

On surfaces with $p_g = q = 2$ and non-birational bicanonical maps

Ciro Ciliberto and Margarida Mendes Lopes

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Abstract. The present paper is devoted to the classification of irregular surfaces of general type with $p_g = q = 2$ and non-birational bicanonical map. The main result is that, if S is such a surface and if S is minimal with no pencil of curves of genus 2, then S is a double cover of a principally polarized abelian surface (A, Θ) , with Θ irreducible. The double cover $S \rightarrow A$ is branched along a divisor $B \in |2\Theta|$, having at most double points and so $K_S^2 = 4$.

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1 Introduction

If a smooth surface S of general type has a pencil of curves of genus 2, i.e. it has a morphism to a curve whose general fibre F is a smooth irreducible curve of genus 2, then the line bundle $\mathcal{O}_S(K_S) \otimes \mathcal{O}_F$ is the canonical bundle on F , and therefore the bicanonical map ϕ of S cannot be birational. Since this property is, of course, of a birational nature, the same remark applies if S has a rational map to a curve whose general fibre is an irreducible curve with geometric genus 2.

We call this exception to the birationality of the bicanonical map ϕ the *standard case*. A *non-standard case* will be the one of a surface of general type S for which ϕ is not birational, but there is no pencil of curves of genus 2. The classification of the non-standard cases has a long history and we refer to the expository paper [8] for information on this problem. We will just mention here the fact that the non-standard cases with $p_g \geq 4$ are all regular.

The classification of non-standard irregular surfaces has been considered by Xiao Gang in [24] and by F. Catanese and the authors of the present paper in [6]. Xiao Gang studied the general problem of classifying the non-standard cases by taking the point of view of the projective study of the image of the bicanonical map. The outcome of his analysis is a list of numerical possibilities for the invariants of the cases which might occur. More precise results have been obtained in [6], where the first significant case $p_g = 3$ has been considered. Indeed in [6] it is shown, among other things, that a minimal irregular surface S with $p_g = 3$ presents the non-standard case

if and only if S is isomorphic to the symmetric product of a smooth irreducible curve of genus 3, thus $p_g = q = 3$ and $K^2 = 6$.

In the present paper we study this problem for surfaces with $p_g = q = 2$ and we prove the following result, which rules out a substantial number of possibilities presented in [24]:

Theorem 1.1. *Let S be a minimal surface of general type with $p_g = q = 2$. Then S presents the non-standard case if and only if S is a double cover of a principally polarized abelian surface (A, Θ) , with Θ irreducible. The double cover $S \rightarrow A$ is branched along a symmetric divisor $B \in |2\Theta|$, having at most double points. One has $K_S^2 = 4$.*

Surfaces with $p_g = q = 2$ are still far from being understood. The list of known examples of surfaces of general type with $p_g = q = 2$ is relatively small (see [25], [26]) and there are several constraints for their existence. Here we only mention that there are various restrictions for the existence of a genus 2 fibration (see [23]) and also that M. Manetti, working on the Severi conjecture, showed in particular that if $p_g = q = 2$, K_S is ample and $K_S^2 = 4$ then S is a double cover of its Albanese image (see [16]).

To prove our classification Theorem 1.1 we first show that the degree of the bicanonical map is 2 for surfaces presenting the non-standard case, then we study the possibilities for the quotient surface by the involution induced by the bicanonical map, and finally we show that the unique case which really occurs is the one described above. We use a diversity of techniques, which may be useful in other contexts.

The paper is organized as follows. In Section 2 we list the properties of surfaces S with $p_g = q = 2$ that we need. In Section 3 we characterize, by a small adaptation of a proof in [6], the surfaces S presenting the non-standard case with $K_S^2 = 9$, and in particular we verify that there is no such surface with $p_g = q = 2$. In Section 4 we establish some properties of the paracanonical system and then we use these results in Section 5 to conclude that for the non-standard cases S with $p_g = q = 2$ the degree of the bicanonical map is 2. Thus there is an involution i induced by the bicanonical map on S . We consider the quotient surface $\tilde{S} := S/\langle i \rangle$ and the projection map $p : S \rightarrow \tilde{S}$. In Section 6 we discuss the various possibilities for \tilde{S} , showing that the only one which can really occur is that \tilde{S} is a minimal surface of general type with $p_g(\tilde{S}) = 2$, $q(\tilde{S}) = 0$, $K_{\tilde{S}}^2 = 2$ and with 20 nodes. Moreover we show that the double cover p ramifies exactly over the 20 nodes. Finally in Section 7, using this description, and some results on Prym varieties contained in [10], we finally prove Theorem 1.1.

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We dedicate this paper to the memory of Paolo Francia, with whom we first started on this subject.

Notation and conventions. We work over the complex numbers. All varieties are assumed to be compact and algebraic. We do not distinguish between line bundles and divisors on a smooth variety, using the additive and the multiplicative notation interchangeably. Linear equivalence is denoted by \equiv and numerical equivalence by \sim . A *node* on a surface is an ordinary double point (i.e. a singularity of type A_1). The exceptional divisor of a minimal desingularization of a node is a rational irreducible curve A with $A^2 = -2$, usually called a (-2) -curve.

As already mentioned, we will say that a surface S of general type *presents the non-standard case*, or that it is a *non-standard case*, if S has no pencil of curves of geometric genus 2 and the bicanonical map of S is not birational.

The remaining notation is standard in algebraic geometry.

2 Some properties of surfaces with $p_g = q = 2$

The minimal surfaces S of general type with $p_g = q = 2$ have various interesting properties (cf. [25], [26]). In this section we only mention those that we will need further on.

Proposition 2.1. *Let S be a minimal surface of general type with $p_g = q = 2$. Then:*

- i) $4 \leq K_S^2 \leq 9$;
- ii) *if $K_S^2 = 8, 9$, there are no rational smooth curves on S (in particular $\mathcal{O}_S(K_S)$ is ample), and, if $K_S^2 = 7$ and $\mathcal{O}_S(K_S)$ is not ample, then S contains either one irreducible (-2) -curve, or two forming an A_2 configuration. Furthermore if $K_S^2 = 9$, S does not contain elliptic curves.*

Proof. i) The first inequality follows from the inequality $K^2 \geq 2p_g$ for minimal irregular surfaces (see [11]), and the second from the inequality $K^2 \leq 3c_2$.

ii) follows from Miyaoka's and Sakai's inequalities (see [18] and [22]) for the number of rational or elliptic curves on a non-ruled minimal surface. □

We will also need to consider the Albanese image of these surfaces. First we recall the following facts which we will use repeatedly:

Lemma 2.2 (see [2], p. 343; [1], p. 97). *Let S be a minimal surface and let $f : S \rightarrow B$ be a genus $b := g(B)$ pencil of curves of genus $g \geq 2$. Then*

- i) $K_S^2 \geq 8(g - 1)(b - 1)$,
- ii) $c_2(S) \geq 4(g - 1)(b - 1)$ and
- iii) $q \leq g + b$.

Furthermore if equality holds in i) then the curves of the pencil have constant modulus,

if equality holds in ii) every fibre of f is smooth, and if equality holds in iii) S is birationally equivalent to a product of B with the general fibre of f .

Using this lemma we obtain the following:

Proposition 2.3. *Let S be a minimal surface of general type with $p_g = q = 2$ for which the Albanese morphism $a : S \rightarrow A := \text{Alb}(S)$ is not surjective. Then $a(S) = B$ is a genus 2 curve, the Albanese pencil $a : S \rightarrow B$ has smooth, connected fibres F of genus 2 with constant modulus and $K_S^2 = 8$.*

Proof. Since $q(S) = 2$, the Albanese image of S is a genus 2 curve B . Then the remainder of the assertion is a consequence of Lemma 2.2 and $\chi(\mathcal{O}_S) = 1$. \square

Corollary 2.4. *Let S be a minimal surface of general type with $p_g = q = 2$. If ω, ω' are two 1-forms which generate $H^0(S, \Omega_S^1)$ and $\omega \wedge \omega' \equiv 0$, then the Albanese morphism $a : S \rightarrow A := \text{Alb}(S)$ is not surjective, the Albanese pencil $a : S \rightarrow B$ has smooth, connected fibres F of genus 2 with constant modulus and $K_S^2 = 8$.*

Proof. The assertion follows from the theorem of Castelnuovo–De Franchis (see e.g. [1], p. 123) and the previous proposition. \square

Finally we notice that, if the surface S of general type with $p_g = q = 2$ has a genus 2 fibration, then the canonical system is not composed with the genus 2 fibration (see [23], Theorem 2.1, p. 16, and Theorem 5.1, p. 71). As a consequence we have:

Proposition 2.5. *Let S be a minimal surface of general type with $p_g = q = 2$ and write $|K_S| = |M| + Z$, where $|M|$ is the moving part of $|K_S|$ and Z the fixed part. Then the general curve in $|M|$ is irreducible.*

Proof. Assume otherwise. Then $|M|$ is composed with an irrational pencil \mathcal{P} . If F is a generic fibre of \mathcal{P} , $|M| = aF$ where $a \geq 2$, and furthermore $F^2 = 0$. Since F is not a genus 2 curve, $K_S \cdot F \geq 4$. Since $K_S^2 \leq 9$, we see that either $K_S \cdot Z = 1$, $K_S^2 = 9$ or $K_S \cdot Z = 0$, $K_S^2 = 8$. This cannot occur. Indeed, in the former case S would contain a curve θ with $K_S \cdot \theta = 1$, hence θ would be rational or elliptic, whereas in the latter case S would contain a (-2) -curve. In either case we would have a contradiction to Proposition 2.1, ii). \square

3 The case $K_S^2 = 9$

In [21] I. Reider proved that if S is a minimal surface of general type with $K_S^2 \geq 10$ and the bicanonical map is not birational, then S presents the standard case. In Proposition (1.1) of [6], it is proven that the same holds if $K_S^2 = 9$ and $p_g \geq 3$, unless $p_g = 6$, $K_S^2 = 9$, and S is the Du Val–Bombieri surface described in [12] and in [3], p. 193. In fact this result can be extended:

Proposition 3.1. *Let S be a minimal surface of general type with $K_S^2 = 9$ such that the bicanonical map is not birational. Assume that S presents the non-standard case. Then $p_g = 6$, $q = 0$ and S is the Du Val–Bombieri surface.*

Proof. To prove the assertion it suffices to use the proof of Proposition (1.1) of [6]. There, the assumption $p_g \geq 3$ is only necessary for the proof of Claim 4. But Claim 4 can be proved without using the assumption on p_g . In fact, since $K_S - D \sim 2D$ is big and nef, Mumford’s vanishing theorem (see [19], p. 250) yields $h^1(S, \mathcal{O}_S(2K_S - D)) = 0$. Thus the map $H^0(S, \mathcal{O}_S(2K_S)) \rightarrow H^0(D, \mathcal{O}_D(2K_S))$ is surjective, which in turn implies that D is hyperelliptic. \square

4 The paracanonical system in the case $p_g = q = 2$

Let S be a minimal irregular surface of general type. If $\eta \in \text{Pic}^0(S)$ is a point, we can consider the linear system $|K_S + \eta|$. A curve in $|K_S + \eta|$ is a *paracanonical curve* on S .

Assume that the Albanese image of S is a surface. Given a general point $\eta \in \text{Pic}^0(S)$, one has, by [14], Theorem 1, $h^1(S, \mathcal{O}_S(\eta)) = 0$ and $\dim|K_S + \eta| = \chi(\mathcal{O}_S) - 1$.

For $\eta \in \text{Pic}^0(S)$, let C_η be the general curve in $|K_S + \eta|$. The curves C_η describe, for $\eta \in \text{Pic}^0(S)$ a general point, a continuous system \mathcal{H} of curves on S , of dimension $q + \dim|K_S + \eta| = q + \chi(\mathcal{O}_S) - 1 = p_g$. This is what we will call the *main paracanonical system* of S .

Assume now that S is a minimal surface of general type with $p_g = q = 2$, for which the Albanese map $a : S \rightarrow A := \text{Alb}(S)$ is surjective. The main paracanonical system of S has dimension 2 and, for $\eta \in \text{Pic}^0(S)$ a general point, the curve $C_\eta \in |K_S + \eta|$ is linearly isolated. We write $C_\eta = F + M_\eta$, where F is the fixed part of the continuous system \mathcal{H} and M_η the movable part, and we denote by \mathcal{M} the continuous, 2-dimensional system described by the curve $M := M_\eta$. This system is parametrized by a surface P which is birational to $\text{Pic}^0(S)$.

Lemma 4.1. *Let S be a minimal surface of general type with $p_g = q = 2$ presenting the non-standard case. Let $C_\eta = F + M_\eta$ be the general paracanonical curve. Then either:*

- (i) $M := M_\eta$ is irreducible and $M^2 \geq 3$, or
- (ii) $F = 0$ and M is reducible as $M = M_1 + M_2$, with M_1 and M_2 irreducible each varying in two 1-dimensional systems of curves $\mathcal{M}_1, \mathcal{M}_2$. The following possibilities can occur:
 - (a) $M_1^2 = M_2^2 = 0$, $M_1 \cdot M_2 = 4$, $K_S^2 = 8$
 - (b) $M_1^2 = M_2^2 = M_1 \cdot M_2 = 2$, $M_1 \sim M_2$, $K_S^2 = 8$.

Proof. Suppose that M is irreducible. Then $M^2 > 0$, otherwise \mathcal{M} is a pencil, whereas we know it has dimension 2. The case $M^2 = 1$ is excluded by Proposition (0.14, iii) of [6]. The case $M^2 = 2$ is also excluded by Theorem (0.20) of [6]. This proves (i).

Suppose that M is reducible. Since \mathcal{M} is a two-dimensional system parametrized by $\text{Pic}^0(S)$, M must consist of two distinct irreducible components $M = M_1 + M_2$.

Suppose $M_i^2 = 0$ for one of $i = 1, 2$. Then M_i varies in a pencil \mathcal{M}_i of curves of genus at least 3 and so $K_S \cdot M_i \geq 4$. If instead $M_i^2 > 0$, then, by Proposition (0.18) of

[6], $M_i^2 \geq 2$ and one has again $K_S \cdot M_i \geq 4$, by the 2-connectedness of the paracanonical curves. In both cases

$$K_S^2 = K_S \cdot F + K_S \cdot M_1 + K_S \cdot M_2 \geq 8,$$

and, so, by Proposition 3.1, one has $K_S^2 = 8$ and $K_S \cdot M_1 = K_S \cdot M_2 = 4$, $K_S \cdot F = 0$. Since S does not contain (-2) -curves, one has $F = 0$ and we have the two numerical possibilities listed in (ii). □

Lemma 4.2. *Let S be a minimal surface of general type with $p_g = q = 2$ presenting the non-standard case and let $C_\eta = F + M_\eta$ be as in case i) of Lemma 4.1. Then:*

- i) if $F \neq 0$, then $F \cdot M_\eta = 2$ and F is 1-connected;
- ii) if $F \neq 0$ and η is general, the image of the restriction map

$$H^0(S, \mathcal{O}_S(2K_S)) \rightarrow H^0(M_\eta, \mathcal{O}_{M_\eta}(2K_S))$$

has codimension at most 1 in $H^0(M_\eta, \mathcal{O}_{M_\eta}(2K_S))$.

Proof. i) Let $M := M_\eta$. If $F \cdot M = 2$, the 2-connectedness of the canonical divisors and Lemma (A.4) of [9] implies that F is 1-connected. To show that $F \cdot M = 2$ first we claim that $F \cdot M \leq 4$. Indeed, Proposition 3.1 yields $K_S^2 \leq 8$ and Lemma 4.1, i) yields $M^2 \geq 3$. Therefore

$$8 \geq K_S^2 \geq K_S \cdot M = M^2 + F \cdot M \geq 3 + F \cdot M.$$

So $F \cdot M$ being even implies $F \cdot M \leq 4$. Now we show that $F \cdot M = 4$ cannot occur. Suppose otherwise. Then from $8 \geq K_S^2 = M^2 + 8 + F^2$ and $K_S \cdot M = M^2 + M \cdot F \geq 7$ we have the possibilities:

- a) $K_S^2 = 7, K_S \cdot F = 0, F^2 = -4$,
- b) $K_S^2 = 8, K_S \cdot F = 1, F^2 = -3$ or
- c) $K_S^2 = 8, K_S \cdot F = 0, F^2 = -4$.

The first possibility implies that F contains two disjoint (-2) -curves, whilst the second and third imply that F contains a smooth rational curve. This is impossible by Proposition 2.1, ii). Therefore $F \cdot M = 2$ and so F is 1-connected.

ii) Note that, since $H^1(S, \mathcal{O}_S(2K)) = 0$, the codimension of the image of the restriction map

$$H^0(S, \mathcal{O}_S(2K)) \rightarrow H^0(M_\eta, \mathcal{O}_{M_\eta}(2K))$$

is exactly $h^1(S, \mathcal{O}_S(K_S + F - \eta))$, which by duality is equal to $h^1(S, \mathcal{O}_S(\eta - F))$. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_S(\eta - F) \rightarrow \mathcal{O}_S(\eta) \rightarrow \mathcal{O}_F(\eta) \rightarrow 0$$

which yields the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(S, \mathcal{O}_S(\eta - F)) &\rightarrow H^0(S, \mathcal{O}_S(\eta)) \rightarrow H^0(F, \mathcal{O}_F(\eta)) \\ &\rightarrow H^1(S, \mathcal{O}_S(\eta - F)) \rightarrow H^1(S, \mathcal{O}_S(\eta)) \rightarrow \dots \end{aligned}$$

Since $h^0(S, \mathcal{O}_S(\eta)) = 0$ and $h^1(S, \mathcal{O}_S(\eta)) = 0$, for η general (by [14], Theorem 1), we see that $h^1(S, \mathcal{O}_S(\eta - F)) = h^0(F, \mathcal{O}_F(\eta))$. Now $\mathcal{O}_F(\eta)$ has degree 0 on every component of F . By the first part of the lemma, F is 1-connected and so by Corollary (A.2) of [9], $h^0(F, \mathcal{O}_F(\eta)) \leq 1$ (with equality holding if and only if $\mathcal{O}_F(\eta) \simeq \mathcal{O}_F$). \square

5 The degree of the bicanonical map

In the present section we prove the following result:

Proposition 5.1. *Let S be a minimal surface of general type with $p_g = q = 2$. Assume that S presents the non-standard case. Then the degree σ of the bicanonical map is 2.*

Remark 5.2. For completeness let us point out that if S has a genus 2 fibration then the degree σ of the bicanonical map is either 2 or 4, (see [23]) and $\sigma = 4$ does occur (cf. Remark 7.2).

First of all we treat the case $K_S^2 = 8$, adapting a proof which appears in [17].

Proposition 5.3. *Let S be a minimal surface of general type with $p_g = q = 2$ and $K_S^2 = 8$ presenting the non-standard case. Then the degree σ of the bicanonical map is 2.*

Proof. Let ϕ be the bicanonical map of S . Notice that $(2K_S)^2 = 4K_S^2 = 32$ and $h^0((S, \mathcal{O}_S(2K_S)) = K_S^2 + 1 = 9$. Then the degree of $\Sigma =: \phi(S)$ is $\frac{32}{\sigma} \geq 7$, hence σ is either 2 or 4.

Suppose $\sigma = 4$. In this case Σ is a surface of degree 8 in \mathbb{P}^8 . The list of such surfaces is known (see [20], Theorem 8). Since $|2K_S|$ is a complete linear system, Σ can be one of the following:

- a) the Veronese embedding in \mathbb{P}^8 of a quadric in \mathbb{P}^3 ;
- b) a Del Pezzo surface, i.e. the image of \mathbb{P}^2 by the rational map associated to the linear system $|3L \otimes \mathcal{I}_{x|\mathbb{P}^2}|$, where L is a line and x is a point of \mathbb{P}^2 ;
- c) a cone over an elliptic curve of degree 8 in \mathbb{P}^7 .

We are going to prove the result by showing that none of these cases can occur. First we consider case c). Take the pull back F of a line in the cone. Then $2K_S \cdot F = 4$, hence $K_S \cdot F = 2$. The index theorem then yields $F^2 = 0$, and therefore we would have a genus 2 pencil on S .

In case a) $2K_S \equiv 2H$, where H is the pull back of the hyperplane section of Σ . Then $\eta = H - K_S$ is a nontrivial 2-torsion element in $\text{Pic } S$, since $p_g(S) = 2$ whereas $h^0(S, \mathcal{O}_S(K_S + \eta)) = 4$. The étale double cover $\pi : Y \rightarrow S$ given by $2\eta \equiv 0$ has invari-

ants $\chi(Y) = 2, K_Y^2 = 16$. In addition $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(K_S + \eta)) = 6$ so that $q(Y) = 5$. Then, since $q(S) = 2$, the subspace V^- of $H^0(Y, \Omega_Y^1)$ containing the anti-invariant 1-forms by the involution ι determined by $\pi : Y \rightarrow S$ has dimension 3. Since the image of $\bigwedge^2 V^-$ in $H^0(Y, \Omega_Y^2)$ is contained in the subspace of invariant 2-forms which is 2-dimensional, we conclude that there are two independent 1-forms $\omega, \omega' \in V^-$ such that $\omega \wedge \omega' \equiv 0$ and so by the theorem of Castelnuovo–De Franchis there exists a fibration $g : Y \rightarrow B$ with $b := g(B) \geq 2$ (cf. also [5], Corollary (4.8)).

Let f be the genus of a general fibre F of g . Suppose $f = 2, b \geq 3$. Then the curve $F' = \iota(F)$ cannot dominate B via g . Hence F' is again a curve of the pencil $g : Y \rightarrow B$. It cannot be the case that $F' = F$, otherwise $\pi(F)$ would be a moving curve of genus 0 or 1 on S , a contradiction. In conclusion $F \neq F'$ and $\pi(F) = \pi(F')$ would be a curve of genus 2 on S varying in a pencil, a contradiction.

Now, by Lemma 2.2, i), we have $K_Y^2 = 16 \geq 8(f - 1)(b - 1)$ and, by Lemma 2.2, ii), $5 = q(Y) \leq f + b$. This forces $f = 3, b = 2$ or viceversa and so Y is birational to $B \times F$ (see again Lemma 2.2, ii)). Hence Y has a pencil of curves of genus 2, whose image on S , by what we observed above, is again a genus 2 pencil, against our hypothesis. Thus also case a) does not occur.

Finally we consider case b). We abuse notation and we denote by L the image on Σ of a line of \mathbb{P}^2 . Let $2L + L_0$ be the hyperplane section of Σ . We have $2K_S \equiv \phi^*(2L + L_0)$, and so $\phi^*(L_0) \equiv 2(K_S - \phi^*L)$.

Choose L_0 such that $\phi^*(L_0)$ is a smooth irreducible curve and consider the double cover Y of S branched over $\phi^*(L_0)$ and determined by $K_S - \phi^*(L)$. The double cover formulas give $\chi(Y) = 3, K_Y^2 = 24, p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - \phi^*L)) = 7$, so that $q(Y) = 5$.

Notice that $|\phi^*(L_0)|$ is a genus 3 pencil on S . The pull back of it to Y is either a rational pencil of curves of genus 5, or a genus 3 pencil. In the former case Y would be birational to the product of \mathbb{P}^1 by a curve of genus 5 (see again Lemma 2.2), which is not possible. In the other case let b be the genus of the base curve of the pencil. As before $b \geq 2$, because $b + 3 \geq q(Y) = 5$. On the other hand Lemma 2.2, i) yields $K_Y^2 = 24 \geq 16(b - 1)$. Hence $b = 2$ and as above we conclude that Y is birational to a product of a genus 2 and a genus 3 curve, which is impossible because $p_g(Y) = 7$. □

Before continuing towards the proof of Proposition 5.1 we need to recall some facts about continuous systems of curves on a surface. For the basic definitions, we refer the reader to [6], §0. Given an irreducible, continuous system \mathcal{C} of curves of dimension r on a surface S , the *index* $v := v_{\mathcal{C}}$ of \mathcal{C} is the number of curves of \mathcal{C} passing through r general points of S . Of course $v \geq 1$. A system \mathcal{C} is called an *involution* if its index is $v = 1$. Typical examples of involutions are:

- (i) the linear systems;
- (ii) pencils, or, more generally systems *composed with pencils*. This means that there is a pencil $f : S \rightarrow B$ and an involution of divisors on B such that the curves of \mathcal{C} are pull-backs via f of divisors of an involution on B .

The classical theorem of Castelnuovo–Humbert tells us that these are essentially the only involutions.

Theorem 5.4 (Castelnuovo–Humbert, see [7], §5). *Let S be a smooth, irreducible, projective surface and let \mathcal{C} be an r -dimensional involution on S which has no fixed divisor and whose general divisor C is reduced. Then either \mathcal{C} is a linear system or it is composed with a pencil.*

We will use this theorem to prove the following basic result:

Proposition 5.5. *Let S be a minimal surface of general type with $p_g = q = 2$. Assume that S presents the non-standard case. Let $C_\eta = F + M_\eta$ be the general paracanonical curve and suppose that $M := M_\eta$ is irreducible. Then the restriction of the bicanonical map ϕ to M is a birational map of M onto its image.*

Proof. First we consider the case $F = 0$. Then the arithmetic genus g of M is $g = K_S^2 + 1$. Since, by [14], Theorem 1, $h^1(S, \mathcal{O}_S(K_S + \eta)) = 0$ for a general point $\eta \in \text{Pic}^0(S)$, $|2K_S|$ cuts out on $M = M_\eta$ a non-special, base point free complete g_{2g-2}^{g-2} . We will argue by contradiction and we will suppose from now on that this series is composed with an involution $\tau := \tau_M$ of degree $\delta \geq 2$. Then we must have $2g - 2 \geq \delta(g - 2)$ which yields $\delta \leq 2 + \frac{2}{g-2} = 2 + \frac{2}{K_S^2 - 1}$. Since, by Proposition 2.1, one has $K_S^2 \geq 4$, we see that $\delta = 2$. This means that $\phi(M)$ is a linearly normal curve of degree $g - 1$ in \mathbb{P}^{g-2} , whose arithmetic genus is 1. Notice that two distinct points x, x' are conjugated in τ_M if and only if $\phi(x) = \phi(x')$.

Claim 1: *Let M, M' be general curves in \mathcal{M} , then $M \cap M'$ does not contain four distinct points x, y, x', y' such that $\phi(x) = \phi(x')$ and $\phi(y) = \phi(y')$.*

Otherwise we would have $h^0(M, \mathcal{O}_M(M')) \geq h^0(M, \mathcal{O}_M(x + x' + y + y')) = 2$. On the other hand, since $h^1(S, \mathcal{O}_S(\eta)) = 0$ for $\eta \in \text{Pic}^0(S)$ a general point, $|M'|$ cuts out a complete linear series on M . Since M' is linearly isolated, we find a contradiction.

Let x be a point on S . We denote by \mathcal{M}_x the system of curves in \mathcal{M} passing through x .

Claim 2: *Let x and x' be general points on M conjugated in τ , i.e. such that $\phi(x) = \phi(x')$. Every irreducible component of \mathcal{M}_x is a 1-dimensional system of curves. Consider the union of all of these components containing M . Every curve in such a union contains x' .*

Let M'' be the general curve in a component \mathcal{M}' of the union in question and let $x_{M''}$ be the point conjugated to x in the involution $\tau_{M''}$ on M'' . Since $\phi(x_{M''}) = \phi(x)$ and ϕ is generically finite, $x_{M''}$ belongs to a finite set when M'' varies in \mathcal{M}' , and therefore it stays fixed when M'' varies in \mathcal{M}' . Since $x_M = x'$ we have $x_{M''} = x_M = x'$, proving the claim.

It is appropriate to denote by $\mathcal{M}_{M,x,x'}$ the union of all components of \mathcal{M}_x containing M . Since \mathcal{M} is parametrized by a surface P birational to $\text{Pic}^0(S)$, the system $\mathcal{M}_{M,x,x'}$ corresponds to a reduced curve $D_{M,x,x'}$ on P . This curve might be reducible, but all of its irreducible components pass, by definition, through the point m of $\text{Pic}^0(S)$ corresponding to M .

Claim 3: When M and x, x' vary, $D_{M,x,x'}$ varies in a 2-dimensional system \mathcal{D} of curves on P with no base point. There is only one curve of \mathcal{D} containing two general points of P , i.e. \mathcal{D} has index 1, hence it is an involution.

Let M be a general curve in \mathcal{M} , thus corresponding to a general point m of $\text{Pic}^0(S)$. Of course m belongs to a 1-dimensional system of curves $D_{M,x,x'}$, when $x, x' \in M$ are conjugated by τ . This proves that \mathcal{D} is 2-dimensional. A base point of \mathcal{D} would correspond to a curve \bar{M} of \mathcal{M} which belongs to $D_{M,x,x'}$ for the general curve M and every pair of points x, x' conjugated in τ on M . But then \bar{M} would have every pair of points x, x' on M conjugated in τ in common with M , a contradiction. The final assertion follows by Claim 1.

Claim 4: \mathcal{D} is not a linear system.

Suppose \mathcal{D} is a linear system. Consider the morphism $\phi_{\mathcal{D}} : P \rightarrow \mathbb{P}^2$ determined by \mathcal{D} , which has degree at least 2. This means that, given a general curve M , corresponding to $m \in P$, there is a curve $M' \neq M$ corresponding to $m' \in P$ with $m' \neq m$, such that for every curve $D \in \mathcal{D}$ containing m , it also contains m' . Therefore for every pair of points x, x' conjugated in τ on M the curve $D_{M,x,x'}$, which contains m , also contains m' , and this implies that M' has x and x' in common with M . As x, x' vary on M staying conjugated in τ , we see that M and M' have infinitely many points in common, a contradiction.

Claim 5: \mathcal{D} is not composed with a pencil.

Suppose \mathcal{D} is composed with a pencil. By the very definition of a family $\mathcal{M}_{M,x,x'}$, we have that the general curve $D_{M,x,x'}$, if reducible, has all of its components containing the point $m \in P$ corresponding to M . On the other hand, by the definition of a system composed with a pencil, the general curve of such a system may have a singular point only at the base points of the pencil, which are fixed. Hence the general curve of a system composed with a pencil is not singular at a moving point. Thus we see that $D_{M,x,x'}$ must be irreducible. Since we are assuming that \mathcal{D} is composed with a pencil, this would imply that \mathcal{D} itself is a pencil, which contradicts the fact that \mathcal{D} has dimension 2.

In conclusion Claims 4 and 5 above contradict the Castelnuovo–Humbert theorem above, which concludes our proof in case $F = 0$.

Next we consider the case $F \neq 0$. By Lemma 4.2, F is 1-connected, $F \cdot M_{\eta} = 2$ and the linear system $|2K_S|$ cuts out on M a base point free linear series g_{2g}^r , with $r \geq g - 1$. Suppose that this series is composed with an involution $\tau := \tau_M$ of degree $\delta \geq 2$. Then we must have $2g \geq \delta(g - 1)$. This yields $\delta = 2$. Otherwise we would have $g \leq 3$, whereas $M^2 \geq 3$, (see Lemma 4.1), which implies $g \geq 5$.

If $r = g$, then M is hyperelliptic and $|2K_S|$ cuts on M the g -fold multiple of the g_2^1 . In this situation, Claim 2 above still holds. On the other hand, by arguing as in Claim 1 above, we see that, if x, x' are two general points on M conjugated in the hyperelliptic involution, then M is the unique curve in \mathcal{M} containing them. Putting these two things together, we reach a contradiction.

If $r = g - 1$ either M is hyperelliptic and we can argue as before, or $\phi(M)$ has arithmetic genus 1, and then we can argue as in the case $F = 0$. \square

Now we are ready to give the

Proof of Proposition 5.1. Proposition 5.3 is the statement for $K_S^2 = 8$ so we can assume that $K_S^2 \leq 7$. Then, by Lemma 4.1, the general curve $M := M_\eta$ in \mathcal{M} is irreducible and, by Proposition 5.5, ϕ is birational on M . Set $M' = M_{-\eta}$. Since M' is also a general curve in \mathcal{M} , ϕ is also birational on M' . Let $x \in M$ be a general point and let $x' \notin M$ be another point of S such that $\phi(x) = \phi(x')$. By the generality of $x \in M$, the point x' is also a sufficiently general point on S , hence it does not lie on F . Since $M + M' + 2F \in |2K_S|$ we have $x' \in M'$. Again by the generality of x' and of M' , there is no other point $x'' \in M'$ such that $\phi(x'') = \phi(x) = \phi(x')$. So the degree of ϕ has to be 2.

6 The bicanonical involution

Let S be a surface with $p_g = q = 2$ presenting the non-standard case. By Proposition 5.1 the bicanonical map $\phi : S \rightarrow \Sigma$ has degree 2.

In general if the bicanonical map of a surface S has degree 2 we can consider the bicanonical involution $i : S \rightarrow S$.

The involution i is biregular, since S is minimal of general type, and the fixed locus of i is the union of a smooth curve R' and of isolated points P_1, \dots, P_t . Let $\tilde{\Sigma}$ be the quotient of S by i and let $p : S \rightarrow \tilde{\Sigma}$ be the projection onto the quotient. The surface $\tilde{\Sigma}$ has nodes at the points $Q_i := p(P_i)$, $i = 1, \dots, t$, and is smooth elsewhere. Of course the bicanonical map of S factors through p .

If $R' \neq \emptyset$, the image via p of R' is a smooth curve B'' not containing the singular points Q_i , $i = 1, \dots, t$.

Let now $f : V \rightarrow S$ be the blow-up of S at P_1, \dots, P_t and set $R = f^*R'$, $E_i = f^{-1}(P_i)$, $i = 1, \dots, t$. The involution i induces a biregular involution \tilde{i} on V whose fixed locus is $R + \sum E_i$. The quotient $W = V / \langle \tilde{i} \rangle$ is smooth and one has a commutative diagram

$$\begin{array}{ccc}
 V & \xrightarrow{f} & S \\
 \pi \downarrow & & \downarrow p \\
 W & \xrightarrow{g} & \tilde{\Sigma}
 \end{array} \tag{6.1}$$

where $\pi : V \rightarrow W$ is the projection onto the quotient and $g : W \rightarrow \tilde{\Sigma}$ is the minimal desingularization map. Of course also the bicanonical map of V factors through π . Notice that $A_i := g^{-1}(Q_i)$ is an irreducible (-2) -curve for $i = 1, \dots, t$. The map π is flat, since it is finite and W is smooth. Set $B' = g^*B''$. Thus there exists a line bundle L on W such that $2L \equiv B := B' + \sum A_i$ and $\pi_*\mathcal{O}_V = \mathcal{O}_W \oplus L^{-1}$. \mathcal{O}_W is the invariant

and L^{-1} the antiinvariant part of $\pi_*\mathcal{O}_V$ under the action of \tilde{i} . Since π is a double cover, the invariants of V and W are related by

$$\begin{aligned} K_V^2 &= 2(K_W + L)^2, \\ \chi(\mathcal{O}_V) &= 2\chi(\mathcal{O}_W) + \frac{1}{2}L \cdot (K_W + L), \\ p_g(V) &= p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)). \end{aligned} \tag{6.2}$$

Since V is the blow-up of S at t points, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_V)$ and $K_S^2 = K_V^2 + t$. In this case, because we are considering double covers through which the bicanonical map factors, we can be more precise:

Proposition 6.1. *Let S be a minimal surface of general type with $p_g(S) \geq 1$ and bicanonical map of degree 2. Then, keeping the above notation, one has:*

- i) $h^0(W, \mathcal{O}_W(2K_W + L)) = 0, h^0(W, \mathcal{O}_W(2K_W + B)) = h^0(S, \mathcal{O}_S(2K_S));$
- ii) *either $p_g(W) = 0$ and $h^0(W, \mathcal{O}_W(K_W + L)) = p_g(S)$, or $p_g(W) = p_g(S)$ and $h^0(W, \mathcal{O}_W(K_W + L)) = 0;$*
- iii) $|2K_V| = \pi^*|2K_W + B'| + \sum_i E_i, f^*|2K_S| = \pi^*|2K_W + B'|$ and furthermore $\mathcal{O}_W(2K_W + B')$ is nef and big;
- iv) $(2K_W + B')^2 = 2K_S^2;$
- v) $\chi(\mathcal{O}_W(2K_W + L)) = 0;$
- vi) $K_W \cdot (K_W + L) = \chi(\mathcal{O}_W) - \chi(\mathcal{O}_S).$

Proof. i), ii) By the projection formulas for double covers, one has

$$H^0(V, \mathcal{O}_V(K_V)) = H^0(W, \mathcal{O}_W(K_W)) \oplus H^0(W, \mathcal{O}_W(K_W + L))$$

and

$$H^0(V, \mathcal{O}_V(2K_V)) = H^0(W, \mathcal{O}_W(2K_W + B)) \oplus H^0(W, \mathcal{O}_W(2K_W + L)).$$

In both the above decompositions, the first summand is the invariant, the second the anti-invariant part under the action of the involution \tilde{i} . The fact that the bicanonical map of V factors through π implies the vanishing of one of the two summands in each of the decompositions. Thus assertion ii) follows immediately. Since $p_g(S) \geq 1$, either the invariant or the anti-invariant part of $H^0(V, \mathcal{O}_V(K_V))$ is non-zero. Hence the invariant part of $H^0(V, \mathcal{O}_V(2K_V))$ is certainly non-zero, and therefore i) also holds.

iii) Recall that $B = B' + \sum A_i$. Part i) implies that $|2K_V| = \pi^*|2K_W + B|$. Since $|2K_S|$ is base point free (see [8]), the fixed part of $|2K_V|$ is $2 \sum_i E_i$. More precisely, one

has $|2K_V| = f^*|2K_S| + 2 \sum_i E_i$. Thus one has $f^*|2K_S| = \pi^*|2K_W + B'|$ and therefore $\mathcal{O}_W(2K_W + B')$ is nef and big because $\mathcal{O}_S(2K_S)$ is nef and big.

iv) follows immediately from $f^*|2K_S| = \pi^*|2K_W + B'|$ because $f^*(2K_S)^2 = 4K_S^2$ and π is a double cover.

v) Since $2(K_W + L) \equiv (2K_W + B') + \sum A_i$ and $\mathcal{O}_W(2K_W + B')$ is nef and big by iii), we can apply the Kawamata–Viehweg vanishing theorem to the divisor $K_W + L$ (see [13], Corollary 5.12, c), pp. 48–49) obtaining:

$$h^i(W, \mathcal{O}_W(2K_W + L)) = 0 \quad i = 1, 2.$$

By i) $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$, thus $\chi(\mathcal{O}_W(2K_W + L)) = 0$.

vi) By the Riemann–Roch theorem and by the formulas (6.2) we have

$$\begin{aligned} \chi(\mathcal{O}_W(2K_W + L)) &= \frac{1}{2}(2K_W + L) \cdot (K_W + L) + \chi(\mathcal{O}_W) \\ &= K_W \cdot (K_W + L) + \frac{1}{2}L \cdot (K_W + L) + \chi(\mathcal{O}_W) \\ &= K_W \cdot (K_W + L) + \chi(\mathcal{O}_S) - \chi(\mathcal{O}_W). \end{aligned}$$

Then the assertion follows from part v). □

If S is a minimal surface of general type with $p_g = q = 2$ and bicanonical map of degree 2, we can be more specific.

Lemma 6.2. *Let S be a minimal surface of general type with $p_g = q = 2$ for which the Albanese map is surjective. Suppose the bicanonical map of S has degree 2 and let W be as above. Then either*

- i) $p_g(W) = 2, q(W) = 2$, or
- ii) $p_g(W) = 0, q(W) = 1$ or
- iii) $p_g(W) = 2, q(W) = 0$.

Proof. By ii) of Proposition 6.1 we know that either $p_g(W) = 2$ or $p_g(W) = 0$. By the projection formulas for double covers, one has

$$2 = q(S) = h^1(V, \mathcal{O}_V(K_V)) = h^1(W, \mathcal{O}_W(K_W)) + h^1(W, \mathcal{O}_W(K_W + L))$$

and therefore $q(W) \leq 2$ with equality holding if and only if $h^1(W, \mathcal{O}_W(K_W + L)) = 0$.

Assume that $q(W) = 2$. Then $H^0(V, \Omega_V^1)$ is generated by two 1-forms ω, ω' which are invariant under the bicanonical involution and therefore $\omega \wedge \omega'$ is an invariant element of $H^0(V, \Omega_V^2)$. Since, by Corollary 2.4, $\omega \wedge \omega' \neq 0, p_g(W) \neq 0$ and so $p_g(W) = 2$.

Assume now that $q(W) = 1$. Then $H^0(V, \Omega_V^1)$ has invariant and antiinvariant subspaces both of dimension 1. If ω^+ and ω^- are generators of such subspaces, they form a basis of $H^0(V, \Omega_V^1)$. Since, as before, $\omega^+ \wedge \omega^- \neq 0$, $\omega^+ \wedge \omega^-$ is a nonzero antiinvariant element of $H^0(V, \Omega_V^2)$. So $p_g(W)$ is not 2 and therefore $p_g(W) = 0$.

Suppose now that $q(W) = 0$. Then $H^0(V, \Omega_V^1)$ is generated by two 1-forms ω, ω' which are antiinvariant under the bicanonical involution and therefore $\omega \wedge \omega'$ is an invariant element of $H^0(V, \Omega_V^2)$. As in the preceding paragraphs we conclude that $p_g(W) = 2$. □

We keep the same assumptions as in Lemma 6.2, and we analyse the possibilities given by the lemma.

Lemma 6.3. *The case $q(W) = 2$ cannot occur.*

Proof. Suppose otherwise. By Proposition 6.1, vi) we have $K_W \cdot (K_W + L) = 0$ and so $K_W \cdot (2K_W + B') = 0$. Therefore, since $|K_W|$ is a pencil we get a contradiction to the fact that $2K_W + B'$ is nef and big (see Proposition 6.1, iv)). □

Lemma 6.4. *Keep the assumptions in Lemma 6.2 and assume furthermore that S has no genus 2 pencils. Then the case $q(W) = 1$ does not occur.*

Proof. We notice first that $k(W) < 0$ and thus W is a ruled surface. In fact suppose otherwise. Then some multiple of K_W is an effective divisor. By Proposition 6.1, vi) we have $K_W \cdot (K_W + L) = -1$, and so $K_W \cdot (2K_W + B') < 0$, which contradicts $2K_W + B'$ being nef and big.

In this case we have, by Proposition 6.1, ii), $h^0(W, \mathcal{O}_W(K_W + L)) = 2$ and thus we can write $|K_W + L| = |Y| + Z$, where $|Y|$ is the moving part and Z is the fixed part. Since for each (-2) -curve A_i we have $A_i \cdot (K_W + L) = -1$, we infer that $Z \neq 0$. Notice that $\pi^*(|Y|)$ is exactly the moving part of $|K_V|$ and therefore by Proposition 2.5 the general curve Y in $|Y|$ is irreducible. Furthermore, since W is not rational, and $|Y|$ is a linear system of dimension 1, the geometric genus of a general curve $Y \in |Y|$ is at least 1.

Claim 1: *for every effective, non-zero divisor $N < K_W + L$, one has $h^0(N, \mathcal{O}_N) + p_a(N) \leq 2$.*

By the Riemann–Roch theorem, we have

$$\begin{aligned} & h^0(W, \mathcal{O}_W(K_W + N)) + h^2(W, \mathcal{O}_W(K_W + N)) \\ &= \frac{1}{2}(K_W \cdot N + N^2) + h^1(W, \mathcal{O}_W(K_W + N)). \end{aligned} \tag{*}$$

Now notice that $h^0(W, \mathcal{O}_W(K_W + N)) = 0$. If not, since $N < K_W + L$, we would have

$h^0(W, \mathcal{O}_W(2K_W + L)) \neq 0$, a contradiction to Proposition 6.1. Since N is effective we have also $h^2(W, \mathcal{O}_W(K_W + N)) = 0$, and so $(*)$ can be written as

$$p_a(N) - 1 + h^1(W, \mathcal{O}_W(K_W + N)) = 0.$$

Since, by duality, $h^1(W, \mathcal{O}_W(K_W + N)) = h^1(W, \mathcal{O}_W(-N))$ and one has $h^1(W, \mathcal{O}_W(-N)) \geq h^0(N, \mathcal{O}_N) - 1$ for any effective divisor N , we obtain $p_a(N) - 1 + h^0(N, \mathcal{O}_N) - 1 \leq 0$, proving the claim.

Claim 2: If T is a general ruling of W , then $Y \cdot T = 1$.

As we noticed already, the geometric genus of a general curve $Y \in |Y|$ is at least 1, and of course $h^0(Y, \mathcal{O}_Y) = 1$. By Claim 1 we conclude that Y is smooth and elliptic. Claim 1 implies also that each irreducible component θ of Z is rational and such that $\theta \cdot Y \leq 1$. Consider the pencil $|Y|$. By the Riemann–Roch theorem

$$h^0(W, \mathcal{O}_W(Y)) = Y^2 + h^1(W, \mathcal{O}_W(Y))$$

and so $0 \leq Y^2 \leq 2$. We claim that $|Y|$ has no multiple fibres. If $Y^2 > 0$ the claim is trivial, since $Y^2 \leq 2$. Assume $Y^2 = 0$ and notice that $Y \cdot L > 0$ because otherwise we would have a pencil of curves of genus 1 on V , which is impossible. Hence $Y \cdot Z = Y^2 + Y \cdot Z = Y \cdot (K_W + L) = Y \cdot L > 0$ and thus there exists an irreducible curve θ in Z such that $Y \cdot \theta = 1$. So also in this case $|Y|$ has no multiple fibres.

We can consider now the relatively minimal fibration $h : \tilde{W} \rightarrow \mathbb{P}^1$ associated to $|Y|$, i.e. we blow up the base points of $|Y|$, if any, and contract the (-1) -curves contained in fibres of $|Y|$. Since $|Y|$ has no multiple fibres and $\chi(\mathcal{O}_{\tilde{W}}) = 1$, we have by [1], corollary V.12.3, p. 162, $K_{\tilde{W}} \equiv -2F$, where F is a general fibre of h .

Let now T be a general ruling of W and \tilde{T} the corresponding ruling of \tilde{W} . Since $K_{\tilde{W}} \cdot \tilde{T} = -2$, we conclude that $F \cdot \tilde{T} = 1$ and therefore also $Y \cdot T = 1$ proving Claim 2.

Now we can finish our proof. Let T be the general ruling of W . Since each component of Z is rational, $TZ = 0$, and so we have $(K_W + L) \cdot T = (Z + Y) \cdot T = Z \cdot T + Y \cdot T = 1$. Since $K_W \cdot T = -2$, we have $L \cdot T = 3$. This implies that, by pulling back to V the ruling of W , we obtain a pencil of curves of genus 2, against our hypothesis. □

Finally we come to the case $q(W) = 0$.

Proposition 6.5. *Keep the assumptions as in Lemma 6.2 and assume furthermore that S has no genus 2 pencils. If $q(W) = 0$ then $B' = 0$, W is a minimal surface of general type with $p_g(W) = 2$, $K_W^2 = 2$ and $p : S \rightarrow \tilde{\Sigma}$ is ramified only at 20 nodes of $\tilde{\Sigma}$. Furthermore, if C and C' are the general curves in $|K_W|$ and $|K_S|$ respectively, C and C' are smooth, irreducible and non-hyperelliptic.*

Proof. We keep the notation as in the beginning of the section. Let $a : S \rightarrow A := \text{Alb}(S)$ be the Albanese map. We can define a morphism $\tilde{a} : \tilde{\Sigma} \rightarrow A$ by associating to

each point $x \in \tilde{\Sigma}$ the sum of the Albanese images of the two points in the cycle $p^*(x)$. Since $q(\tilde{\Sigma}) = 0$ this map is constant and, up to a translation, we may assume that its image is the point $0 \in A$. Hence if $p^*(x) = y_1 + y_2$ we have $a(y_1) = -a(y_2)$. Thus we can define a morphism $\alpha : \tilde{\Sigma} \rightarrow K(A)$, where $K(A)$ is the Kummer surface of A , by associating to $x \in \tilde{\Sigma}$ the point $y \in K(A)$ corresponding to $a(y_1) = -a(y_2)$.

Given any point x_0 in the branch locus, we have $p^*(x_0) = y_0 + y_0$, so $a(y_0)$ is a 2-torsion point in A . In particular the ramification divisor R must be contracted by the Albanese map and so also B'' is contracted by α . Notice that $K_{\tilde{\Sigma}} = \alpha^*(K_{K(A)}) + D$, where D is the divisor where the differential of α drops rank, in particular D contains all the curves contracted by α . Since $K(A)$ is a $K3$ surface, we see that there is an effective canonical divisor on $\tilde{\Sigma}$ containing the smooth curve B'' . Hence also K_W can be written as $B' + \Delta$, where Δ is an effective divisor.

Notice that by the classification of surfaces W is either elliptic or of general type. Let $|K_W| = |Y| + Z$, where Z is the fixed part and $|Y|$ the movable part of $|K_W|$. Since $p_g(W) = 2$, the system $|Y|$ is a pencil. Since W is regular, by Bertini's theorem the general curve of $|Y|$ is irreducible.

Remember that the bicanonical map of V has degree 2 to its image, and factors through π and through the map defined by the linear system $|2K_W + B'|$ on W . This implies that the linear series cut out by $|2K_W + B'|$ on the general curve $Y \in |Y|$ determines a birational map on Y . In particular it has projective dimension at least 2. Since W is not ruled, one has $(2K_W + B') \cdot Y \geq 3$, with equality being possible only if $g(Y) = 1$, which in turn is only possible if $Y^2 = K_W \cdot Y = 0$.

By the formulas (6.2) and by Proposition 6.1, we have $K_W \cdot (K_W + L) = 2$, hence $(2K_W + B') \cdot K_W = 4$.

Since $2K_W + B'$ is nef we have $(2K_W + B') \cdot Y \leq (2K_W + B') \cdot K_W = 4$. Since Y is nef, one has $B' \cdot Y \geq 0$, hence we obtain $K_W \cdot Y \leq 2$. By the adjunction formula $Y \cdot Z$ is even, hence either $Y \cdot Z = 0$ or $Y \cdot Z = 2$. On the other hand we have seen above that we can write $K_W = B' + \Delta$, where Δ is an effective divisor, and so $2K_W + B' = 3B' + 2\Delta$, hence $3 \leq 3B' \cdot Y + 2Y \cdot \Delta \leq 4$, so either $B' \cdot Y = 0$ or $B' \cdot Y = 1$ and $g(Y) = 1$. This is impossible because then $Y^2 = Y \cdot K_W = 0$, thus $0 = Y \cdot K_W = Y \cdot B' + Y \cdot \Delta = 1 + Y \cdot \Delta$ and the nef divisor Y would be such that $Y \cdot \Delta = -1$, a contradiction. Thus the only possibility is $Y \cdot B' = 0$, $Y \cdot \Delta = 2$ and therefore $K_W \cdot Y = 2$. Since $Y \cdot Z$ is even and non-negative, we have that either $Y^2 = 0$, $Y \cdot Z = 2$ or $Y^2 = 2$, $Y \cdot Z = 0$.

In the first case $Y^2 = 0$, $Y \cdot Z = 2$, we get $\mathcal{O}_Y(2K_W + B') \simeq \mathcal{O}_Y(2K_Y)$. This is impossible because in this case $|Y|$ is a genus 2 pencil and so $|2K_W + B'|$ would determine a non-birational map on W .

If $Y^2 = 2$, $Y \cdot Z = 0$, then we have $2K_W + B' \equiv 2Y + (2Z + B')$ and $2Y \cdot (2Z + B') = 0$. Since $2K_W + B'$ is nef and big, the only possibility is that $2Z + B' = 0$. So $B' = Z = 0$, hence $K_W \equiv Y$ is nef and therefore W is minimal. Moreover $K_W^2 = Y^2 = 2$. Furthermore $2K_W + B' = 2K_W$ and, by Proposition 6.1, iv) we have $K_S^2 = 4$. In addition, by the formulas (6.2) and by Proposition 6.1, we have $(K_W + L)^2 = -8$ and so $K_V^2 = -16$. Hence $t = 16 + K_S^2 = 20$, where t is, as before, the number of isolated fixed points of the bicanonical involution. Thus p is ramified exactly over 20 nodes.

By the above the general curve C in the linear system $|K_W| = |Y|$ is irreducible and non-hyperelliptic, because the bicanonical map of W is birational. Since $K_W^2 = 2$ and $|K_W|$ is a rational pencil, C is necessarily smooth. The assertion for the general curve C' in $|K_S|$ is then obvious. \square

7 The main theorem

In the previous sections we saw that if the bicanonical map ϕ of a surface S with $p_g = q = 2$ is not birational and S has no pencil of curves of genus 2, then ϕ has degree $\sigma = 2$ and we have described in Proposition 6.5 some properties of the quotient of S by the involution induced by the bicanonical map.

In this section we will classify these quotients. Let us start by presenting an example, which was first pointed out by F. Catanese (cf. [8], Example (c), page 70, and Remark 3.15, page 72).

Example 7.1. Let A be an abelian surface with an irreducible symmetric principal polarization Θ , and suppose that A contains no elliptic curves. Let $h : S \rightarrow A$ be the double cover branched on a smooth divisor $B \in |2\Theta|$ so that $h_*\mathcal{O}_S = \mathcal{O}_A \oplus \mathcal{O}_A(-\Theta)$. Since $K_S = h^*(\Theta)$, the invariants of the smooth surface S are $p_g(S) = 2$, $q(S) = 2$, $K_S^2 = 4$. Notice that the map $h : S \rightarrow A$ factors through the Albanese map $a : S \rightarrow \text{Alb}(S)$. Since h has degree 2 and $\text{Alb}(S)$ is a surface, we see that $\text{Alb}(S) \simeq A$. In addition we observe that S has no genus b pencil of curves of genus 2. Indeed, by Lemma 2.2, ii) and by the assumption that $A \simeq \text{Pic}^0(S)^*$ contains no elliptic curve, one should have $b = 2$, and by part i) of the same lemma we would find $K_S^2 \geq 8$, a contradiction.

Remark now that B is symmetric with respect to the involution j of A determined by the multiplication by -1 . Hence j can be lifted to an involution i on S that acts as the identity on $H^0(S, \mathcal{O}_S(K_S))$. We denote by $p : S \rightarrow \tilde{S} := S/\langle i \rangle$ the projection onto the quotient. We observe that $p_g(\tilde{S}) = 2$, $q(\tilde{S}) = 0$, $K_{\tilde{S}}^2 = 2$ and the only singularities of the surface \tilde{S} are 20 nodes. Since $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2K_{\tilde{S}})) = \chi(\mathcal{O}_{\tilde{S}}) + K_{\tilde{S}}^2 = 5 = h^0(S, \mathcal{O}_S(2K_S))$, the bicanonical map of S factors through $p : S \rightarrow \tilde{S}$. Since \tilde{S} has no pencil of curves of genus 2, we have the situation described in Proposition 6.5.

For the sake of completeness, we want to point out the following alternative description of \tilde{S} . One embeds, as usual, the Kummer surface $\text{Kum}(A)$ of A as a quartic surface in $\mathbb{P}^3 = \mathbb{P}(H^0(A, 2\Theta)^*)$. The surface \tilde{S} is a double cover of $\text{Kum}(A)$ branched along the smooth plane section H of $\text{Kum}(A)$ corresponding to B and on 6 nodes, corresponding to the six points of order 2 of A lying on Θ . The ramification divisor R of $\tilde{S} \rightarrow \text{Kum}(A)$ is a canonical curve isomorphic to H , and thus it is not hyperelliptic.

Remark 7.2. The same construction can also be done with a reducible polarization Θ on A . Then A is isomorphic to the product $E_1 \times E_2$ of two elliptic curves and the surface S constructed as above has two elliptic pencils of genus 2 curves. In this case the bicanonical map of S has degree 4 (see [23], Theorem 5.6).

We are finally going to prove our classification theorem:

Theorem 7.3. *Let S be a minimal surface of general type with $p_g = q = 2$, presenting the non-standard case. Then S is as in Example 7.1.*

For the proof we need a preliminary lemma and some notation. Let X and Y be smooth, projective surfaces and $f : X \rightarrow Y$ be a surjective map. Let R be the ramification curve on X , i.e. the subscheme of X where f drops rank. Let C be a smooth, irreducible curve on X not contained in R . Set $\Gamma := f(C)$ and $f^*(\Gamma) = C + D$. Notice that C and D have no common component. For every point $p \in C$, denote by r_p [resp. by d_p] the coefficient of p in the divisor cut out on C by R [resp. by D]. Set $\delta_p = r_p - d_p$ and $p' := f(p)$. Then:

Lemma 7.4. *With the above notation, if Γ is smooth at p' , then $\delta_p \geq 0$.*

Proof. Use local coordinates (s, t) centered at p' in such a way that Γ has equation $t = 0$. Use local coordinates (x, y) centered at p in such a way that C has equation $x = 0$ and $\phi(x, y) = 0$ is the equation of D . Then f has local equations $s = \psi(x, y)$ and $t = x\phi(x, y)$. Therefore R has equation

$$\phi \frac{\partial \psi}{\partial y} + x \frac{\partial(\phi, \psi)}{\partial(x, y)} = 0$$

whence the assertion follows immediately. □

Now we can prove our classification theorem:

Proof of Theorem 7.3. The main step in our proof is to show that the Albanese map $a : S \rightarrow A := \text{Alb}(S)$ has degree $v = 2$. This is what we are going to prove first.

As we saw in Section 6, the bicanonical map of S factors through the degree 2 finite cover $p : S \rightarrow \tilde{\Sigma}$ branched only at the 20 nodes of $\tilde{\Sigma}$. By Proposition 6.1, iii), $K_{\tilde{\Sigma}}$ is a nef and big line bundle on $\tilde{\Sigma}$. More precisely, from Proposition 6.5 it follows that $|K_{\tilde{\Sigma}}|$ is a pencil with no fixed component and with two base points which do not occur at any of the nodes of $\tilde{\Sigma}$. Hence $(p : S \rightarrow \tilde{\Sigma}, K_{\tilde{\Sigma}})$ is a good generating pair in the sense of [10].

Let C be a general curve in $|K_{\tilde{\Sigma}}|$ and let $C' := p^*(C)$. Since C does not contain any of the nodes of $\tilde{\Sigma}$, the cover $p : C' \rightarrow C$ is an étale double cover. Theorem (6.1) of [CPT] yields that the Prym variety $P := \text{Prym}(C', C)$ related to the double cover $p : C' \rightarrow C$ is isomorphic to the Albanese surface A . Therefore A is principally polarized, and we denote by Θ its principal polarization. Furthermore, after having identified A with P , the Abel–Prym map $\alpha : C' \rightarrow P$ coincides, up to translation, with the restriction to C' of the Albanese map $a : S \rightarrow A$. Notice that C' is not hyperelliptic and set $\Gamma := a(C')$. By the results in [15], chapter 12, the map $a|_C : C' \rightarrow \Gamma$ is an isomorphism and therefore Γ is smooth. Furthermore Γ is in the class of 2Θ by Welters’ criterion (see again [15], chapter 12).

Let us set $a^*(\Gamma) = C' + D$ and let us denote by R the ramification curve of a . By Lemma 7.4 we have $K_S \cdot D = C' \cdot D \leq C' \cdot R = C' \cdot K_S = 4$, with equality holding if

and only if $D \sim K_S$. By the index theorem we have $D^2 \leq 4$. Thus $\nu \cdot 8 = \nu \cdot (2\Theta)^2 = a^*(\Gamma)^2 = (C' + D)^2 \leq 16$. This proves that $\nu = 2$ and, in addition, that $D \sim K_S$.

Now we can finish our proof by showing that the branch curve B of $a: S \rightarrow A$ is a divisor in the class of 2Θ . This immediately follows from the fact that $16 = 2R \cdot (C' + D) = 2B \cdot \Gamma$, hence $B \cdot \Theta = 4$, so that B is numerically equivalent to 2Θ .

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C. Ciliberto, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy
Email: cilibert@mat.uniroma2.it

M. M. Lopes, CMAF, Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal
Email: mmlopes@lmc.fc.ul.pt