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A group-free characterization of the *P*-geometry for Co_2

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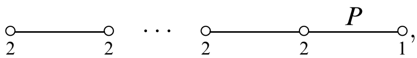
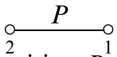
Abstract. It is shown under minor extra assumptions that a *P*-geometry which has as its point residue the rank three *P*-geometry for the group M_{22} is the rank four *P*-geometry which has Co_2 as its group of automorphisms.

Key words. diagram geometry, sporadic groups.

1 Introduction

A *P*-geometry is a geometry that belongs to the diagram



where the edge  denotes the geometry of edges and vertices of the Petersen graph. Flag-transitive *P*-geometries were classified in a series of papers by Ivanov and Shpectorov. A survey of this classification can be found in [6]. It was shown that there exist exactly eight such geometries, all of them related to sporadic simple groups or non-split extensions of sporadic groups with one of their modules over GF(3). In fact, this relation between *P*-geometries and sporadic simple groups was the principal motivation for the study of *P*-geometries: the classification of flag-transitive *P*-geometries was meant to be a contribution to the geometric theory of finite simple groups.

The classification of [6] makes heavy use of the flag-transitivity condition, that is, it is essentially group-theoretic. This is, of course, very far from a purely geometric theory. From this point of view, it is desirable to develop methods to study *P*-geometries in a "group-free" way. Ideally, the classification of *P*-geometries must be reproved under purely geometric assumptions. However, this goal seems to be too ambitious at present. The principal complication is that if the flag-transitivity condition is dropped then the number of examples increases astronomically. To illustrate

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this point, let us consider one of the flag-transitive P-geometries, a rank four geometry with the full automorphism group $3^{23} \cdot \mathrm{Co}_2$. Factorizing this geometry over the action of any subgroup of the normal subgroup 3^{23}, one always gets again a P-geometry. Needless to say, the number of examples obtained in this way is huge. In rank five, there is another P-geometry with the automorphism group $3^{4371} \cdot \mathrm{BM}$, leading to even more impressive numbers.

One possible solution of the above problem would be to classify only the 2-simply connected P-geometries. They can be considered as the generic examples, because, arguably, every P-geometry can be obtained from some 2-simply connected P-geometry by factorizing it over a suitable subgroup of the automorphism group. Only five known P-geometries are 2-simply connected, so this looks like a meaningful project. However, at present it is unclear how the condition of 2-simple connectedness can be utilized, and so new ideas are needed.

Of course, even though a complete classification is beyond reach, we can try and characterize particular examples of P-geometries by some geometric conditions. Of particular interest is a series of three P-geometries of ranks 3, 4 and 5 related to groups M_{22}, Co_2 and BM. A purely geometric characterization of the geometry of M_{22} was obtained by Hall and Shpectorov [4], who proved the following.

Theorem 1. *Suppose that $\mathscr{G}$ is a rank three P-geometry such that*

(1) *any two lines intersect in at most one point, and*

(2) *any three pairwise collinear points belong to a plane.*

Then $\mathscr{G}$ is either the geometry of M_{22} *or its triple cover, the geometry of* $3 \cdot \mathrm{M}_{22}$.

(Here *points*, *lines* and *planes* are the types corresponding to the first three nodes in the diagram above.) In the present paper we do the second step and obtain the following characterization of the geometry of Co_2.

Theorem 2. *Suppose that $\mathscr{G}$ is a rank four P-geometry such that*

(1) *any two lines intersect in at most one point,*

(2) *any three pairwise collinear points belong to a plane, and*

(3) *the residue of every point is the geometry of* M_{22}.

Then $\mathscr{G}$ is the geometry of Co_2.

Notice that it is the condition (3) that eliminates the geometry of $3^{23} \cdot \mathrm{Co}_2$ and its numerous quotients. Incidentally, (3) also eliminates the flag-transitive rank four P-geometry of the group J_4. The fourth (and last) example of flag-transitive P-geometries of rank four, the geometry of M_{23}, is eliminated by the condition (1).

Our proof of Theorem 2 proceeds in several stages. First we study the collinearity graph Γ of $\mathscr{G}$ and establish its distance-distribution diagram (not surprisingly, the same diagram describes the collinearity graph of the geometry of Co_2). This gives us a wealth of information about $\mathcal{G}$, in particular, the total number of points and the possible relations between points. At stage two we recover two classes of subgeometries in $\mathcal{G}$, symplecta (or symps) related to $Sp(6,2)$ and subgeometries isomorphic to the dual polar space of $U_6(2)$. We call the latter subgeometries unita. The unita, numbered 2300, become at stage three the vertices of a new graph Σ. We study the local structure of Σ and show that it is the same as that of the well-known rank three graph on 2300 points with the automorphism group Co_2. The latter graph was characterized by its local structure by Cuypers [3], so we can invoke his result to identify Σ. It remains, at stage four, to recover Γ and $\mathcal{G}$ from the known Σ.

For an introduction to diagram geometries we recommend [7]. In [5] one can find a wealth of information about the known P-geometries and a related class of sporadic geometries, called tilde geometries.

2 The collinearity graph of $\Gamma(M_{22})$

Let $\mathcal{H}$ be the rank three P-geometry of M_{22} and let Δ be the collinearity graph of $\mathcal{H}$. In this section we collect a number of results on Δ.

First, we review the construction of the geometry $\mathcal{H}$. Recall that the Witt design $\mathcal{W} = S(3,6,22)$ is a block design $(P,\mathcal{B})$, where $P = \{1,2,\dots,22\}$ is the point set of $\mathcal{W}$ and $\mathcal{B}$ is the set of blocks, that is, subsets of P. We will refer to the elements of P as to the Witt points. Every block consists of six Witt points, which explains why the blocks of $\mathcal{W}$ are called *hexads*. The property that makes $\mathcal{W}$ unique up to isomorphism among all block designs on 22 points is the following: any three Witt points are contained in a unique hexad. The automorphism group of $\mathcal{W}$ coincides with $\mathrm{Aut}\, M_{22}$.

The *points* of $\mathcal{H}$ (or $\mathcal{H}$*-points*) are the 231 pairs of Witt points. The *lines* of $\mathcal{H}$ ($\mathcal{H}$-lines) can be described as follows. Every line consists of three points (*i.e.*, three pairs of Witt points) that are pairwise disjoint and whose union is a hexad. Thus, two points are collinear whenever they are disjoint as pairs of Witt points and the 4-set they form is contained in a hexad. It follows from this description that the geometry $\mathcal{H}$ is a partial linear space, that is, any two collinear points belong to a unique line. Indeed, any three Witt points are contained in a unique hexad and hence any four Witt points are contained in at most one hexad.

Let Δ be the collinearity graph of $\mathcal{H}$. The graph Δ is locally the 2-clique extension (see the definition in the next section) of the line graph of the Petersen graph. In particular, every point x is collinear with 30 other points. The points outside $x^\perp$ (as usual, $x^\perp$ denotes the set of points collinear with x, including x itself) split into two groups. Let $\Delta_2^1(x)$ be the set of points y such that x and y share a Witt point. Let $\Delta_2^2(x)$ denote the set of points $y \notin x^\perp$ such that x and y are disjoint as pairs of Witt points. We have that $|\Delta_2^1(x)| = 2 \cdot 20 = 40$ and hence $|\Delta_2^2(x)| = 231 - 1 - 30 - 40 = 160$.

Figure 1 shows the decomposition of Δ with respect to a point, including information about the embedding of lines. The information about lines is encoded in the diagram in the following way. Suppose $\{y,u,v\}$ is a line on a point y. If the edge $\{y,u\}$ is represented by a valency n next to the box for y then $\{y,v\}$ is represented either by the same n (in which case n must be even), or by $\bar{n}$.

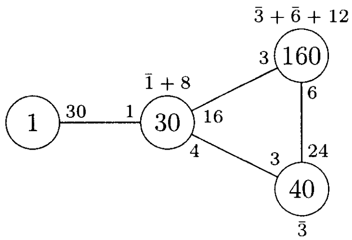

Figure 1. Decomposition of Δ with respect to a point

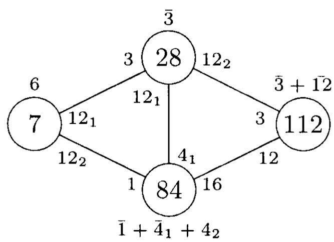

Figure 2. Decomposition of Δ from a plane

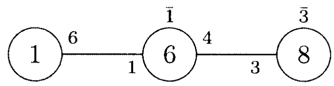

Figure 3. Decomposition of a quad from a point

Maximal sets of pairwise collinear points of $\mathscr{H}$ are called *planes* (or $\mathscr{H}$-planes). Every plane consists of seven points. The seven pairs of Witt points that are the points of a plane partition the complement of an octad. *Octads* are certain 8-element subsets of P inherited from the largest Witt design $S(5,8,24)$. (Octads are the blocks of the latter design that fall into the point set of $\mathscr{W}$.) Planes bijectively correspond to octads. Every plane is closed with respect to lines, that is, a line that contains two points of a plane is fully contained in it. It follows that a plane of $\mathscr{H}$ contains exactly seven lines, which turn it into a projective plane of order two, a Fano plane. Figure 2 shows the decomposition of Δ with respect to a plane.

The geometry $\mathscr{H}$ contains *quads*. Those are point-line subgeometries of $\mathscr{H}$ isomorphic to the generalized quadrangle of order $(2,2)$. We will often identify the quad with the the subgraph in Δ induced on the points of the quad. Figure 3 represents the decomposition of (the collinearity graph of) a quad with respect to one of its points.

The quads are in a one-to-one correspondence with the hexads. The quad Σ corresponding to a hexad X consists of all points (*i.e.*, pairs of Witt points) that are contained in X and all lines (*i.e.*, triples of pairs of Witt points) that partition X. Notice that quads are closed with respect to lines, and every line is contained in a unique quad. In fact, since two hexads meet in at most two Witt points (in fact, zero or two

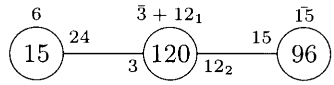

Figure 4. Decomposition of Δ from a quad

Witt points), two quads are either disjoint or they meet in just one point. Therefore, two points are contained in at most one quad. If x and y, $x \neq y$, are contained in a quad then that quad will be denoted by $Q(x, y)$.

Two edges of the Petersen graph are called *opposite* if they are at maximal distance, three. If x is a point of a quad Q then the three lines on x in Q are opposite in the residue of x. (Recall that the lines on x correspond to the edges and planes on x correspond to the vertices of the Petersen graph.) Reversely, if x is a point and L_1, L_2 and L_3 are three lines on x pairwise opposite in the residue of x then there is a quad Q containing the lines L_i.

If $y \in \Delta_2^1(x)$ then $x \cup y$ is a set of three Witt points. Since every triple of points is contained in a hexad, $Q(x, y)$ is defined for such x and y. Reversely, if x and y are two non-collinear points in a quad $\Sigma = Q(x, y)$ then $y \in \Delta_2^1(x)$. Comparing Figure 3 with Figure 1, we see that quads are geodesically closed subgraphs. In Figure 4 we present the decomposition of Δ with respect to a quad.

In the remainder of this section we record some properties of Δ, related to Figures 1, 2 and 4. Recall that $\Delta(x, y)$ is the set of vertices of Δ adjacent to x and y (*i.e.*, points collinear with x and y). If x and y are non-adjacent then the subgraph induced on $\Delta(x, y)$ is called a *μ-graph* of Δ.

Lemma 2.1. (1) *If $y \in \Delta_2^1(x)$ then $\Delta(x, y)$ is a coclique of size three. It is fully contained in $Q(x, y)$.*

(2) *If $y, z \in \Delta_2^1(x)$ and y and z are collinear then $Q(x, y) = Q(x, z)$.*

(3) *If $y \in \Delta_2^2(x)$ then $\Delta(x, y)$ is a line.*

Proof. For (1), let $x = \{a, b\}$ and $y = \{a, c\}$ and let X be the hexad containing a, b and c. Suppose $z = \{d, e\}$ is a point collinear with x and y and let X_1 and X_2 be the hexads containing $x \cup z$ and $y \cup z$, respectively. Notice that both X_1 and X_2 contain a, d and e. Hence $X_1 = X_2 = X$ and (1) follows.

Similarly, if z is a point collinear with y and $z \in \Delta_2^1(x)$ then $z = \{b, d\}$ for some Witt point d. Again the hexad containing y and z contains a, b and c, and hence it coincides with X. This implies (2). Notice that (1) and (2) confirm two of the valencies shown in Figure 1.

For (3), observe that x is contained in 10 planes, every one of those planes contains four lines L not containing x, and each of those lines is contained in a unique second plane which, in turn, contains four points not on L. Thus, there are $10 \cdot 4 \cdot 4$ points $y \in \Delta_2(x)$ such that $\Delta(x, y)$ contains a line. Together with Figure 1 and (1), this proves (3). □

Recall that a *geometric hyperplane* is a set of points such that every line of the geometry is either contained in the set, or meets it in exactly one point.

Lemma 2.2. *If Θ is a plane or a quad then the set of points collinear with a point of Θ (this includes the points of Θ itself) is a geometric hyperplane of $\mathscr{H}$. In particular, every line of $\mathscr{H}$ contains a point that is either contained in Θ or is collinear with a point of Θ.*

Proof. Follows from Figures 2 and 4. □

Corollary 2.3. *If Θ_1 and Θ_2 are two planes, or two quads, or a plane and a quad, then $\Theta_1 \cup \Theta_2$ induces a connected subgraph of Δ.*

Proof. Indeed, Θ_2 contains a line, and so it is at distance at most one from Θ_1. □

Recall that an *ovoid* is a set of points such that every line contains exactly one point from this set.

Lemma 2.4. *Suppose Θ is a quad in Δ.*

(1) *If x is a point collinear with a point of Θ then x is collinear with exactly three points from Θ, and these three points form a line.*

(2) *If x is a point not collinear with a point of Θ then there are exactly five points y in Θ such that $y \in \Delta_2^1(x)$. The points y form a coclique (an ovoid) in Θ.*

Proof. Let X be the hexad corresponding to the quad Θ, and let $x = \{a,b\}$ be a point, not in Θ. (Thus, $\{a,b\} \not\subset X$.) Suppose first that X and $\{a,b\}$ are disjoint. For $c \in X$ let Y be the hexad containing a, b and c. Since $X \cap Y \neq \emptyset$, we have $|X \cap Y| = 2$, say, $X \cap Y = \{c,d\}$. Clearly, $y = \{c,d\}$ is a point of Θ adjacent to x. Since c was arbitrary, we obtain that x is adjacent to three points from Θ. Furthermore, the corresponding three pairs of Witt points partition X and so the three points form a line.

Suppose now that a (but not b) is contained in X. Let Y be an arbitrary hexad containing $\{a,b\}$. Then $|X \cap Y| \geqslant 1$ and hence $|X \cap Y| = 2$, say, $X \cap Y = \{a,c\}$. This shows, first, that x is not adjacent to a point from Θ. Secondly, since for every $c \in X \backslash \{a\}$, the set $\{a,b,c\}$ is contained in a hexad, we also obtain that the points of Θ that are contained in a common quad with x are exactly the five points $\{a,c\}$, $c \in X \backslash \{a\}$. Clearly, these five points form a coclique in Θ, in fact, an ovoid. □

3 Local structure

For a graph Ξ, its *k-clique extension* $k.\Xi$ is defined as the graph on the set $\{(x,i) \mid x \in \Xi, i \in \{1,\ldots,k\}\}$, where distinct vertices (x,i) and (y,j) are adjacent whenever x and y are adjacent or equal. Let π be the mapping from $k.\Xi$ to Ξ defined by the projection $(x,i) \mapsto x$. Then every fiber of π is a k-clique. Two such k-cliques induce a

$2k$-clique or a disjoint union of two k-cliques, depending on whether or not the two vertices of Ξ form an edge.

For two graphs Γ and Ξ we say that Γ is *locally* Ξ if for every vertex $x \in \Gamma$ the neighborhood $\Gamma(x)$ of x in Γ induces a subgraph isomorphic to Ξ. In particular, the valency of Γ must be $|\Xi|$.

Recall that Δ is the collinearity graph of the rank three P-geometry $\mathscr{H}$ for the group M_{22}.

Proposition 3.1. *If $\mathscr{G}$ is a geometry satisfying the assumptions of Theorem* 2, *then its collinearity graph* Γ *is locally the* 2-*clique extension of* Δ. *Conversely*, *every graph* Γ *that is locally the* 2-*clique extension of* Δ *is the collinearity graph of some P-geometry* $\mathscr{G}$ *satisfying the assumptions of Theorem* 2.

Proof. First suppose $\mathscr{G}$ is a geometry satisfying the assumptions of Theorem 2 and let Γ be its collinearity graph. Let $x \in \mathscr{G}$ be a point. By assumption, the residue of x is isomorphic to $\mathscr{H}$, and the lines on x play the role of the points of $\mathscr{H}$ ($\mathscr{H}$-points; similarly, we will be using the terminology '$\mathscr{H}$-lines' and '$\mathscr{H}$-planes' wherever ambiguity may otherwise arise). If $y \in \mathscr{G}$ is a point collinear with x then define $\pi(y)$ to be the $\mathscr{H}$-point corresponding to the line xy. This line is unique due to Assumption (1) in Theorem 2, and so π is well-defined. Furthermore, Assumption (2) of Theorem 2 implies that two neighbors y and z of x in Γ are collinear if and only if the lines xy and xz are coplanar, that is, if and only if $\pi(y)$ and $\pi(z)$ are equal or collinear. Clearly, every fiber of π consists of two points, since each line of $\mathscr{G}$ has three points. Thus, Γ is locally $2.\Delta$.

Let now Γ be an arbitrary graph that is locally $2.\Delta$. For a vertex $x \in \Gamma$ let π_x denote the mapping from $\Gamma(x)$ onto Δ that exists due to the assumption that Γ is locally $2.\Delta$. Recall that all maximal cliques of Δ are of size 7 and they correspond to the $\mathscr{H}$-planes. Two $\mathscr{H}$-planes intersect in a $\mathscr{H}$-line, or in a single $\mathscr{H}$-point, or in an empty set. Define a geometry $\mathscr{G}$ related to Γ as follows. The points of $\mathscr{G}$ will be the vertices of Γ; other elements of $\mathscr{G}$ on a point x will be defined as the full preimages under π_x of non-empty intersections of maximal cliques from Δ. Thus, besides points, we will have in $\mathscr{G}$ elements of size $3 = 1 + 2 \cdot 1$, which we will call lines, of size $7 = 1 + 2 \cdot 3$, which we will call planes, and of size $15 = 1 + 2 \cdot 7$, which we will call 3-spaces. Notice that lines, planes and 3-spaces are defined symmetrically with regard to the points contained in them. Indeed, 3-spaces are simply all maximal cliques of Γ, while all other elements are non-empty intersections of maximal cliques.

Incidence on $\mathscr{G}$ is defined by inclusion. Notice that every edge of Γ is contained in a unique line, and $\{x, y, z\}$ is a line if and only if $\pi_x(y) = \pi_x(z)$. Since Γ is connected, so is $\mathscr{G}$. Clearly, $\mathscr{G}$ has a linear (string) diagram. Furthermore, it follows from the definition of $\mathscr{G}$ that the residue of a point is isomorphic to $\mathscr{H}$. Thus, in order to establish that $\mathscr{G}$ is a P-geometry it remains to show that the points and lines in a plane of $\mathscr{G}$ form a Fano plane. However, this is immediate, since planes consist of seven points, lines have size three, and any two points in a plane are contained in a unique line. We have shown that $\mathscr{G}$ has the diagram of a P-geometry. Since $\mathscr{G}$ is connected and all its residues of rank at least two are connected, too, we conclude that $\mathscr{G}$ is a geometry, indeed, a P-geometry.

Manifestly, Γ is the collinearity graph of $\mathscr{G}$. We have already shown that every edge of Γ is contained in a unique line, so that the condition (1) from Theorem 2 is satisfied. Since Γ is locally $2.\Delta$, two points, y and z, both collinear with x, are collinear with each other if and only if x, y and z are contained in a plane. □

Proposition 3.1 shows that Theorem 2 is equivalent to the following.

Theorem 3.2. *If Γ is a connected graph that is locally $2.\Delta$ then Γ is isomorphic to the collinearity graph of $\mathscr{G}(\mathrm{Co}_2)$.*

In the remainder of this paper Γ is a connected graph that is locally $2.\Delta$. When we prefer to use geometric terminology we view it as the collinearity graph of a geometry $\mathscr{G}$ satisfying the assumptions of Theorem 2.

4 The diagram of Γ

In this section we see how Γ decomposes with respect to a point. Since Γ is locally $2.\Delta$, for every point x there is a mapping π_x sending points collinear to x onto $\mathscr{H}$-points, as described in the previous section. We can also view π_x as a mapping sending the lines on x onto the $\mathscr{H}$-points.

Let us now fix a point x of $\mathscr{G}$. Since the local structure of Γ is known we can immediately start with the points in $\Gamma_2(x)$. By a 2-*string* in Γ we understand a 2-path yzt such that y and t are not adjacent. We say that a 2-string yzt is of *type* 1 if $\pi_z(y)$ and $\pi_z(t)$ are contained in a quad in Δ (that is, $\pi_z(t) \in \Delta_2^1(\pi_z(y))$), and we say that yzt is of *type* 2 otherwise. Recall that, for $z \in \Gamma_2(x)$, the *μ-graph* of x and z is defined as the subgraph induced on $\Gamma(x,z) = \Gamma(x) \cap \Gamma(z)$.

Lemma 4.1. *Suppose xyz is a 2-string and let Θ be the connected component of $\Gamma(x,z)$ containing y. Then the following holds*:

(1) *If xyz is of type 1 then π_x establishes an isomorphism between Θ and a quad in Δ. Furthermore, if $y' \in \Theta$ then $xy'z$ is of type 1.*

(2) *If xyz is of type 2 then Θ is a plane, and every $xy'z$ with $y' \in \Theta$ is of type 2.*

Proof. First notice that, as follows from Lemma 2.1 (1) and (3), if xyz is of type 1 then $\{y\} \cup \Theta(y)$ is a union of three lines on y and the lines are pairwise non-coplanar, while if xyz is of type 2 then $\{y\} \cup \Theta(y)$ is a plane. In particular, if xyz is of type 2 and $y' \in \Theta(y)$ then $xy'z$ is also of type 2, and (2) follows by connectivity.

Suppose now that xyz is of type 1. By connectivity, if $y' \in \Theta$ then $xy'z$ is again of type 1. Thus, locally Θ is a union of three lines. Let the three lines on y be L_1, L_2 and L_3, and let P_i be the plane containing x and L_i. Let L be the line through x and y. Since the geometry $\mathscr{G}$ associated with Γ is a P-geometry, the planes and the 3-spaces containing L correspond to edges and vertices of the Petersen graph, respectively. In the residue of y we observe that $\pi_y(P_i \backslash \{y\})$ are three $\mathscr{H}$-lines forming the neighborhood of $\pi_y(x)$ in a quad. This means that P_i's correspond in the residue of L to three pairwise opposite edges of the Petersen graph. (See the discussion in Section 2.) The same logic used in reverse and applied to x instead of y, allows us now to conclude that $\pi_x(L_i) = \pi_x(P_i \backslash \{x\})$ are three $\mathscr{H}$-lines on $\pi_x(y)$, all contained in some quad. Since this is true for every vertex y of Θ and since every $\mathscr{H}$-line is contained in a unique quad, we conclude that $\pi_x(\Theta)$ is a quad in Δ, establishing (1). □

Lemma 4.2. *Every μ-graph $\Gamma(x,z)$ in Γ is connected.*

Proof. Suppose Θ_1 and Θ_2 are two connected components of $\Gamma(x,z)$. Then $\pi_x(\Theta_i)$ is either a quad or an $\mathscr{H}$-plane. Now the claim follows from Lemma 2.3. □

Lemmas 4.1 and 4.2 imply that there are two kinds of points in $\Gamma_2(x)$. We define $\Gamma_2^1(x)$ (respectively, $\Gamma_2^2(x)$) as the set of those points z in $\Gamma_2(x)$ such that the μ-graph $\Gamma(x,z)$ is a copy of a quad (respectively, a plane). Clearly, $z \in \Gamma_2^i(x)$ if and only if for an arbitrary point $y \in \Gamma(x,z)$ the 2-string xyz is of type i.

For a vertex y in $\Gamma(x)$ we have $2 \cdot 40$ (cf. Figure 1) extensions of xy to a 2-string of type 1, and $2 \cdot 160$ extensions to a 2-string of type 2. Thus $|\Gamma_2^1(x)| = \frac{462 \cdot 80}{15} = 2464$ and $|\Gamma_2^2(x)| = \frac{462 \cdot 320}{7} = 21120$.

Lemma 4.3. *If z is in $\Gamma_2^1(x)$ then z has* 15 *neighbors in* $\Gamma(x)$, $\overline{15}$ *neighbors in* $\Gamma_2^1(x)$, $2 \cdot 120$ *neighbors in* $\Gamma_2^2(x)$, *and* $2 \cdot 96$ *neighbors in* $\Gamma_3(x)$.

Remark. We use the same bar notation as in the diagrams of graphs to indicate the embedding of lines.

Proof. Since $z \in \Gamma_2^1(x)$ we have that $\pi_x(\Gamma(x,z))$ is a quad. If $y \in \Gamma(x,z)$ and t is the third point on the line yz then clearly $t \in \Gamma_2^1(x)$. This accounts for fifteen lines on z. According to Figure 4, there are further 120 lines $\{z,u,v\}$ such that u and v are collinear with some (in fact, three—cf. Lemma 2.4(1)) points from $\Gamma(x,z)$. Clearly this implies that u and v are at distance two from x. We contend that $u, v \in \Gamma_2^2(x)$. Indeed, let $y \in \Gamma(x,z)$ be a point collinear with u and v. Consider Figure 1 as describing the decomposition of Δ with respect to $\pi_y(x)$. Then $\pi_y(z)$ must be in $\Delta_2^1(\pi_y(x))$, since xyz is of type 1. As follows from Figure 1, every $\mathscr{H}$-line on $\pi_y(z)$ containing no $\mathscr{H}$-point collinear with $\pi_y(x)$ has two $\mathscr{H}$-points in $\Delta_2^2(\pi_y(x))$. This means that u and v are in $\Gamma_2^2(x)$.

We claim that the remaining 96 lines on z have two points at distance three from x. Indeed, suppose $\{z,u,v\}$ is one of those 96 lines. Clearly, the distance between x and u is at least two. Suppose it is two. Since z and u have no common neighbors in $\Gamma(x)$ (this follows from Figure 4), the μ-graphs $\Gamma(x,z)$ and $\Gamma(x,u)$ must be disjoint. In particular, $\pi_z(u)$ is not adjacent to the quad $\pi_z(\Gamma(x,z))$. By Lemma 2.4(2), we have that uzt is of type 1 for exactly five points $t \in \Gamma(x,z)$, and these five points form an ovoid in $\Gamma(x,z)$. On the other hand, $\Gamma(x,u)$ has a point w at distance one from $\Gamma(x,z)$. By Lemma 2.4(1), w is collinear with three points a, b, and c, forming a line in $\Gamma(x,z)$. Since w and z are non-collinear, uza, uzb, and uzc must be of type 1, which places a, b and c in an ovoid in $\Gamma(x, z)$. This contradiction completes the proof of the lemma. □

Lemma 4.4. *If z is in $\Gamma_2^2(x)$ then z has* 7 *neighbors in* $\Gamma(x)$, 28 *neighbors in* $\Gamma_2^1(x)$, $\bar{7} + \overline{28} + 2 \cdot 84$ *neighbors in* $\Gamma_2^2(x)$, *and* $2 \cdot 112$ *neighbors in* $\Gamma_3(x)$.

Proof. Recall that $\Gamma(x, z)$ is a plane, *i.e.*, $|\Gamma(x, z)| = 7$. If $\{z, u, v\}$ is a line with $u \in \Gamma(x, z)$ then, clearly, $v \in \Gamma_2^2(x)$. This takes care of the sets of neighbors shown above as 7 and $\bar{7}$. Consider Figure 2 as describing the decomposition of Δ with respect to the $\mathscr{H}$-plane $\pi_z(\Gamma(x, z))$. We see that for $28 + 84$ lines $\{z, u, v\}$ not meeting $\Gamma(x, z)$ the points u and v are collinear with some points from $\Gamma(x, z)$. More in particular, for the lines in the box 28 we have that u and v are adjacent with three points (clearly, forming a line) from $\Gamma(x, z)$, while for the lines in the box 84 we have that u and v are collinear with a unique point from $\Gamma(x, z)$.

It follows from Lemma 4.3 that there are exactly $|\Gamma_2^1(x)| \cdot 120$ lines $\{z, u, v\}$ with $u \in \Gamma_2^1(x)$ and $z \in \Gamma_2^2(x)$. For each such line we have that $v \in \Gamma_2^2(x)$. It follows from Figure 4 that z, u and v have three common neighbors in $\Gamma(x)$. Hence all such lines fall (with respect to z) into the box 28. Since $|\Gamma_2^2(x)| \cdot 28$ is equal to $|\Gamma_2^1(x)| \cdot 120 \cdot 2$, we conclude that all the lines from the box 28 are of the above type. It now follows also that if $\{z, u, v\}$ is from the box 84 then both u and v must be in $\Gamma_2^2(x)$.

Finally, suppose $\{z, u, v\}$ is one of the 112 lines, for which u and v have no neighbors in $\Gamma(x, z)$. It follows from Figure 2 that z, u and v are collinear with a point w in $\Gamma_2^1(x)$. Then π_w sends $\{z, u, v\}$ to an $\mathscr{H}$-line with one point, $\pi_w(z)$, in the box 120 (see Figure 4) and the other two points, $\pi_w(u)$ and $\pi_w(v)$, in the box 96. Indeed, suppose that $\pi_w(u)$ is in the box 120. Let a and b be points in $\Gamma(x, w)$ which are adjacent to z and u, respectively. Notice that z and u have no common neighbors in $\Gamma(x)$ by our assumption. Hence $a \neq b$. Since $\Gamma(x, w)$ is isomorphic to a quad (say, via p_w) and since, by Lemma 2.4(1), z is collinear with three points on a line in $\Gamma(x, w)$, we can choose a and b to be collinear. Notice that a and u (and likewise, b and z) are not collinear. Hence $\pi_w(a)$, $\pi_w(b)$, $\pi_w(z)$ and $\pi_w(u)$ are all contained in a common quad Θ. (This follows from Lemma 2.1(3) and (1).) However, this means that the quads Θ and $\pi_w(\Gamma(x, w))$ meet in two points, namely, $\pi_w(a)$ and $\pi_w(b)$. This contradiction proves that $\pi_w(u)$ (and similarly, $\pi_w(v)$) is in the box 96. Lemma 4.3 now yields that u and v are in $\Gamma_3(x)$. □

We now switch to the points at distance three from x.

Lemma 4.5. *If $z \in \Gamma_3(x)$ then z has* 21 *neighbors in* $\Gamma_2^1(x)$, 210 *neighbors in* $\Gamma_2^2(x)$, *and* $\overline{21} + \overline{210}$ *neighbors in* $\Gamma_3(x)$.

Proof. It follows from Lemmas 4.3 and 4.4 that if $\{z, u, v\}$ is a line and $u \in \Gamma_2(x)$ then $v \in \Gamma_3(x)$. This means that π_z isomorphically maps $\Theta = \Gamma_2(x) \cap \Gamma(z)$ onto an induced subgraph of Δ. We will see that that subgraph is in fact the entire Δ.

Suppose $u \in \Theta \cap \Gamma_2^1(x)$. We claim that u has 30 neighbors in Θ (recall that the valency of Δ is 30) and that they are all in $\Theta \cap \Gamma_2^2(x)$. Indeed, let $\{z, u', v'\}$ be one of the 30 lines on z that are coplanar with zu. (Equivalently, π_z sends u' and v' to an $\mathscr{H}$-point collinear with $\pi_z(u)$.) It follows from Figure 4 that π_u sends the line zu' onto a line in Δ having two points in the box 96 and one point, $\pi_u(u')$ or $\pi_u(v')$, in the box 120. Thus, according to Lemma 4.3, either u', or v' is in $\Gamma_2^2(x)$; the claim follows.

Suppose now that $u \in \Theta \cap \Gamma_2^2(x)$. Then we claim that again u has 30 neighbors in Θ, out of which 3 are in $\Gamma_2^1(x)$ and the remaining 27 are in $\Gamma_2^2(x)$. Indeed, according to Figure 2, there are, respectively, three and twelve lines on u that are coplanar with zu and that fall under π_u into the boxes, respectively, 28 and 84. According to Lemma 4.4, these fifteen lines produce 30 neighbors of u and z, that are in $\Gamma_2(x)$. (Hence they are in Θ.) More in particular, 3 of these neighbors are in $\Gamma_2^1(x)$, while the remaining $\bar{3} + 24$ of them are in $\Gamma_2^2(x)$. We have proved the claim.

Since Δ is connected of valency 30, we now have that π_z maps Θ isomorphically onto Δ. Since every point in $\Theta \cap \Gamma_2^1(x)$ is adjacent to 30 points in $\Theta \cap \Gamma_2^2(x)$, while every point from $\Theta \cap \Gamma_2^2(x)$ is adjacent to 3 points from $\Theta \cap \Gamma_2^1(x)$, we obtain that the points from Θ split between $\Gamma_2^1(x)$ and $\Gamma_2^2(x)$ in the proportion one to ten. Since $|\Delta| = 231$, we finally obtain that $|\Theta \cap \Gamma_2^1(x)| = 21$ and $|\Theta \cap \Gamma_2^2(x)| = 210$. □

Remark. It was shown in this proof that $\Gamma(z) \cap \Gamma_2^1(x)$ is a coclique.

As an immediate consequence of (4.5) we have

Corollary 4.6. $\Gamma_{<3}(x) = \{x\} \cup \Gamma(x) \cup \Gamma_2^1(x) \cup \Gamma_2^2(x)$ *is a geometric hyperplane of* $\mathscr{G}$.

It follows from Lemmas 4.3 and 4.5 that $|\Gamma_3(x)| = \frac{|\Gamma_2^1(x)| \cdot 2 \cdot 96}{21} = 22528$. Since every neighbor of $z \in \Gamma_3(x)$ is shown to be in $\Gamma_2(x) \cup \Gamma_3(x)$, Γ has diameter three. The total number of points in Γ is therefore $1 + 462 + 2464 + 21120 + 22528 = 46575$. We collect most of the information at hand in the diagram in Figure 5. Not coincidentally, this diagram coincides with the diagram of the collinearity graph of the P-geometry for Co_2.

Since x was an arbitrary point from Γ, Γ_2^1, Γ_2^2 and Γ_3 can be understood as binary relations on Γ. It easily follows from the definition of these relations that they are symmetric.

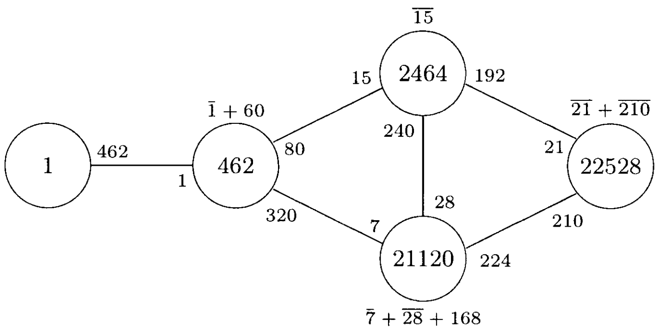

Figure 5. Decomposition of Γ from a point

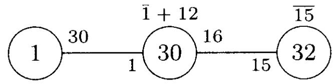

Figure 6. Decomposition of a symp from a point

5 Subgeometries

In this section we establish the existence and some properties of two kinds of subgeometries in $\mathcal{G}$.

By a symplecton (or a *symp*, for short) we mean a point-line subgeometry in $\mathcal{G}$ isomorphic to the (rank three) polar space for $\mathrm{Sp}(6,2)$, as well as the subgraph in Γ induced on the points of the subgeometry. The diagram of a symp is shown in Figure 6.

The existence of symps in Γ can be derived from Cooperstein's lemma [1].

Lemma 5.1. *The geodesic closure of any two points in relation Γ_2^1 is a symp. Two non-collinear points that are contained in a common symp are necessarily in relation Γ_2^1.*

Proof. Observe that every μ-graph in Γ is either a clique, or it is (isomorphic to) a quad. Notice that the 3-cliques in the 15-point μ-graphs are true lines, and so the assumptions of Cooperstein's lemma [1] are satisfied. Hence the conclusion of that lemma must hold, that is, the geodesic closure of any pair of points in relation Γ_2^1 is a subgeometry isomorphic to a non-degenerate polar space of rank three. Since the μ-graphs in the collinearity graph of that polar space are isomorphic to quads, we conclude that that subgeometry is a symp. The second claim also follows. □

The symp defined by two points x and y in relation Γ_2^1 will be denoted $S(x,y)$. Notice that symps are subspaces, *i.e.*, they are closed with respect to lines. We will make two further comments: First, it follows from Lemma 5.1 that every symp coincides with the geodesic closure of any pair of its non-collinear points. Second, suppose Σ is a symp and $x \in \Sigma$. Then $\pi_x(\Sigma(x))$ is a quad. Vise versa, every quad Θ in Δ arises as $\pi_x(\Sigma(x))$ for some (unique) symp Σ on x. Indeed, if y and z are chosen in $\Gamma(x)$ so that $\pi_x(y)$ and $\pi_x(z)$ are non-collinear $\mathcal{H}$-points in Θ then y and z are in relation Γ_2^1 and Σ must be chosen as $S(y,z)$.

We now prove two lemmas outlining some properties of symps.

Lemma 5.2. *Every plane is contained in a unique symp. In particular, the intersection of two symps is empty, or it is a point, or a line.*

Proof. Suppose P is a plane and $x \in P$. Since $\pi_x(P \backslash \{x\})$ is a line, it is contained in a unique quad. This implies the first claim. Clearly, the intersection of two symps cannot contain non-collinear points. Since symps are subspaces, the intersection is either empty, or a point, or a line. □

Lemma 5.3. *If y and z are in $\Gamma_2^1(x)$ and y and z are adjacent then $S(x,y) = S(x,z)$.*

Proof. Comparing Figures 5 and 6, we see that all neighbors of y in $\Gamma_2^1(x)$ are contained in $S(x, y)$. □

Recall that a *subspace* is a set of points that is closed with respect to lines. That is, it contains every line that it meets in two points. For example, symps are subspaces. Also, every geometric hyperplane is a subspace. If Ω is a set of points then $\langle\Omega\rangle$ denotes the subspace *generated* by Ω, *i.e.*, the smallest subspace containing Ω. The following lemma introduces a new class of subgeometries.

Lemma 5.4. *Let Σ be a symp, x and y be non-collinear points of Σ, and $\mathcal{O} = \{u_1, u_2, \ldots, u_5\}$ be an ovoid in $\Sigma(x, y)$. Then $\langle x, y, \mathcal{O}\rangle$ is a subgeometry of Σ isomorphic to the generalized quadrangle for $\mathrm{O}_6^-(2)$.*

Proof. The automorphism group of Σ, $\mathrm{Sp}(6, 2)$, is transitive on pairs of non-collinear points. The stabilizer of two such points (say, x and y) is isomorphic to $\mathrm{Sp}(4, 2) \cong \mathrm{S}_6$. It acts transitively on the set of all ovoids in $\Sigma(x, y)$. Hence $\mathrm{Sp}(6, 2)$ acts transitively on pairs $(\{x', y'\}, \mathcal{O})$, where x' and y' are non-collinear and $\mathcal{O}$ is an ovoid in $\Sigma(x', y')$. So it suffices to show that one such pair generates an $\mathrm{O}_6^-(2)$ generalized quadrangle.

Observe now that $\mathscr{S}$ does contain an $\mathrm{O}_6^-(2)$ generalized quadrangle as a geometric hyperplane. Let $\mathscr{S}_0$ be such a subspace. Let x' and y' be non-collinear points of $\mathscr{S}_0$. Then x' and y' have five common neighbors in $\mathscr{S}_0$ (see the diagram of the collinearity graph of $\mathscr{S}_0$ in Figure 7), and these five points form a coclique in $\Sigma(x', y')$, hence an ovoid $\mathcal{O}$. The 16 points in $\mathscr{S}_0$ that are non-collinear with x' form a connected subgraph. Hence the seven points $\{x', y'\} \cup \mathcal{O}$ generate the entire $\mathscr{S}_0$. □

The following lemma will be useful.

Lemma 5.5. *Let Σ be a symp and Θ be an $\mathrm{O}_6^-(2)$ subgeometry in it. Let u be a point of Θ and let uv_i be the five lines in Θ on u. Then the following hold.*

(1) *The five $\mathscr{H}$-points $\pi_u(v_i)$ share a Witt point a.*

(2) *If w is a neighbor of u outside Θ, such that $\pi_u(w)$ contains a, then w is in relation Γ_2^1 to all the points of Θ that are collinear with u, and in relation Γ_3 to all the points of Θ that are not collinear with u.*

(3) *The point u is the only neighbor of w in Σ.*

Proof. Claim (1) holds because the points v_i are pairwise in relation Γ_2^1 (since they are in a symp). The first part of (2) holds by the choice of w. Let t be a point of Θ that is not collinear with u. Since $t \in \Gamma_2^1(u)$, we have that $\pi_u(\Gamma(u, t))$ is a quad (say, Φ)

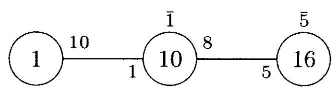

Figure 7. Decomposition of the $\mathrm{O}_6^-(2)$ generalized quadrangle from a point

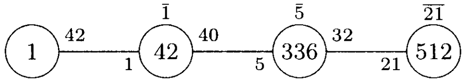

Figure 8. Decomposition of a uniton from a point

and, furthermore, the union of the pairs $\pi_u(v_i)$ is the hexad X that corresponds to that quad Φ. By the definition of w, the pair $\pi_u(w)$ meets X in just a. This means (see Lemma 2.4 and its proof) that $\pi_u(w)$ is at distance two from Φ. Now Lemma 4.3 yields that $w \in \Gamma_3(t)$, proving (2). Since $\pi_u(w)$ meets the hexad X in just one Witt point, we observe that w is not adjacent to any vertex of Σ adjacent to u. Also, since Σ is geodesically closed, w cannot be adjacent to any vertex of Σ that is not adjacent to u. Hence (3) holds. □

By a *uniton* in $\mathscr{G}$ we mean a point-line subgeometry that is closed with respect to lines (*i.e.*, it is a subspace) and isomorphic to the dual polar space for $U_6(2)$. It will be convenient to view a uniton as the subgraph of Γ induced on the points of the subgeometry. Since the μ-graphs in a uniton are disconnected, two points at distance two in a uniton are necessarily in relation Γ_2^1. However, as one can see from the diagram of a uniton (Figure 8) μ-graphs in the uniton are not the full μ-graphs in Γ. So unita are not geodesically closed. It means we will have to work harder in order to construct them.

For points x and z with $z \in \Gamma_3(x)$ set $P(x,z) = \Gamma(x) \cap \Gamma_2^1(z)$. According to Figure 5, the size of $P(x,z)$ is 21. Moreover, we remarked after Lemma 4.5 that this set of 21 points is a coclique. Also, define $\Omega(x,z) = \{x,z\} \cup P(x,z) \cup P(z,x)$ and $D(x,z) = \langle \Omega(x,z) \rangle$. It is immediate from the definition that $D(x,z) = D(z,x)$.

Proposition 5.6. *Suppose x and z are points with $z \in \Gamma_3(x)$. Then $D(x,z)$ is a uniton.*

We will prove this proposition in a series of lemmas. First we will study the set $\Omega(x,z)$. For a point $y \in P(z,x)$ we will denote by $L(y)$ the set $P(x,z) \cap \Gamma(y)$.

Lemma 5.7. *If $y \in P(z,x)$ then $L(y) = S(x,y) \cap P(x,z)$ is of size five. Furthermore, $\pi_x(L(y))$ is an ovoid in the quad $\pi_x(\Gamma(x,y))$.*

Proof. Let $\Theta = \pi_y(\Gamma(x,y))$. Observe that $\pi_y(z)$ cannot be collinear with an $\mathscr{H}$-point from Θ. Indeed, if it is, then z is adjacent to a point from $\Gamma(x,y)$, which means that the distance between x and z is at most two, a contradiction. Now the claim of the lemma follows from Lemma 2.4(2). □

Lemma 5.8. *The set of points $P(x,z)$ endowed with the lines $L(y)$, $y \in P(z,x)$, is a projective plane of order* 4.

Proof. We claim that any two points u and v in $P(x,z)$ have at most one common neighbor in $P(z,x)$. Indeed, suppose $y, w \in P(z,x)$ are collinear with both u and v.

Then $S(x, y) = S = S(x, w)$, where $S = S(u, v)$. However this means that both y and w are in S, which implies that $z \in S$, since S is geodesically closed. This is a contradiction since the distance between x and z is three. Thus, u and v have at most one common neighbor in $P(z, x)$.

We now have that the geometry in question is a partial linear space. Since it has 21 points, each incident to five lines, and 21 lines, each incident to five points, we obtain that our geometry is a projective plane. □

For a Witt point a let C_a denote the set of 21 $\mathscr{H}$-points $\{a, b\}$, $b \neq a$. Notice that C_a is a coclique in Δ, in which any two $\mathscr{H}$-points are in relation Δ_2^1.

Lemma 5.9. *The coclique $P(x, z)$ is bijectively mapped by π_x onto C_a for some Witt point a.*

Proof. Let $u, v \in P(x, z)$. It follows from Lemmas 5.7 and 5.8 that u and v are in relation Γ_2^1. This means that the $\mathscr{H}$-points $\pi_x(u)$ and $\pi_x(v)$ share a common Witt point. Since this is true for all $u, v \in P(x, z)$, the claim of the lemma follows. □

We will construct the subgraph $D(x, z)$ "layer-by-layer". Let $D_1(x, z) = \langle x, P(x, z) \rangle$. For a point w and a Witt point b, let $\mathscr{C}(w, b)$ be the union of 21 lines on w, whose points (other than w) are mapped by π_w into C_b. We will call $\mathscr{C}(w, b)$ a *line claw*.

Lemma 5.10. *The subspace $D_1(x, z)$ is the union of the* 21 *lines xu, $u \in P(x, z)$. That is, $D_1(x, z)$ is the line claw $\mathscr{C}(x, a)$, where a is as in Lemma* 5.9.

Proof. According to Lemma 5.9, $P(x, z)$ is mapped by π_x onto C_a, and it is a coclique, so the claim follows. □

It follows that $D_1(x, z)$ has size $1 + 21 \cdot 2 = 43$. Clearly, $D_1(x, z) \subset D(x, z)$. We will see later that $D_1(x, z)$ contains, in fact, the entire neighborhood of x in $D(x, z)$.

For a line claw $D_1 = \mathscr{C}(x, a)$, we will call a symp Σ on x *compatible* with D_1 if the hexad corresponding to the quad $\pi_x(\Sigma(x))$, contains a. There are exactly 21 symps compatible with D_1, and each of them contains exactly five lines from D_1. (If we view the lines in D_1 as points, and the intersections of D_1 with compatible symps as lines, then the geometry that results is a projective plane of order four, cf. Lemma 5.8.)

Next, we extend $D_1(x, z)$ to a larger subgraph $D_2(x, z)$. For $y \in P(z, x)$ define $O^-(x, y) = \langle x, y, L(y) \rangle$. According to Lemma 5.7, $L(y)$ is an ovoid in $\Gamma(x, y)$ and hence, by Lemma 5.4, $O^-(x, y)$ is an $O_6^-(2)$ subgeometry in the symp $S(x, y)$. The latter symp is one of the 21 symps compatible with $D_1(x, z)$. Define $D_2(x, z)$ as the union of the twenty one subgraphs $O^-(x, y)$, $y \in P(z, x)$. Since symps are geodesically closed, $S(x, y) \neq S(x, y')$ whenever $y, y' \in P(z, x)$, $y \neq y'$. (Indeed, z is a common neighbor of y and y', and $z \in \Gamma_3(x)$.) Hence $S(x, y) \cap S(x, y')$ is a line, namely, one of the lines in $D_1(x, z)$ (cf. Lemmas 5.2 and 5.8). Thus, the intersection of $S(x, y)$ with $D_2(x, y)$ coincides with $O^-(x, y)$. Clearly, $O^-(x, y)$ and $O^-(x, y')$ meet in the same line $S(x, y) \cap S(x, y')$. It follows that $D_2(x, z)$ contains, besides $D_1(x, z)$, exactly $21 \cdot 16 = 336$ points and all these points belong to $\Gamma_2^1(x)$.

Lemma 5.11. *Suppose $y \in P(z, x)$ and let z' be the third point on the line zy. Then $P(x, z') \cap P(x, z) = L(y)$. Furthermore, $D_1(x, z') = D_1(x, z)$ and $D_2(x, z') = D_2(x, z)$.*

Proof. First of all, $L(y)$ is contained in $P(x, z') \cap P(x, z)$. This means that the line claws $D_1(x, z')$ and $D_1(x, z)$ share five lines, yielding $D_1(x, z') = D_1(x, z)$. In particular, the same 21 symps are compatible with $D_1(x, z')$ and $D_1(x, z)$. Let S be one of those symps and let t (respectively, t') be the neighbor of z (respectively, z') in S. Let $T = P(x, z') \cap \Gamma(t')$ and $\Theta = \langle x, t', T \rangle$. To prove that $D_2(x, z') = D_2(x, z)$ it suffices to show that $\Theta = O^-(x, t)$. If $t = y$ then also $t' = y$ and $T = L(y)$, so $\Theta = O^-(x, t)$. Suppose now that $t \neq y$. Let u be the unique point in $L(y) \cap L(t)$. Since $u \in L(y)$, we have that $u \in P(x, z')$. It now follows that t' is collinear with u. We claim that furthermore t' is the third point on the line ut. Indeed, let $S' = S(u, z)$. Notice that $y \in S'$ and hence also $z' \in S'$, because z' is on the line zy. Since $z' \in S'$, also $t' \in S'$. On the other hand, $t' \in S$. Thus, t' is contained in $S \cap S'$ which coincides with the line ut. (Notice also that $t \neq t'$ because every point of ut is collinear with a unique point on zy. This can be seen in the symp S'.) Thus indeed, t' is the third point on the line ut. This shows that $t' \in O^-(x, t)$. Since also $T \leqslant D_1(x, z') \cap S \leqslant O^-(x, t)$, we conclude that $\Theta \leqslant O^-(x, t)$, yielding equality because of the equal size. We have proved that $D_2(x, z') = D_2(x, z)$.

Finally, the fact that t' is the third point on ut implies that u is the only common point in T and $L(t)$. This shows that $P(x, z')$ and $P(x, z)$ have no common points outside $L(y)$. □

We can now establish that $D_2(x, z)$ is a subspace.

Lemma 5.12. *The following hold.*

(1) *$D_2(x, z)$ is a subspace, and hence $D_2(x, z) = \langle x, P(x, z), P(z, x) \rangle$.*

(2) *If $u \in D_1(x, z)$ then u is on exactly 21 lines in $D_2(x, z)$ and these lines form a line claw $\mathscr{C}(u, a)$ for some Witt point a.*

Proof. Suppose $u, v \in D_2(x, z)$ and they are collinear. We claim that u and v must belong to the same $O^-(x, y)$. Suppose not. Then in view of Lemma 5.10 either u or v is not in $D_1(x, y)$. Without loss of generality, $u \notin D_1(x, y)$. Then $u \in O^-(x, y)$ for a unique $y \in P(z, x)$. If $v \in S(x, y)$ then $v \in O^-(x, y)$ (indeed, $S(x, y) \cap D_2(x, z) = O^-(x, y)$), a contradiction. So $v \notin S(x, y)$. If $v \in D_1(x, z)$ then Lemma 5.5(2) implies that u and v are in relation Γ_3, a contradiction. Hence also $v \notin D_1(x, z)$ and Lemma 5.3 yields a contradiction, proving our claim and (1).

Turning to (2), let $u \in D_1(x, z)$. In the case $u = x$ the Claim (2) follows from Lemma 5.9, so suppose $u \neq x$. If u is not in $P(x, z)$ then choose $y \in P(z, x)$ that is noncollinear with u and substitute z with the third point z' on the line zy. According to Lemma 5.11, $u \in P(x, z')$ and $D_2(x, z') = D_2(x, z)$. Thus, without loss of generality we may assume that $u \in P(x,z)$. Observe that u is contained in $O^-(x,y)$ for five points $y \in P(z,x)$ (see Lemma 5.8). Hence u is on $1 + 5 \cdot 4 = 21$ lines in $D_2(x,z)$. Let us take a closer look at these 21 lines. One of them is the line ux and five more of them are the lines uy, $y \in P(z,x) \cap \Gamma(u)$. According to Lemma 5.9, the points y form a coclique (hence, an ovoid) in the quad $\Gamma(u,z)$. This means that the $\mathscr{H}$-points $\pi_u(y)$ are in C_a for some (unique) Witt point a. Since $x \in \Gamma_2^1(y)$ for all $y \in P(z,x) \cap \Gamma(u)$, we have that also $\pi_u(x) \in C_a$. Each $O^-(x,y)$ gives us three more lines uv on u. Since v is in relation Γ_2^1 to both x and y, we obtain that all $\mathscr{H}$-points $\pi_u(v)$ are contained in the same C_a, and (2) follows. □

Lemma 5.13. *We have that* $z \notin \Gamma_2^2(t)$ *for all* $t \in D_2(x,z)$.

Proof. Clearly, $t \in \Theta = O^-(x,y)$ for some $y \in P(z,x)$. Observe that the points in $L(y)$ are pairwise in relation Γ_2^1. Also, z is in relation Γ_2^1 with all the points from $L(y)$ (since $L(y) \subset P(x,z)$). Therefore, $\pi_y(L(y) \cup \{z\})$ is contained in C_a for some Witt point a. Now the claim follows from Lemma 5.5(2). □

Let us now introduce two equivalence relations on $\Gamma_3(x)$. For $z, z' \in \Gamma_3(x)$ and $i \in \{1,2\}$ we write $z \sim_i z'$ if and only if $D_i(x,z) = D_i(x,z')$. Let Σ be a symp compatible with D_1. For a point $y \in \Sigma_2(x)$, let $L(y) = D_1 \cap \Gamma(y)$. This extends the notation introduced before Lemma 5.7. The set $L(y)$ consists of five points and is a coclique. Since $L(y) \subset \Sigma$, the points in $L(y)$ are pairwise in relation Γ_2^1. Hence $\pi_y(L(y))$ is contained in C_b for some unique b, *i.e.*, $L(y)$ is contained in a line claw $\mathscr{C}(y,b)$. This unique b will be denoted by b_y.

We can now characterize the equivalence classes defined by $\sim_i$, $i = 1,2$. Let $[z]_i$ denote the $\sim_i$ equivalence class containing z.

Lemma 5.14. *Let* $D_1 = D_1(x,z)$ *and let* Σ *be a compatible symp. Then* $z' \in [z]_1$ *if an only if* z' *is collinear with a point* $y \in \Sigma_2(x)$ *and* $z' \in \mathscr{C}(y,b_y)\backslash\Sigma$. *In particular, every equivalence class of* $\sim_1$ *consists of* 1024 *points.*

Proof. If $z' \in [z]_1$ then $\Sigma = S(x,y)$ for some $y \in P(z',x)$. Furthermore, the points in $L(y)$ are in relation Γ_2^1 to z'. Hence $z' \in \mathscr{C}(y,b_y)$. Clearly, $z' \notin \Sigma$, since $z' \in \Gamma_3(x)$.

Reversely, let y be an arbitrary point from $\Sigma_2(x)$. (There are 32 such points y.) Let z' be one of the 32 points from $\mathscr{C}(y,b_y)\backslash\Sigma$. It follows from Lemma 5.4 that x, y and the five points in $L(y)$ generate an $O_6^-(2)$ subgeometry Θ in Σ. By Lemma 5.5 we have that $z' \in \Gamma_3(x)$. Notice now that $D_1(x,z')$ contains the five lines xu, $u \in L(y)$, and hence $D_1 = D_1(x,z')$. Therefore, $z' \in [z]_1$.

Now a counting gives us that $[z]_1$ consists of $32 \cdot 32 = 1024$ points (compare Lemma 5.5(3)). □

We now turn to the second equivalence, $\sim_2$.

Lemma 5.15. *Let* $D_2 = D_2(x,z)$, *and let* Θ *be one of the* $O_6^-(2)$ *generalized quadrangle subgeometries forming* D_2. *Then* $z' \in [z]_2$ *if an only if* z' *is collinear with a point*

$y \in \Theta_2(x)$ and $z' \in \mathscr{C}(y, b_y) \backslash \Theta$. In particular, every equivalence class of $\sim_2$ consists of 512 *points.*

Proof. Let $D_1 = D_1(x, z)$ and let Σ be the symp containing Θ. Then Σ is compatible with D_1.

If $z' \in [z]_2$ then, clearly, z' is collinear with a point $y \in \Theta$ (which must be in $\Theta_2(x)$, because $z' \in \Gamma_3(x)$). We have that $L(y) \subset \Gamma_2^1(z')$, which implies that $z' \in \mathscr{C}(y, b_y)$. Manifestly, $z' \notin \Theta$.

Reversely, let $y \in \Theta_2(x)$, and let $z' \in \mathscr{C}(y, b_y) \backslash \Theta$. Observe that $\Sigma \cap \mathscr{C}(y, b_y) \subset \Theta$. Hence, $z' \in \Gamma_3(x)$ and $D_1(x, z') = D_1$ (cf. Lemma 5.14). Furthermore, $y \in P(z', x)$ and Θ is contained in $D_2(x, z')$. Let Σ' be any other symp compatible with D_1. Let $\Phi = D_2 \cap \Sigma'$ and $\Phi' = D_2(x, z') \cap \Sigma'$. To establish that $D_2(x, z') = D_2$ it suffices to show that $\Phi = \Phi'$. Let xu be the line that is the intersection of Σ with Σ'. By Lemma 5.12(2), we have that u together with its neighborhood in D_2 forms a line claw $\mathscr{C}(u, a)$, and similarly for $D_2(x, z')$ we get a second line claw $\mathscr{C}(u, b)$. Since these two line claws share the five lines in Θ, we have $a = b$. Thus Φ and Φ' share a point w at distance two from x. Since $D_1 = D_1(x, z')$ we now conclude that $\Phi = \langle x, w, L(w) \rangle = \Phi'$. Thus, $D_2(x, z') = D_2$, that is, $z' \in [z]_2$.

For the last claim, observe that $|\Theta_2(x)| = 16$ and that $|\mathscr{C}(y, b_y) \backslash \Theta| = 32$ for $y \in \Theta_2(x)$, since $\Sigma \cap \mathscr{C}(y, b_y)$ is fully contained in Θ. □

Corollary 5.16. *Every $\sim_1$ class is a union of two $\sim_2$ classes.*

Proof. Clearly, $\sim_2$ is a refinement of $\sim_1$. The rest follows from Lemmas 5.14 and 5.15. □

Lemma 5.17. *The set $D_2(x, z) \cup [z]_2$ is a subspace.*

Proof. Let $D = D_2(x, z) \cup [z]_2$. Suppose u and v are collinear points in D. Since $D_2(x, z)$ is a subspace (see Lemma 5.12), we will assume that $u \notin D_2(x, z)$. If $v \in D_2(x, z)$ then according to Lemma 5.15, the entire line uv is contained in D. Thus, we will also assume that $v \notin D_2(x, z)$. This means that $u, v \in \Gamma_3(x)$ and $D_2(x, u) = D_2(x, v)$. In particular, by Lemma 5.13, no point of $D_2(x, u)$ is contained in $\Gamma_2^2(v)$.

Consider now the 21 lines uy on u, where y runs through $P(u, x)$. If uv is one of those lines then, clearly, the claim of the lemma holds true. So suppose the third point on uv is not in $P(u, x)$. According to Lemma 5.9, the 21 $\mathscr{H}$-points $\pi_u(y)$, $y \in P(u, x)$, share a Witt point a. Consequently, $\pi_u(v)$ does not contain a, say, $\pi_u(v) = \{b, c\}$. It remains to notice that the $\mathscr{H}$-point $\{a, d\}$ is in $\Delta_2^2(\{b, c\})$, whenever d is chosen outside the hexad containing a, b and c. Therefore, for some $y \in P(x, u)$ we have $y \in \Gamma_2^2(v)$. This contradiction completes the proof of the lemma. □

Finally, we can determine $D(x, z)$.

Lemma 5.18. $D(x, z) = D_2(x, z) \cup [z]_2$.

Proof. Let $D_1 = D_1(x,z)$, $D_2 = D_2(x,z)$ and $D = D(x,z)$. Since $D_2 \cup [z]_2$ is a subspace, we clearly have that $D \subset D_2 \cup [z]_2$. So it remains to show that $[z]_2 \subset D$.

As in the proof of Lemma 5.15, let Θ be one of the $O_6^-(2)$ generalized quadrangles forming D_2. Suppose $z' \in D \cap \Gamma_3(x)$ and y' is the point in Θ that is collinear with z'. Suppose further that $y'' \in \Theta_2(x)$ is collinear with y'. Then we claim that there exists $z'' \in D \cap \Gamma_3(x)$ that is collinear with y''. Indeed, since Θ is a generalized quadrangle, the line $y'y''$ contains a point $u \in D_1$. Consider the $O_6^-(2)$ subgeometry Θ' in $S(u,z')$, generated by $\{u,z'\} \cup (P(z',x) \cap \Gamma(u))$. On the one hand, $\Theta' \subset D$. On the other hand, every line in Θ' on z' is of the form $z'w$ for $w \in P(z',x) \cap \Gamma(u)$. Pick $w \neq y'$. Then, since $y'' \in \Theta'$, we have that y'' is collinear with a point z'' on $z'w$. Clearly, $z'' \neq w$. Thus, $z'' \in \Gamma_3(x)$ and the claim follows.

Starting from the point $y \in \Theta$ collinear with z, and using connectivity of the subgraph induced on $\Theta_2(x)$, we can conclude that every point in $\Theta_2(x)$ is collinear with a point from $D \cap \Gamma_3(x)$.

Let now y' be an arbitrary point from $\Theta_2(x)$, and let z' be a point from $D \cap \Gamma_3(x)$ that is collinear with y'. Since $z' \in [z]_2$, Lemma 5.15 yields that $z' \in \mathscr{C}(y', b_{y'})$. (See the definitions before Lemma 5.14.) Observe that the $O_6^-(2)$ subgeometries $\langle u, z', P(z',x) \cap \Gamma(u)\rangle$, where $u \in L(y') = P(x,z') \cap \Gamma(y)$, are fully contained in D and their union covers the entire line claw $\mathscr{C}(y', b_{y'})$. Thus, $\mathscr{C}(y', b_{y'}) \subset D$ and hence $[z]_2 \subset D$ by Lemma 5.15. □

Corollary 5.19. *The subgeometry* $\Psi = D(x,z)$ *contains exactly* $1 + 42 + 336 + 512 = 891$ *points.*

Next we exhibit a certain symmetry in the generation of the subgeometries $D(x,z)$.

Lemma 5.20. *Let* $x' \in D(x,z)$. *Then* $D(x,z) \cap \Gamma_3(x') \neq \emptyset$. *Furthermore, for every* $z' \in D(x,z) \cap \Gamma_3(x')$ *we have that* $D(x',z') = D(x,z)$.

Proof. Let $D = D(x,z)$. We first deal with the case $x' = x$. In this case the first claim is obvious. Let $z' \in D \cap \Gamma_3(x)$. Then $z' \in [z]_2$, which implies that $D_2(x,z') = D_2(x,z)$ and $[z']_2 = [z]_2$. So Lemma 5.18 implies that $D(x,z') = D$. Notice that since x is arbitrary,

$$D(u,v) = D(u,w), \qquad (*)$$

whenever $v, w \in \Gamma_3(u)$ and $v \in D(u,w)$.

Next suppose that $x' \in D$ is collinear with x. If $x' \notin P(x,z)$ then $z \in \Gamma_3(x')$, proving the first claim. Furthermore, we have $D = D(z,x) = D(z,x') = D(x',z)$ (the second equality due to $(*)$). Therefore, again by $(*)$, $D(x',z') = D$ for all $z' \in D \cap \Gamma_3(x')$, so the second claim holds, too. Suppose now that $x' \in P(x,z)$. Choose $y \in P(z,x)$ such that y is not collinear with x'. Also choose $u \in P(z,x) \cap \Gamma(x')$ and set $\Theta = O^-(x,u)$. In view of Lemma 5.8, there is a point $w \in P(x,z)$ that is collinear with both y and u. Observe that $x', w \in \Theta$ (by definition of Θ) and that x' and w are not collinear. Also observe that, by Lemma 5.12, w and its neighbors in D form a line claw. It now follows from Lemma 5.5 that $y \in \Gamma_3(x')$. (In particular, the first claim of the lemma holds for x'.) Furthermore, by Corollary 4.6, the third point v on the line zy is in $\Gamma_3(x) \cap \Gamma_3(x')$. Hence, $D = D(x, v) = D(v, x) = D(v, x') = D(x', v)$ (use $(*)$ for the first and third equality). Now the second claim of lemma follows for x' again from $(*)$.

Now let us fix D and vary x. Notice that x is arbitrary subject to the condition that $D = D(x, z)$ for some $z \in \Gamma_3(x)$. Take any such $x \in D$ and let x' be its neighbor in D. It was shown above that $D \cap \Gamma_3(x')$ is nonempty and that $D = D(x', z')$ for every $z' \in D \cap \Gamma_3(x')$. In particular, again $D = D(x', z')$ for some $z' \in \Gamma_3(x')$. This means we can now look at the neighbors of x' in D and so on. Since D is obviously connected, the claims of the lemma hold for all $x' \in D$. □

Corollary 5.21. *Let $\Psi = D(x, z)$ for $z \in \Gamma_3(x)$. Then the following hold.*

(1) *If $x' \in \Psi$ then $\Psi(x')$ is a line claw. In particular, Ψ has valency* 42.

(2) *Ψ is an isometric subgraph of Γ, that is, the distance in Ψ between two points x' and y' coincides with the distance in Γ between them.*

(3) *If $x' \in \Psi$ and L is a line in Ψ then L contains a unique point closest to x'.*

(4) *If $x', u' \in \Psi$ and $d(x', u') = 2$ then the geodesic closure of x' and y' in Ψ is isomorphic to the generalized quadrangle for $O_6^-(2)$.*

Proof. In view of Lemma 5.20, we can take $x' = x$ in all cases. Then (1) follows from Lemma 5.10, while (2) follows from Lemma 5.18. For (3), consider a line L contained in Ψ. If L contains a point at distance three from x (say, $z \in L$) then, according to (1), L is the line zy for some $y \in P(z, x)$. Furthermore, y is the only point on L closest to x. Suppose now that L contains a point at distance two from x, say a point y. We can additionally assume that L is not one of the 16 lines on y which contain points in $\Gamma_3(x)$. Hence L is one of the five lines contained in $O^-(x, y)$. Each of those lines containes a unique point in $\Gamma(x)$. Finally, if L contains no points from $\Gamma_2(x) \cup \Gamma_3(x)$ then L is contained in $D_1(x, z)$ and therefore $x \in L$. This proves (3). To prove (4), consider a point $y \in \Psi$ at distance two from x and let $\Theta = O^-(x, y)$. Notice that $\Psi(x) \cap \Psi(y) = L(y)$ is of size five. In view of Lemma 5.20, the same is true for every pair of points in Ψ at distance two from each other. This means that Θ is geodesically closed in Ψ. Manifestly, Θ is the geodesic closure of x and y in Ψ. □

We are now ready to complete the proof of Proposition 5.6.

Lemma 5.22. *$\Psi = D(x, z)$ is a uniton.*

Proof. It follows from Lemma 5.21 that Ψ is a near-hexagon with quads. Let Θ be a quad of Ψ, *i.e.*, $\Theta = O^-(x', y')$ for some $x', y' \in \Psi$. Let w be a point of Ψ, not contained in Θ. We claim that w is adjacent to a point of Θ. Indeed, there is nothing to prove if $w \in \Psi(x')$. If the distance between x' and w is two then it follows from Lemma 5.8 that w is adjacent to a point in $\Theta(x')$. Finally, if the distance between x'

and w is three then w is adjacent to a point in $\Theta_2(x')$ by Lemma 5.15. This proves that w is adjacent to a point u of Θ. Invoking now Lemma 5.5(2), we obtain that u is the only neighbor of w in Θ and that Θ is gated through u. (This means that for every vertex v of Th there is a shortest path between w and v passing through u.) Thus, Ψ is near-classical and so by [2] it is a dual polar space. By consideration of the orders for dual polar spaces with three points on a line it then follows that $D(x,z)$ is of type $\mathrm{U}_6(2)$. □

6 A graph on the set of unita

Define $\mathscr{D} = \{D(x,z) \mid (x,z) \in \Gamma_3\}$, the set of unita. We will first study how unita can intersect.

Lemma 6.1. *Let $\Psi \in \mathscr{D}$. Then the following hold.*

(1) *For each $x \in \Psi$ there is a unique $\Psi' \in \mathscr{D}$ such that $x \in \Psi'$ and $\Psi(x) = \Psi'(x) = \Psi \cap \Psi'$.*

(2) *For each $\Psi' \in \mathscr{D}$ such that $\Psi \cap \Psi' \neq \emptyset$, there exists $x \in \Psi \cap \Psi'$ such that $\Psi(x) = \Psi'(x) = \Psi \cap \Psi'$.*

Proof. Let $x \in \Psi$ and let $z \in \Psi_3(x)$. Then by Lemma 5.20 we have $\Psi = D(x,z)$. Define the equivalence relations $\sim_1$ and $\sim_2$ as in the preceding section. Pick a point $z' \in [z]_1 \backslash [z]_2$ and define $\Psi' = D(x,z')$. Clearly, $\Psi(x) = \Psi'(x)$. Furthermore, it follows from Lemma 5.18 that $\Psi \cap \Psi'$ contains no points at distance three from x. Suppose $\Psi \cap \Psi'$ contains a point y at distance two from x. Then Ψ and Ψ' share a quad $\Theta = \langle x, y, \Psi(x) \cap \Gamma(y) \rangle$. However, Lemma 5.21(1) implies that z and z' belong to the line claw on y defined by the five lines on y in Θ. Lemma 5.15 now forces a contradiction with the fact that $[z']_2 \neq [z]_2$. Thus, $\Psi \cap \Psi' = \Psi(x)$.

If now Ψ' is an arbitrary uniton such that $\Psi \cap \Psi' = \Psi(x)$ then, clearly, $\Psi' = D(x,z')$ where $z' \in [z]_1 \backslash [z]_2$. Corollary 5.16 implies the uniqueness of such a Ψ', proving (1).

For (2), let Ψ' be a uniton such that $\Psi \cap \Psi' \neq \emptyset$, say, $y \in \Psi \cap \Psi'$. Observe that the unita containing y are in a one-to-one correspondence with the equivalence classes of $\sim_2$ (where the latter is defined with respect to y—on $\Gamma_3(y)$). It follows from Lemma 5.15 and Figure 5 that there are exactly 44 unita containing y. One of them is Ψ. Also, by (1), for each $x \in \{y\} \cup \Psi(y)$, there is a uniton Ψ'' such that $\Psi \cap \Psi'' = \Psi(x)$. Since each of these unita contains y and since $|\{y\} \cup \Psi(y)| = 43$, we obtain (2). □

For a uniton Ψ and a point $x \in \Psi$ let Ψ_x denote the unique uniton such that $\Psi \cap \Psi_x = \Psi(x)$. Define a graph Σ on the set of unita. Two unita are adjacent in Σ if and only if they have a nonempty intersection.

Lemma 6.2. *Let Ψ be a uniton. Then the subgraph induced on the neighbors of Ψ in Σ is isomorphic to the distance 1 or 2 graph of Ψ.*

Proof. According to Lemma 6.1, the neighbors of Ψ are the 891 unita Ψ_x, $x \in \Psi$. If $x, y \in \Psi$ and the distance between them is at most two then, clearly, Ψ_x and Ψ_y have a point in common, and hence they are adjacent in Σ. It remains to show that if the distance between x and y is three then Ψ_x and Ψ_y are disjoint. Suppose by contradiction that that Ψ_x and Ψ_y meet, namely, $\Psi_x \cap \Psi_y = \Psi_x(z) = \Psi_y(z)$ is a line claw at a certain point z. Since the distance between x and y is three, we have that $\Psi \cap \Psi_x \cap \Psi_y = \emptyset$. This in turn implies that z is at distance three from both x and y. Consequently, $\Psi_x = D(x,z)$ and $\Psi_y = D(y,z)$.

Choose $a \in P(x,y)$ and $b \in P(y,x) \cap \Gamma(a)$. Clearly, the symp $S(x,b)$ is compatible with $\Psi(x) = \Psi_x(x)$ and $S(y,a)$ is compatible with $\Psi(y) = \Psi_y(y)$. Consider $Q_1 = S(x,b) \cap \Psi_x$ and $Q_2 = S(y,a) \cap \Psi_y$, which are $O^-(6,2)$ quads in Ψ_x and Ψ_y, respectively. By Lemma 5.5(3), $\Gamma(z) \cap S(x,b)$ is a point c contained in Q_1. In a similar fashion, $\Gamma(z) \cap S(y,a) = \{d\}$ for a point $d \in Q_2$.

Observe that $c \neq d$. Indeed, if $c = d$ then this point is contained in $S(x,b) \cap S(y,a)$, and this intersection coincides with the line ab (cf. Lemma 5.2). Since ab is contained in Ψ, we obtain that c is contained in all three unita, Ψ, Ψ_x, and Ψ_y, a contradiction. Since $c \neq d$, we have that $d \notin Q_1$ and $c \notin Q_2$.

Since c and the line ab are contained in the symp $S(x,b)$, the line ab contains a point e collinear with c. Notice that e is at distance two from z. According to Corollary 4.6, $\Gamma_{<3}(z)$ is a geometric hyperplane. Consequently, $S(y,a) \cap \Gamma_{<3}(z)$ is a geometric hyperplane of $S(y,a)$, which clearly coincides with the neighborhood of d in $S(y,a)$. This yields that e is collinear with d.

By Lemma 5.5, e is the only point in $S(x,b)$ collinear with d and e lies in Q_1. (We use here that $d \in \Psi_x$.) Similarly, e lies in Q_2. This means that e is contained in all three unita Ψ, Ψ_x and Ψ_y, a contradiction. □

Theorem 6.3. *The graph Σ is isomorphic to the graph on* 2300 *vertices on which* Co_2 *acts as a rank three group.*

Proof. This follows from Lemma 6.2 and the result of [3]. □

Before we prove our main theorem we require one further lemma. Towards that end we first introduce some notation. Let $\mathcal{U}$ be the point-line incidence geometry of the dual polar space of type $\mathrm{U}_6(2)$. Let Π denote the collinearity graph of $\mathcal{U}$ and $\hat{\Pi}$ be the graph on the set of vertices of Π, where two vertices are adjacent if and only if they are at distance one or two in Π. (Notice that Π is isomorphic to every uniton in Γ and that the graph Σ is locally $\hat{\Pi}$.)

Lemma 6.4. *Let C be a maximal clique in $\hat{\Pi}$ with $|C| \geqslant 43$. Then $|C| = 43$ and there is a point a of $\mathcal{U}$ such that $C = \{a\} \cup \Pi(a)$.*

Proof. First note that C cannot be fully contained in a quad because a quad contains only 27 points. Let Q be an arbitrary quad and let $z \in C \backslash Q$. Then Q contains a unique point z' collinear with z and, furthermore, $Q \cap \Pi_{\leqslant 2}(z) = Q \cap (\{z'\} \cup \Pi(z'))$. In particular, every point in $C \cap Q$ is equal to z' or collinear with it. This vertex z' is called the *gate* for z.

Suppose first that there are collinear points a and b in C. Let c be the third point on the line $L = ab$ and choose a quad Q which contains L. Let z be any point of $C \backslash Q$. Then by the above, a and b are collinear with the gate z' for z. This yields that $z' \in L$ and, in particular, c is collinear with z'. Consequently, c is at distance at most two from z (which was arbitrary in $C \backslash Q$) and also at distance at most two from every element from $C \cap Q$, as all these elements are collinear with z'. Since C is maximal, we conclude that $c \in C$, which means that C is closed with respect to lines. Furthermore, we see from the above that if L' is any line in C and $t \in C \backslash L'$ then t is collinear with a point on L'.

Without loss of generality we may suppose that $z' = a$. Set $L' = az$ and let Q' be the quad containing L and L'. If $t \in C \backslash Q'$ then the corresponding gate t' must be on both L and L' and hence $t' = a$. It follows that every point of C is collinear with a, which means that the conclusion of the lemma holds.

Thus, we may assume that C contains no collinear points. Choose a quad Q with a largest possible intersection $C \cap Q$. Considering a point $z \in C \backslash Q$, we see that every point in $C \cap Q$ is collinear with the gate z'. Since there are only five lines on z' in Q, we have that $|C \cap Q| \leqslant 5$.

Clearly, $|C \cap Q| \geqslant 2$. Let R be the set of points of Q that are collinear with all points in $C \cap Q$. It is easy to see that $|R| = 5$ if $|C \cap Q| = 2$, $|R| \leqslant 3$ if $|C \cap Q| = 3$, and $|R| \leqslant 2$ if $|C \cap Q| = 4$ or 5.

Let us now do counting. If $|C \cap Q| = 2$ then every one of the at least 41 points from $C \backslash Q$ is collinear with a single point from R. Since $|R| = 5$, there exists $r \in R$ that is collinear with at least nine points from $C \backslash Q$. Also r is collinear with the two points in $C \cap Q$. Hence $|C \cap \Pi(r)| \geqslant 11$. Notice that the lines and the quads containing r form a projective plane of order four. From this it easily follows that one of the quads on r contains three points from C, a contradiction with maximality of $|C \cap Q|$. Therefore, $|C \cap Q| \neq 2$. Similarly, if $|C \cap Q| = 3$ then some $r \in R$ is collinear with at least $3 + \frac{40}{3}$ points from C, yielding that $|C \cap \Pi(r)| \geqslant 17$. This cannot be unless one of the quads on r contains at least four points from C; a contradiction. Hence, $|C \cap Q| \geqslant 4$ and again some point $r \in R$ is collinear with at least $4 + \frac{38}{2} = 23$ points from C. This is a final contradiction as the number of lines on r is only 21. □

We can now complete our proof of Theorems 3.2 and 2.

Proof. Observe that every point $x \in \Gamma$ produces a clique $\hat{C}(x)$ in Σ, consisting of the 44 unita containing x. Reversely, suppose $\hat{C}$ is a clique in Σ of size 44. Let Ψ be one of the unita from $\hat{C}$. Since the subgraph induced on $\Sigma(\Psi)$ is isomorphic to $\hat{\Pi}$ and since $C = \hat{C} \backslash \{\Psi\}$ is a clique of size 43 in $\Sigma(\Psi)$, Lemma 6.4 implies that there is $x \in \Psi$ such that $C = \{\Psi_y \mid y \in \{x\} \cup \Psi(x)\}$. So clearly $\hat{C}$ is the clique produced by x. Therefore, the vertices of Γ are in a one-to-one correspondence with the 44-cliques from Σ.

It remains to see how the adjacency of vertices x and y translates in terms of $\hat{C}(x)$ and $\hat{C}(y)$. If x and y are adjacent in Γ then y belongs to exactly two line claws on x. Hence y is contained in exactly four unita from $\hat{C}(x)$, that is, $|\hat{C}(x) \cap \hat{C}(y)| = 4$.

If $y \in \Gamma_2^1(x)$ then there are exactly six line claws on x compatible with $S(x, y)$.

For each compatible line claw, only one of the corresponding two unita contains y. Therefore, if $y \in \Gamma_2^1(x)$ then $|\hat{C}(x) \cap \hat{C}(y)| = 6$. If $y \in \Gamma_2^2(x)$ then clearly $\hat{C}(x) \cap \hat{C}(y) = \emptyset$. Finally, if $y \in \Gamma_3(x)$ then $D(x, y)$ is the only uniton containing both x and y. Hence in this last case $|\hat{C}(x) \cap \hat{C}(y)| = 1$. We have proved that x and y are adjacent if and only if $|\hat{C}(x) \cap \hat{C}(y)| = 4$. This shows that the structure of Γ is uniquely determined by Σ, which is in turn unique by Theorem 6.3. This completes the proof. □

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