1.4). *Let $\mathscr{S}$ be a non-singular polar space. If $\mathscr{S}$ is a Tallini set of a projective space $\mathbb{P}(K)$ over a skew-field K, then K is a field and $\mathscr{S}$ is a quadric.*

Proof. By Lemma 3.3, we can consider a finite-dimensional subspace X of $\mathbb{P}(K)$ which is not a singular subspace of the polar space $\mathscr{S}$ and such that the polar space $\mathscr{S}_X$ induced by $\mathscr{S}$ on X is a non-singular polar space of finite rank at least two whose set of points is a proper subset of X spanning X. Clearly, $\mathscr{S}_X$ is a Tallini set of X, thus, by Theorem 1.1 of Buekenhout, K is a field and $\mathscr{S}_X$ is a non-singular quadric of X. It follows that we can extend to $\mathbb{P}(K)$ the orthogonal polarity defining the quadric $\mathscr{S}_X$ of X as in the proof of Theorem 3.7, and the statement is proved.

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