

Extended near hexagons and line systems

Hans Cuypers

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Abstract. In this paper we study extended near hexagons, and classify a class of line systems in which two lines are either perpendicular, or make an angle α with $\cos \alpha = \pm 1/3$. Among the examples we encounter a set of 2300 lines in \mathbb{R}^{23} related to the second Conway group Co_2 and a set of 2048 lines in \mathbb{R}^{24} related to the group $2^{1+11} : M_{24}$. These line systems carry the structure of an extended near hexagon. The other line systems and extensions of near hexagons under consideration in this paper are all subsystems of these.

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1 Introduction

In [32], Shult and Yanushka studied a class of *line systems* in \mathbb{R}^n , equipped with the standard inner product $(\cdot | \cdot)$, in which any pair of distinct lines is either perpendicular or makes an angle α with $\cos \alpha = \pm 1/3$. A nice and simple example of such a line system is the set of 4 lines in \mathbb{R}^3 passing through the vertices of a regular tetrahedron centered at the origin. Other examples considered by Shult and Yanushka include a system of 2300 lines in \mathbb{R}^{23} related to the Leech lattice and the second Conway group Co_2 and a system of 2048 lines in \mathbb{R}^{24} related to the binary Golay code and the group $2^{1+11} : M_{24}$. These line systems are all *tetrahedrally closed*, i.e., given three lines of the system passing through three of the four vertices of a regular tetrahedron centered at the origin, then also the line passing through the fourth vertex of the tetrahedron is present in the line system.

The study of tetrahedrally closed line systems led Shult and Yanushka to the notion of a *near polygon*. Consider a tetrahedrally closed line system \mathcal{L} , and denote by Σ the set of norm 3 vectors on the lines in \mathcal{L} . (Here the norm of a vector v equals $(v|v)$.) Let $v \in \Sigma$. Then by $\Sigma_i(v)$ we denote the set of vectors in Σ having inner product i with v . Here the relevant values of i are of course $\pm 3, \pm 1$ and 0. For each pair of vectors u and w in $\Sigma_{-1}(v)$ that have inner product -1 , there is a unique tetrahedron with the three vectors v, u and w as vertices. The fourth vertex $-(v + u + w)$ of this tetrahedron is then also a vector in $\Sigma_{-1}(v)$. If one takes the vectors in $\Sigma_{-1}(v)$ as points, and the triples of vectors in $\Sigma_{-1}(v)$ that together with v are the four vertices of

a regular tetrahedron on v as lines, then under some additional conditions on connectedness this point-line geometry satisfies the following axiom:

(NP) given a line l and point p , there is a unique point on l closest to p ;

which is the defining axiom of a *near polygon*.

The additional connectedness conditions forcing NP are:

(C) Every pair of vertices v and w with inner product -1 is contained in a tetrahedron.

Suppose T is a tetrahedron on v , whose vertices are in Σ , and x is an element of $\Sigma_{-1}(v)$, having inner product 0 or 1 with the three vertices of T distinct from v . Then there is a vector $y \in \Sigma_{-1}(v) \cap \Sigma_{-1}(x)$ having inner product -1 with at least 1 of the three vertices of T distinct from v .

This is the revised version by Neumaier (see [27]) of Shult and Yanushka's condition to force connectedness of the local point-line geometry.

Indeed, if T is a tetrahedron on v whose three other vertices are in $\Sigma_{-1}(v)$, then for every vertex $x \in \Sigma_{-1}(v)$ we have one of the following possibilities. The vector x is a vertex of T , or it has inner product -1 with exactly one vertex w of T distinct from v and 1 with the remaining two vertices, or it is perpendicular to two vertices in T and has inner product 1 with the third vertex in T distinct from v . In the second case, the vertices v, x and w are again in a tetrahedron.

Notice that two vertices $x, y \in \Sigma_{-1}(v)$ are at distance 1, 2 or 3 inside the geometry induced on $\Sigma_{-1}(v)$ if and only if their inner product equals $-1, 1$ or 0 , respectively.

So, if the line system is tetrahedrally closed and satisfies condition (C), then the vectors in Σ together with the set of tetrahedra with vertices in Σ form an *extended near hexagon* in the terminology of Cameron et al. [13]. This extended near hexagon is a 2-fold cover of the extended near hexagon that we can define on the line system \mathcal{L} , where the *blocks* or *circles* correspond to the sets of 4 lines in \mathcal{L} through the vertices of the tetrahedra. The point graph of the extended near hexagon on Σ is a graph which is locally the near hexagon; the point graph of the extended near hexagon defined on \mathcal{L} , however, is locally the distance 1-or-2 graph of the near hexagon. So the extended near hexagon on the lines in \mathcal{L} satisfies the following condition:

(*) $\{x, y, z\}$ is a triangle in the point graph not contained in a circle, if and only if the distance between y and z in the local near hexagon at x is 2.

To ensure connectivity of the extended near hexagons, one can restrict attention to line systems that are *indecomposable*. That means, there is no non-trivial partition of the set \mathcal{L} in which any two lines from distinct parts of the partition are perpendicular.

The theory of near polygons has been developed by the work of several authors, see for example [7, 6, 12], and even led to the full classification of all near hexagons with 3 points per line in which any two points at distance ≤ 2 are in some quad, a near hexagon in which a point is always collinear to some point of a line, see [6].

The study of extended near polygons started with the work of Buekenhout and Hubaut [11]. Considerable work has been done on extensions of near polygons, in

particular of generalized polygons, see [12, 26, 34, 35, 36]. For an overview of some of these results, the reader is referred to the chapter by Buekenhout and Pasini in the Handbook of Incidence Geometry [10]. However, since the pioneering work of Shult and Yanushka [32], and the article of Neumaier [27], as far as we know, no progress has been made on the study of the indecomposable tetrahedrally closed line systems.

In this paper we study both extensions of near hexagons and their connections with tetrahedrally closed line systems. We call a line system in which two lines either are perpendicular or make an angle α with $\cos \alpha = \pm 1/3$ *regular*, if and only if it is indecomposable, tetrahedrally closed, for each vector v of norm 3 on one of the lines in \mathcal{L} condition (C) is satisfied, and the near hexagon on $\Sigma_{-1}(v)$ is regular. Our main result on line systems reads as follows.

Theorem 1.1. *Let \mathcal{L} be a line system in \mathbb{R}^n , such that any pair of lines is either perpendicular or makes an angle α with $\cos \alpha = \pm 1/3$. If \mathcal{L} is regular, then it is isomorphic to one of the following systems:*

- i. *the system of 10 lines in \mathbb{R}^5 related to $2 \times \text{PSp}_4(2)$;*
- ii. *the system of 16 lines in \mathbb{R}^6 related to $2^{1+4} : \text{Sp}_4(2)$;*
- iii. *the system of 28 lines in \mathbb{R}^7 related to $2 \times \text{PSp}_6(2)$;*
- iv. *the system of 40 lines in \mathbb{R}^{10} related to $2 \times \text{PSU}_4(2)$;*
- v. *the system of 120 lines in \mathbb{R}^{15} related to $2 \times \text{PSp}_6(2)$;*
- vi. *the system of 128 lines in \mathbb{R}^{16} related to $2^{1+7} : \text{G}_2(2)$;*
- vii. *the system of 256 lines in \mathbb{R}^{16} related to $2^{1+8} : \text{Sp}_6(2)$;*
- viii. *the system of 2048 lines in \mathbb{R}^{24} related to $2^{1+11} : \text{M}_{24}$;*
- ix. *the system of 2300 lines in \mathbb{R}^{23} related to $2 \times \text{Co}_2$.*

In fact, all these line systems, except for the sixth one, have been described by Shult and Yanushka [32], who also gave some partial uniqueness results. A description of these systems will also be given in Section 3 of this paper.

Certainly the most interesting example is the set of 2300 lines in \mathbb{R}^{23} related to the second Conway group Co_2 . It contains all the other line systems, except for the one on 2048 lines related to $2^{1+11} : \text{M}_{24}$.

Our proof of Theorem 1.1 starts with the extended near hexagon defined on the line system. As observed above, the point graph of this geometry is a connected graph which is locally the distance 1-or-2 graph of a regular near hexagon. Such geometries are classified in the following result, which is the main theorem of this paper.

Main Theorem 1.2. *Let Γ be an extended near hexagon with regular local near hexagons having 3 points per line. If Γ satisfies (*), then Γ is isomorphic to one of the following:*

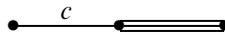
- i. the extended generalized quadrangle on 10 points related to $\text{PSP}_4(2)$;
- ii. the extended generalized quadrangle on 16 points related to $2^4 : \text{Sp}_4(2)$;
- iii. the extended generalized quadrangle on 28 points related to $\text{PSP}_6(2)$;
- iv. the extended near hexagon on 40 points related to $\text{PSU}_4(2)$;
- v. the extended generalized hexagon on 120 points related to $\text{PSP}_6(2)$;
- vi. the extended generalized hexagon on 128 points related to the group $2^7 : \text{G}_2(2)$;
- vii. the extended near hexagon on 256 points related to $2^8 : \text{Sp}_6(2)$;
- viii. the extended near hexagon on 2048 points related to $2^{11} : \text{M}_{24}$;
- ix. the extended near hexagon on 2300 points related to Co_2 .

Theorem 1.1 is obtained as a corollary to the above result. As noticed above, any line system satisfying the conditions of Theorem 1.1 gives rise to an extended near hexagon as in Theorem 1.2. The set Σ of norm 3 vectors of a line system carries also the structure of an extended near hexagon, and is a 2-fold cover of one of the examples in Theorem 1.2. Such covers can be determined, and the set Σ can be recovered as a set of vectors obtained by projecting the points of the cover on an appropriate eigenspace of the adjacency matrix of its collinearity graph. This will be carried out in Section 6.

The *enrichment* of an extended near hexagon, i.e., the geometry of points, pairs of points in a circle and circles (see [13]), is a diagram geometry with Buekenhout diagram



where NH stands for the class of near hexagons. In particular, if the local near hexagon is a generalized hexagon we obtain a diagram geometry with diagram



Moreover, as we will show in Section 4, the regular near hexagons containing quads yield geometries with diagram



The geometries of Theorem 1.2 appear as subgeometries of the Baby monster geometry. The results of Theorem 1.2 have been used by Ivanov, Pasechnik and Sphectorov in their geometric characterization of the Baby Monster, see [25].

The enrichments of all the nine examples of extended near hexagons in the above theorem admit a flag-transitive automorphism group. Weiss [34, 35] has classified the extensions of classical generalized hexagons as diagram geometries admitting a flag-transitive group, and satisfying some additional conditions closely related to condition (*) of Theorem 1.2. In [34], however, he missed the example on 120 points

related to the group $\text{PSp}_6(2)$. Weiss informed us that this example should have appeared in Lemma 4.5 of [34], as the group $G_{1,1,1}$. In [19], the author extended Weiss' result by classifying all flag-transitive extensions of buildings of type G_2 or \mathcal{C}_3^* under the same additional conditions. The Co_2 extended near hexagon has also been characterized by Meixner [26] and by Yoshiara [36] as a flag-transitive extension of the $\text{PSU}_6(2)$ dual polar space.

The case of a circular extension of a near hexagon of order 2 is a special case of an affine extension of a near hexagon. In [4, 5] it is shown that the Examples (v), (vi) and (vii) are special cases of affine extensions of generalized hexagons of type G_2 and dual polar spaces of type C_3 , respectively.

Extensions of regular line thick generalized hexagons whose point graph is locally the distance 1-or-3 graph of the local hexagon are investigated in [20]. There are just four examples of such extended hexagons; they are related to the almost simple groups $G_2(2)$, HJ, Suz and $\text{PSU}_4(3)$, see [20]. These examples are also characterized by Weiss [34, 35] and the author [19] under the assumption of a flag-transitive action of the automorphism group.

In the work of Weiss [34, 35], Yoshiara [36] and the author [19] also circular extensions of finite classical generalized hexagons and dual polar spaces of rank 3 are considered where the number of points per line is not restricted, but arbitrary. The methods we develop in Section 4 lead to the following theorem:

Theorem 1.3. *Suppose Γ is an extended near hexagon, such that for each point p of Γ the local near hexagon Γ_p at p is regular with classical parameters (s, t_2, t) where $s \geq 2$. If Γ satisfies (*), then $s = 2$. In particular, Γ is one of the examples of the conclusion of Theorem 1.2.*

Here, we say that a near hexagon has classical parameters (s, t, t_2) if and only if either $t_2 > 0$ and $t = t_2(t_2 + 1)$ or $t_2 = 0$ and $t \in \{1, \sqrt[3]{s}, s, s^3\}$.

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2 Definitions and notation

In this section we recall some definitions and fix the notation used throughout this paper.

Let Γ be the incidence structure $(\mathcal{P}, \mathcal{C})$ consisting of non-empty sets \mathcal{P} of points and \mathcal{C} of circles (also called blocks), which are subsets of size at least 2 of \mathcal{P} . The point graph (or collinearity graph) of Γ is the graph with vertex set P , and two points adjacent if and only if they are cocircular (also called collinear), i.e., there is a circle containing them. If p and q are points of \mathcal{P} , then we denote by $p \perp q$ that there is a circle containing them. For each subset X of \mathcal{P} we write X^\perp for the set of all points q with $q \perp x$ for all $x \in X$. For points $p \in \mathcal{P}$ we usually write p^\perp instead of $\{p\}^\perp$.

A subspace X of Γ is a subset of \mathcal{P} such that each circle meeting it in at least 2 points is contained in X . A subspace X is often tacitly identified with the incidence

structure $(X, \{C \in \mathcal{C} \mid |C \cap X| \geq 2\})$. A proper subspace of Γ meeting all circles non-trivially is called a *geometric hyperplane* or, for short, *hyperplane* of Γ . A point in a geometric hyperplane is called *deep*, if all the circles on that point are contained in the geometric hyperplane.

We call Γ a *near polygon*, if it satisfies the following condition:

- (NP) given an element $C \in \mathcal{C}$ and $p \in \mathcal{P}$, then in the point graph of Γ there is a unique point $q \in C$ at minimal distance from p .

Here distance is just the usual graph distance. A near polygon Γ is called a *near hexagon*, if there are at least two non-collinear points in Γ and its point graph has diameter at most 3. It is called a *generalized quadrangle*, when its point graph has diameter at most 2. In a near polygon circles are usually referred to as *lines*. If all lines of a near polygon have at least 3 points and two points at distance two have more than one common neighbor, then the geodesic closure of these two points is a subspace of the near polygon, carrying the structure of a generalized quadrangle. This result is known as Yanushka’s Lemma, see [32]. Such subspaces are called *quads*.

A near polygon, whose point graph is of diameter d , is called *regular* with parameters $(s, t_2, \dots, t = t_d)$, if and only if all lines contain $s + 1$ points, for every point p there is a point q at distance d from p , and for each pair of points (p, q) at distance i , where $2 \leq i \leq d$, there are exactly $t_i + 1$ lines through p containing a point at distance $i - 1$ from q .

Consider an arbitrary incidence structure $\Gamma = (\mathcal{P}, \mathcal{C})$. Let p be a point of Γ . Then by \mathcal{P}_p we denote the set of points adjacent to p in the point graph of Γ , and by \mathcal{C}_p the set of all circles containing p . If all circles of Γ contain at least 3 points, then we can consider the incidence system $\Gamma_p = (\mathcal{P}_p, \{C - \{p\} \mid C \in \mathcal{C}_p\})$, the *residue* of Γ at p . We will call Γ an *extended near polygon*, respectively, *extended near hexagon*, if and only if its point graph is connected all circles contain at least 3 points, and Γ_p is a near polygon, respectively, near hexagon, for all points $p \in \mathcal{P}$.

If Γ is an extended near polygon and p is a point of Γ , then Γ_p is also called the *local near polygon* at p .

The *enrichment* $\mathcal{E}(\Gamma)$ of Γ is the incidence system $(\mathcal{P}, \mathcal{E}, \mathcal{C})$, where \mathcal{E} is the set of edges of the point graph of Γ , and incidence is symmetrized inclusion. As was observed in [13], if Γ is an extended near polygon, then this incidence system is a diagram geometry with Buekenhout diagram



where NP is a class of near polygons.

3 Examples of line systems and extended near hexagons

The examples of line systems and extensions of near hexagons appearing in the theorems of the introduction of this paper will briefly be discussed in this section. There is also some discussion of most of these line systems in [32].

We start with the Leech lattice Λ in 24-dimensional real space. This lattice is the unique 24-dimensional even unimodular lattice, see [17]. Its automorphism group is the group $2 \cdot \text{Co}_1$. We scale the lattice in such a way that the vectors of type 2 whose stabilizer in $2 \cdot \text{Co}_1$ is isomorphic to Co_2 have norm 4. All the line systems of Theorem 1.1 are related to the lattice Λ .

Throughout this section we will use the Atlas [16] notation for elements in the groups mentioned below.

3.1 The 2300 lines in \mathbb{R}^{23} related to $2 \times \text{Co}_2$. Fix a norm 4 type 2 vector v of the Leech lattice Λ , and consider all the 4600 type 2 vectors of Λ that have inner product -2 with v . Projection of these 4600 vectors onto the 23-dimensional space v^\perp perpendicular to v yields a set of 4600 vectors that are on 2300 lines. Two of these lines make an angle α with $\cos \alpha = \pm 1/3$ or 0. The setwise stabilizer of $\{v, -v\}$ in $\text{Aut}(\Lambda)$ is isomorphic to $2 \times \text{Co}_2$ and acts on the 4600 vectors.

3.2 The 1408 lines in \mathbb{R}^{22} related to $2 \times \text{PSU}_6(2)$. In the system of 2300 lines obtained above, the 1408 lines perpendicular to a fixed line form also a tetrahedrally closed line system. The local near hexagon of this line system is isomorphic to the Aschbacher near hexagon related to the group ${}^+\Omega_6^-(3)$, see [1, 6]. (Here we follow the notation of [21] for the orthogonal groups.) The group $\text{PSU}_6(2)$ is contained in the automorphism group of this set of lines. This line system is not in the conclusion of Theorem 1.1, as it is not regular. Indeed, the local near hexagon Γ_p is not regular as is shown in [6]. However, we have included it here to make the description of the example of 40 lines in \mathbb{R}^{10} related to $2 \times \text{PSU}_4(2)$ easier.

3.3 The 256 lines in \mathbb{R}^{16} related to the group $2^{1+8} : \text{Sp}_6(2)$. An involution of type $2A$ in the group Co_2 fixes 256 of the 2300 lines that are all contained in the -1 -eigenspace of this involution of dimension 16. The centralizer of such an element is a group $2^{1+8} : \text{Sp}_6(2)$. The line system is tetrahedrally closed, and is locally the near hexagon of the $\text{PSp}_6(2)$ dual polar space. Inside $\text{Aut}(\Lambda)$, the centralizer of an element of type $2A$ is a group isomorphic to $2^{1+8} : \text{O}_8^+(2)$. The embedding of $\text{Sp}_6(2)$ into $\text{O}_8^+(2)$ is the spin representation of $\text{Sp}_6(2)$; the $\text{PSp}_6(2)$ near hexagon is embedded into the $\text{PO}_8^+(2)$ polar space. This will be used later on.

3.4 The 120 lines in \mathbb{R}^{15} related to $2 \times \text{PSp}_6(2)$. If one fixes a line of the 256 lines of the above example, then the 120 lines perpendicular to this fixed line form a tetrahedrally closed system of 120 lines in \mathbb{R}^{15} , which is locally a generalized hexagon of order $(2, 2)$. The group $2 \times \text{PSp}_6(2)$ acts transitively on these lines with line stabilizer $2 \times \text{G}_2(2)$.

3.5 The 128 lines in \mathbb{R}^{16} related to $2^{1+7} : \text{G}_2(2)$. Let σ be an element of order 4 in the normal 2-group of $2^{1+8} : \text{Sp}_6(2)$. Then the centralizer of this element induces a transitive group $2^7 : \text{G}_2(2)$ on the 128 lines fixed by σ . These 128 lines form a closed line system in the 1-eigenspace of dimension 16 of σ , which is locally a generalized hexagon of order $(2, 2)$ related to $\text{G}_2(2)$. Notice that the 128 lines are not contained in the set of 256 lines on which $2^{1+8} : \text{Sp}_6(2)$ acts as in Example 3.3.

3.6 The 40 lines in \mathbb{R}^{10} related to $2 \times \text{PSU}_4(2)$. In the group $\text{PSU}_6(2)$ an element of type 3A is centralized by a group isomorphic to $\text{PSU}_4(2)$. Such an element of type 3A has an eigenspace of dimension 10 in the 22-dimensional representation of $2 \times \text{PSU}_6(2)$ on the 1408 lines in the second example of this section. There are 40 lines inside this eigenspace that form a closed line system which is locally the near hexagon of Hamming type on 27 points.

3.7 The systems C_{10}, C_{16} , and C_{28} related to $2 \times \text{PSP}_4(2)$, $2^{1+4} : \text{Sp}_4(2)$ and $2 \times \text{PSP}_6(2)$, respectively. The maximal sets of respectively 10, 16 or 28 non-perpendicular lines in the Examples 3.6, 3.3 or 3.1 yield tetrahedrally closed systems C_{10}, C_{16} and C_{28} in $\mathbb{R}^5, \mathbb{R}^6$ and \mathbb{R}^7 respectively. These systems are locally a generalized quadrangle of order $(2, t)$, where t is 1, 2 or 4, respectively. Notice that these three extended generalized quadrangles can be obtained from the root lattices of type E_6, E_7 and E_8 , respectively, (rescaled, such that root vectors have norm 4) by the same procedure as in described in 3.1.

All the above systems are contained in the system of 2300 lines associated to the second Conway group Co_2 . The following and last example, however, although closely related to the Leech lattice and the group Co_2 , is not contained in that system.

3.8 The 2048 lines in \mathbb{R}^{24} related to $2^{1+11} : \text{M}_{24}$. Consider an orthonormal basis \mathcal{B} for \mathbb{R}^{24} with respect to the standard inner product. The vectors of the form $\sum_{b \in \mathcal{B}} \pm b$ form a 24-dimensional vector space over the field $\{\pm 1\}$. Inside this vector space we can consider the extended binary Golay code \mathcal{C} as a 12-dimensional subspace. The 4096 vectors inside \mathcal{C} span 2048 lines in \mathbb{R}^{24} . The inner product of 2 codewords (as vectors in \mathbb{R}^{24}) equals ± 24 (if they are the same or opposite), ± 8 or 0. So two distinct lines make an angle α with $\cos \alpha = 0$ or $\pm 1/3$. The system is tetrahedrally closed. The local near hexagon at each vector is isomorphic to the near hexagon on 759 points related to the group M_{24} . The stabilizer of this line system inside the orthogonal group on \mathbb{R}^{24} with its standard inner product, is isomorphic to the group $2^{1+11} : \text{M}_{24}$.

4 The structure and combinatorics of extended near hexagons

In this section we start the investigation of extended near hexagons as in the hypothesis of Theorems 1.2 and 1.3.

Let $\Gamma = (\mathcal{P}, \mathcal{C})$ be an extended near hexagon in which all circles contain at least 4 points. By \mathcal{G} we denote the point graph of Γ , whose edge set will be denoted by \mathcal{E} . We assume that the graph \mathcal{G} is locally the distance 1-or-2 graph of the local near hexagon. So, if p is a point of Γ , then two points q and r in \mathcal{P}_p are adjacent if and only if they are at distance at most 2 in the near hexagon Γ_p . Moreover, for all points p of Γ we assume the near hexagon Γ_p to be regular.

Lemma 4.1. *There are parameters (s, t_2, t) such that for all $p \in \mathcal{P}$, the residue Γ_p is a regular near hexagon with parameters (s, t_2, t) .*

Proof. Fix a point p and suppose that the near polygon Γ_p has parameters (s, t_2, t) . Let q be a point cocircular with p . Suppose that the near hexagon Γ_q has parameters (s', t'_2, t') . As there are $t + 1$ lines on q in Γ_p , there are $t + 1$ circles containing both p and q . So the point p of Γ_q is on $t + 1 = t' + 1$ lines in Γ_q and $t = t'$. As all circles on p contain $s + 2$ points, the lines of Γ_q on p contain $s + 1$ points. Hence $s' = s$.

Finally we will prove that $t'_2 = t_2$. Consider the set $(p^\perp \cap q^\perp) - \{p\}$. This is the set of points at distance at most 2 from q in Γ_p , it has cardinality $1 + s(t + 1) + s(t + 1)st/(t_2 + 1)$. Similarly, $(q^\perp \cap p^\perp) - \{q\}$, which is the set of points in Γ_q at distance at most 2 from p has cardinality $1 + s'(t' + 1) + s'(t' + 1)s't'/(t'_2 + 1)$. But since these two sets are of the same size, we find $t'_2 = t_2$. This implies that Γ_q has the same parameter set as Γ_p . By connectedness of the point graph of Γ we have proved the lemma. \square

From now on we fix integers s, t and t_2 such that for all points p of Γ the local near hexagon Γ_p has order (s, t_2, t) .

Lemma 4.2. *Let C be a circle and p a point of Γ . Then $C \subseteq p^\perp$ or $|p^\perp \cap C| = 0$ or 2 .*

Proof. Suppose p is a point and C a circle meeting p^\perp . Fix a point $q \in C \cap p^\perp$. Inside Γ_q the point p is either at distance 3 from all but one of the points of $C - \{q\}$, in which case $p^\perp \cap C$ consists of 2 points, or p is at distance at most 2 from every point of $C - \{q\}$, and $C \subseteq p^\perp$. \square

Corollary 4.3. *Suppose p and q are points of Γ at mutual distance 2. If $s = 2$, then $p^\perp - q^\perp$, i.e., the complement of $p^\perp \cap q^\perp$ in p^\perp , is a geometric hyperplane of Γ_p .*

Proof. Suppose s to be equal to 2. Let C be a circle on p , then, by 4.2, there are 0 or 2 points of C in q^\perp . So, $C - \{p\}$ is contained in $p^\perp - q^\perp$ or meets it in just one point. Hence $p^\perp - q^\perp$ is a geometric hyperplane of Γ_p . \square

Lemma 4.4. *Two circles meet in 0, 1 or 2 points.*

Proof. Suppose C and D are two distinct circles meeting in at least 3 points. Without loss of generality we may assume that there is a point $q \in D - C$. Fix a point $p \in C \cap D$. Then inside Γ_p the two circles C and D define two lines with at least two points in common, which is well known to be impossible in a near hexagon. \square

Lemma 4.5. *The parameter s and the size of a circle are both even.*

Proof. Let C be a circle and q a point outside of C with $C \subseteq q^\perp$. (Such a point can be found in Γ_r , where r is a point of C .)

If p is a point of C , then we see inside Γ_p that q is collinear to a unique point of $C - \{p\}$. Thus there is a unique circle on q and p meeting C in two points. But that implies that the circles on q , meeting C in two points, partition C into pairs. In particular, $|C| = s + 2$ is even. \square

Lemma 4.6. *Let X be a set of points in a near hexagon such that the distance between any two points of X is at most 2. If a line of the near hexagon is properly contained in X , then X is either contained in a quad or there is a unique point $p \in X$ such that all points of X are collinear with p .*

Proof. Suppose X is a set as in the hypothesis of the lemma containing some line l . We assume that $X \neq l$, so that there is a point x in $X - l$. Since x is at distance at most 2 from any point on l , it is at distance 1 from l . Let p be the unique point on l collinear with x .

So, either every point of X is collinear to p or there is a point in X not collinear to p . We assume that we are in the latter case and show that X is contained in some quad. Let y be a point in X not collinear to p . Then y is at distance ≤ 2 from both p and x , and thus collinear with some point z on the line through p and x . The point y is also at distance one from the line l . Hence there is a point $u \in l$ collinear with y . Thus x and l are contained in some quad Q . As x and l are in at most one quad, Q is the unique quad containing x and l . This quad also contains y and z .

The above implies that every point of X not collinear to p is contained in the quad Q .

Now suppose $v \in X$ is collinear to p . As y is at distance at most 2 from both p and v , it is collinear to a point on the line through p and v . Since Q is geodesically closed, this line and therefore also v is contained in Q . \square

In a regular near hexagon with parameters (s, t_2, t) where $s \geq 2$, Yanuska's Lemma shows that the existence of quads is equivalent with $t_2 > 0$. This will be exploited below.

Lemma 4.7. *If $t_2 \geq 1$, and p is a point of Γ , then for all quads Q of Γ_p , the set $\{p\} \cup Q$ together with the circles contained in it, is a one point extension of a generalized quadrangle.*

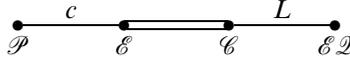
Proof. Fix a point p of Γ , and suppose Q is a quad of the near hexagon Γ_p . Then $E := \{p\} \cup Q$ is a clique in the point graph \mathcal{G} of Γ . Suppose q is a point in E different from p . Then consider the point set $E - \{q\}$ in Γ_q . This is a subset of the near hexagon Γ_q satisfying the hypothesis of the previous lemma. The point p is on $t_2 + 1$ lines in this set. Thus by the above lemma either all points are collinear with p , or $E - \{q\}$ is contained in a quad. Since there are $(1 + s)(1 + st_2)$ points in $E - \{q\}$ of which there are only $s(t_2 + 1)$ points collinear with p , this set $E - \{q\}$ is contained in a quad; it even consists of all points of a quad of Γ_q . This proves the lemma. \square

Corollary 4.8. *If $t_2 \geq 1$, then $s + 2$ divides $2t_2(t_2 + 1)(2t_2 - 1)$.*

Proof. This follows from the above lemma and [13, Theorem 5.1]. \square

The extended generalized quadrangles, as considered in the above lemma, will be called *extended quads* of Γ . By $\mathcal{E}\mathcal{Q}$ we denote the set of all extended quads of Γ . A

consequence of the above lemma is that the extension of a near hexagon with quads (i.e., a near hexagon with $t_2 > 0$) under consideration is a Buekenhout geometry with diagram



Lemma 4.9. *If $t_2 > 0$, then $2 + s(t_2 + 1) \mid (s^2 + 2s + 2)(1 + s)^2 s(t - t_2)$ and $2 + st_2 \mid s^3 \left[\frac{t(t+1) - (t-t_2)(t_2+1)^2}{t_2(t_2+1)} - 1 \right]$.*

Proof. Suppose $t_2 > 0$. Then, as we saw above, Γ contains extended quads. Let EQ be such an extended quad and fix a point $p \in EQ$. Now consider a point q adjacent to p but not in EQ . Denote by Q_p the quad of Γ_p contained in EQ . Within the near hexagon Γ_p we see that q is either collinear to a point of Q_p or at distance ≥ 2 from all points of Q_p and the points of Q_p at distance 2 from q form an ovoid of Q_p . In the first case q is adjacent to $1 + 1 + s(t_2 + 1)$ points of EQ , in the second case to $1 + 1 + st_2$ points.

Suppose $q^\perp \cap EQ$ contains $1 + 1 + s(t_2 + 1)$ points. Let p' be a point of EQ different from p but adjacent to q . Denote by $Q_{p'}$ the quad of $\Gamma_{p'}$ contained in EQ . Then, inside $\Gamma_{p'}$ the number of points of $Q_{p'}$ at distance ≤ 2 from q equals $1 + s(t_2 + 1)$. So, inside $\Gamma_{p'}$ we see that q is at distance 1 from the quad $Q_{p'}$.

Counting pairs of adjacent points x, y with $x \in EQ$ and $y \notin EQ$, but y at distance 1 from the quad Q_x defined by EQ inside Γ_x yields $[1 + (1 + s)(1 + st_2)] \cdot (1 + s) \cdot (1 + st_2)s(t - t_2) = \alpha \cdot (2 + s(1 + t_2))$, where α is the number of points y outside EQ but with $y^\perp \cap EQ$ of cardinality $2 + s(t_2 + 1)$. Hence $2 + s(t_2 + 1) \mid [1 + (1 + s) \cdot (1 + st_2)] \cdot (1 + s)(1 + st_2)s(t - t_2)$ from which we deduce that $2 + s(t_2 + 1) \mid [1 + (1 + s)^2] \cdot (1 + s)^2 s(t - t_2)$.

By counting pairs of adjacent points x, y with $x \in EQ$ and $y \notin EQ$, but y at distance 2 from the quad EQ_x inside Γ_x we obtain the second divisibility condition. \square

Let p and q be two adjacent points of Γ and denote by $X_{p,q}$ the union of all cliques on p and q . This is a clique in the point graph of Γ containing $1 + 1 + s(t + 1)$ points.

Lemma 4.10. *Let p and q be adjacent points in Γ and suppose x is a point of $X_{p,q}$. Then there is a point y in $X_{p,q}$ different from x with $X_{x,y} = X_{p,q}$.*

Proof. If x equals p or q , then the result is obviously true. Thus assume x to be different from both p and q . The set $X_{p,q} - \{x\}$, as subset of Γ_x , satisfies the conditions of Lemma 4.6. Moreover, as $X_{p,q}$ is a maximal clique, $X_{p,q} - \{x\}$ is either a quad in Γ_x or there is a point y of Γ_x such that $X_{p,q} - \{x\}$ consists of points of Γ_x collinear to y . If we are in the first case, then Lemma 4.7 implies that $X - \{p\}$ is also a quad, which, however, is not the case. So we are in the second case and $X_{p,q}$ is contained in $X_{x,y}$ for some point y of Γ_x . But as $X_{p,q}$ and $X_{x,y}$ both have size $2 + s(t + 1)$, we obtain $X_{x,y} = X_{p,q}$. \square

Corollary 4.11. $s + 2$ divides $2t(t + 1)$.

Proof. Let p and q be cocircular points and consider $X = X_{p,q}$. Each point of X is on $t + 1$ circles contained in X . As each circle contains $s + 2$ points, there are $(2 + s + st)(1 + t)/(s + 2)$ circles contained in X . Thus $s + 2$ divides $(2 + s + st)(1 + t)$ and hence also $2t(t + 1)$. \square

Corollary 4.12. If $s = 2$, then the symmetric difference of two circles meeting in two points is again a circle.

Proof. Suppose C and D are two circles meeting in two points p and q . Consider the two points p' and q' on C distinct from p and q . By Lemma 4.10 we find that $X_{p,q} = X_{p',q'}$. In particular, there is a circle E on p' and q' different from C and meeting D non-trivially. Since two circles meet in 0 or 2 points, E meets D in the two points different from p and q . In particular, E is the symmetric difference of the two circles C and D . \square

Lemma 4.13. Let p and q be two points of Γ at distance 2. If r is a point in $p^\perp - q^\perp$, then there are 0 , $(2 + s + st_2)/2$ or $(2 + t)/2$ circles through p and r meeting q^\perp non-trivially. Moreover, the last possibility does occur, only if $s > 2$.

Proof. Consider the set $X_{p,r}$ consisting of all points cocircular with p and r . Suppose there is a point $x \in X_{p,r} \cap q^\perp$ different from p and r . By Lemma 4.10 there is a unique point y in $X_{p,r}$ different from x with $X_{x,y}$ equal to $X_{p,r}$. If q and y are at distance 2 in Γ_x , then q^\perp meets $X_{x,y}$ in $2 + s(t_2 + 1)$ points. If q and y are at distance 3, then q^\perp meets $X_{x,y}$ in $1 + t + 1$ points. Since each circle on p and r has either 0 or 2 points in q^\perp , there are $(2 + s + st_2)/2$ or $(t + 2)/2$ circles on p and q meeting q^\perp non-trivially.

Now suppose $s = 2$. Then $C = \{p, r, x, y\}$ is a circle meeting q^\perp in two points, i.e., in the points x and y . Hence y and q are at distance 2 in Γ_x , and the second part of the lemma follows. \square

As noticed in the introduction, we say that a near hexagon of order (s, t_2, t) has classical parameters if either $t_2 > 0$ and $t = t_2(t_2 + 1)$, or $t_2 = 0$ and $t \in \{1, \sqrt[3]{s}, s, s^3\}$. Combining various of the above results we obtain:

Theorem 4.14. If Γ is an extension of a regular near hexagon with classical parameters (s, t_2, t) , $s \geq 2$, then $s = 2$.

Proof. Let Γ be an extension of a regular near hexagon with parameters (s, t_2, t) , $s \geq 2$ and $t \geq 1$. Suppose p and q are two cocircular points and set $X = X_{p,q}$. A point $x \in \Gamma_p$ but not in X has $t + 2$ neighbors in X if it is at distance 3 from q . Let x be such a point (there are $s^3 t(t - t_2)/(t_2 + 1)$ of them), and suppose y is a neighbor of x in X . On the circle C that contains p, q and y the two neighbors of x are just p and y . So, if z is the unique point in X different from y with $X = X_{y,z}$, then x and z are not ad-

adjacent, and inside Γ , the points x and z are at distance 3. This implies that x^\perp meets each circle in X in just 0 or 2 points.

Hence the number of points adjacent to some point of X but not to all the points of a circle in X equals $(2 + s + st)s^3t(t - t_2)/((t + 2)(t_2 + 1))$. In particular, $t + 2$ divides $(2 + s + st)s^3t(t - t_2)$ and then also $2(s - 2)s^3(t_2 + 2)$.

Suppose $s > 2$. First assume that $t_2 = 0$ and t equals 1, $\sqrt[3]{s}$, s or s^3 . Then Γ is an extension of a generalized hexagon. Since $s + 2$ divides $2t(t + 1)$, we find that t cannot be equal to 1, $\sqrt[3]{s}$ or s , but has to be s^3 . However, this contradicts that $t + 2$ divides $4(s - 2)s^3$.

Next assume that $t_2 > 0$ and $t = t_2(t_2 + 1)$. For each point p of Γ the local near hexagon Γ_p is either of Hamming type and $t_2 = 1$, or is a dual polar space and t_2 equals $\sqrt[3]{s}$, $\sqrt[3]{s^2}$, s , or s^2 . In the latter case s and t_2 are powers of 2, see [12].

Since $s + 2$ divides $2t_2(t_2 + 1)(2t_2 - 1)$, see 4.8, we find that t_2 is at least 2. Moreover, using the above, 4.8 and 4.11, we find that $s + 2$ divides $4(t_2 + 1)(2t_2 - 1)$ and $4(t_2 + 1)(t_2^2 + t_2 + 1)$. Since $4 \cdot (t_2^2 + t_2 + 1) - (2t_2 + 3) \cdot (2t_2 - 1) = 7$, we deduce that $s + 2$ divides $28(t_2 + 1)$.

If $t_2 = \sqrt{s}$, then $s + 2 \mid 28(\sqrt{s} + 1)$ implies that $s = 4$ and $t_2 = 2$. However, then 4.9 implies that $2 + s(t_2 + 1) = 14$ divides $[1 + (1 + s)^2] \cdot (1 + s)^2 s(t - t_2) = 26 \cdot 25 \cdot 4 \cdot 2^2$, a contradiction.

If $t_2 = \sqrt[3]{s^2}$, then $s + 2 \mid 28(\sqrt[3]{s^2} + 1)$ implies that $s = 8$ and $t_2 = 4$.

If $t_2 = s$, then $s + 2 \mid 28(t_2 + 1)$ implies $s + 2 \mid 28$, which contradicts s to be a power of 2.

If $t_2 = s^2$, then $s + 2 \mid 28(t_2 + 1)$ implies $s + 2 \mid 140$. Here we can conclude that $(s, t_2) = (8, 64)$.

Hence, we are left with the cases that (s, t_2) equals $(8, 4)$ or $(8, 64)$. However, in both cases $t + 2$ does not divide $2(s - 2)s^3(t_2 + 2)$. This contradiction proves the theorem. \square

5 Extension of near hexagons with three points per line

In this section we consider extensions of near hexagons as in the hypothesis of Theorem 1.2. Let Γ be such an extended near hexagon. Then, conform Lemma 4.1, there exist t_2 and t such that for each point p of Γ the local near hexagon Γ_p is a regular near hexagon of order $(2, t_2, t)$. By [7, 6], we either have that this near hexagon is a generalized quadrangle of order $(2, t)$, with $t = 1, 2$ or 4, a generalized hexagon of order $(2, t)$, with $t = 1, 2, 8$, or one of the following near hexagons containing quads: the near hexagon of Hamming type on 27 points, the near hexagons on 135, respectively, 891 points associated to the dual polar spaces related to $\text{PSp}_6(2)$ and $\text{PSU}_6(2)$, respectively, the M_{24} -near hexagon on 759 points or the near hexagon on 729 points related to $3^6 : 2 \cdot M_{12}$.

In general, the approach for proving Theorem 1.2 is the following. Suppose Γ is an extension of a near hexagon with 3 points per line. We first consider the point graph \mathcal{G} of Γ . This graph is locally the distance 1-or-2 graph of the local near hexagon. The second neighborhood of a point p , i.e., the subgraph of \mathcal{G} consisting of the vertices at distance 2 from p , will be identified with the use of μ -graphs, or rather their comple-

ments. The subgraph of \mathcal{G} induced on the second neighborhood of p can be recovered by classifying the various μ -graphs and considering the intersections between these μ -graphs. This approach eventually leads to the determination of the isomorphism type of \mathcal{G} . To finish the proof, we have to consider the various ways one can fix a collection of 4-cliques of \mathcal{G} , such that these 4-cliques induce the structure of an extended near hexagon on \mathcal{G} . This program will be carried out for the several possibilities of the local near hexagon Γ_p .

The main tool in the classification of μ -graphs is the following proposition in which we collect some properties of the complement of a μ -graph. For the first two properties, see 4.3 and 4.13; the third property follows easily from the second one.

Proposition 5.1. *Let p and q be points of Γ at mutual distance 2. Then we have the following:*

- i. $p^\perp - q^\perp$ is a geometric hyperplane of Γ_p ;
- ii. if r is a point of the hyperplane $p^\perp - q^\perp$ of Γ_p , then either $2 + t_2$ lines of Γ_p on r meet q^\perp or r is a deep point of $p^\perp - q^\perp$.
- iii. if for all x and y at mutual distance 2 in \mathcal{G} the hyperplane $x^\perp - y^\perp$ of Γ_x does not contain deep points, then the diameter of the graph \mathcal{G} equals 2.

5.1 Extensions of generalized quadrangles. In this subsection we consider extensions of the generalized quadrangles of order 2. These are one point extensions. Such extensions have been studied by several authors, and a proof of the following result can be found at various places, see for example [13].

Proposition 5.2. *Suppose Γ is a one point extension of a generalized quadrangle with $s = 2$, then it is one of the three unique extensions on 10, 16 or 28 points related to the groups $\text{PSp}_4(2)$, $2^4 : \text{Sp}_4(2)$, respectively, $\text{PSp}_6(2)$.*

5.2 Extensions of generalized hexagons. Now we assume that for all p of Γ , the local space Γ_p is a generalized hexagon with 3 points per line. By a result of Haemers and Roos, see [7], we have that t can only take the values 1, 2 or 8. The generalized hexagons with these parameters have been classified, see [15]. There is only one generalized hexagon of order $(2, 1)$, its point graph is the flag-graph of the Fano plane. There are two generalized hexagons of order $(2, 2)$, denoted by \mathcal{H} and \mathcal{H}^* . They are dual to each other and are the classical generalized hexagons related to the group $\text{G}_2(2)$. We distinguish between \mathcal{H} and \mathcal{H}^* in the following way: \mathcal{H} is the $\text{G}_2(2)$ generalized hexagon for which the stabilizer in $\text{G}_2(2)$ of a point is of the shape $4^2 : \text{D}_{12}$, while the stabilizer of a point of \mathcal{H}^* is of the shape $2^{1+4} : \text{S}_3$. Geometrically these two generalized hexagons can be distinguished in the following way. The hexagon \mathcal{H} contains subhexagons of order $(2, 1)$, while \mathcal{H}^* does not contain such hexagons. Related to the group ${}^3\text{D}_4(2)$ there is a generalized hexagon of order $(2, 8)$; it is the unique generalized hexagon with parameters $(s, t) = (2, 8)$.

Lemma 5.3. *For all points p of Γ , the local generalized hexagon Γ_p has order $(2, 2)$.*

Proof. Suppose Γ is locally of order $(2, t)$, with $t \in \{1, 2, 8\}$. Fix a point p of Γ , and let q be a point at distance 2 from p . Then, by Proposition 5.1, the complement of the μ -graph $p^\perp \cap q^\perp$ is a geometric hyperplane of the generalized hexagon Γ_p . Call this hyperplane H . It follows by 5.1 that each point of this hyperplane H is on $t - 1$ lines contained in the hyperplane, or all lines on the point are in the hyperplane.

So, if $t = 1$, then obviously H cannot contain deep points, and we find H to be an ovoid. In particular, counting collinear point pairs (u, v) with $u \in H$ and $v \notin H$ in two ways, we find $4|H| = 2(21 - |H|)$; so H contains 7 points. This implies that \mathcal{G} is a strongly regular graph with parameters $(k, \lambda, \mu) = (21, 12, 14)$. However, these parameters do not satisfy the feasibility conditions for a strongly regular graph, see [7].

Suppose $t = 8$. Then, again by counting collinear point pairs (u, v) with $u \in H$ and $v \notin H$, we have $9(819 - |H|) = 4(|H| - |D|)$, where D is the set of deep points in H . From this it follows that $|H| = 567 + 4|D|/13$.

Fix a line $l = \{u, v, w\}$ of the near hexagon meeting H in the point u . Suppose that u is collinear to x deep points of H . Then H contains 1 point of l , it contains $16 + 14$ points at distance 1 from l , and $536 + 4x$ points at distance 2 from the line l . Hence H contains $567 + 4x$ points, and $x = |D|/13$. (Notice that this implies that x is independent of the choice of the point u .) In particular, $|D| = 13x \leq 13 \cdot 14$. If $x > 0$, then counting the collinear pairs (v, w) in H , with v deep and u not, we obtain that $(|H| - |D|)x \leq |D| \cdot 18$, and hence $|H| \leq |D| + (|D|/x) \cdot 18 \leq 13 \cdot 14 + 13 \cdot 18 < 567$, a contradiction. Thus $x = 0$ and we can assume that H does not contain deep points. But that implies that \mathcal{G} is a strongly regular graph with $k = 819$, $\lambda = 306$ and $\mu = 252$. In particular, the number of points equals 2484.

By 4.10 each point p of Γ is in 819 maximal cliques $X_{p,q}$ of size 20. Let α denote the number of such maximal cliques. Counting the number of pairs $(p, X_{p,q})$ in two ways yields that $\alpha \cdot 20 = 2484 \cdot 819$, which is impossible. This contradiction implies that $t \neq 8$, and the lemma is proved. \square

In the remainder of this subsection we assume Γ to be locally a generalized hexagon of order $(2, 2)$.

Lemma 5.4. *Let p and q be points of Γ . If the distance between p and q is 2, then the complement of $p^\perp \cap q^\perp$ in Γ_p contains 27, 35 or 31 points.*

Proof. Consider two points p and q that are at distance 2 in the graph \mathcal{G} . The complement of $p^\perp \cap q^\perp$ in \mathcal{P}_p is a geometric hyperplane of Γ_p , and will be denoted by H . It follows from Proposition 5.1 that each point in H is either deep, or is on just one line inside the hyperplane. Let D be the set of deep points in H . Then by counting the pairs of collinear points (u, v) with $u \in H$ and $v \notin H$ we find $(|H| - |D|)4 = (63 - |H|)3$. Fix a point $u \in H - D$, that is collinear with x deep points, $x = 0, 1$ or 2 . Then there are 2 points at distance 1 from u in H , there are $4x + 8$ points in H at distance 2 from u , and, finally, there are 16 points at distance 3 from u in H . Hence $|H| = 27 + 4x = 27 + 4|D|/7$. So every point $u \in H - D$ is adjacent to $|D|/7$ deep points, and $|D| = 0, 7$ or 14 . In particular, $|H| = 27, 31$ or 35 . \square

Lemma 5.5. *For all points p and q of Γ , the generalized hexagons Γ_p and Γ_q are isomorphic.*

Proof. Suppose that there is a point p in Γ with Γ_p isomorphic to \mathcal{H}^* .

Fix a point q at distance 2 from p . The previous lemma implies that the complement H of $p^\perp \cap q^\perp$ is a geometric hyperplane of Γ_p containing 27, 35 or 31 points. By [23], \mathcal{H}^* does not contain hyperplanes on 27 or 35 points. So H contains 31 points. Moreover, the results of [23] imply that H consists of all the points in Γ_p at distance at most 2 from some fixed point of Γ_p .

The distance 1-or-2 graphs induced on the set of points at distance 3 in the two generalized hexagons \mathcal{H} and \mathcal{H}^* are non-isomorphic. So, as $p^\perp \cap q^\perp$ is isomorphic to such a graph in $\Gamma_p \simeq \mathcal{H}^*$, we can conclude that Γ_q is also isomorphic to \mathcal{H}^* .

Let r be a point in p^\perp . Fix a point r' in Γ_p collinear to r . Now let q' be a point in Γ_r at distance 3 from p and r . Then inside Γ the distance between q' and p and r is 2.

The above implies that $\Gamma_r \simeq \Gamma_{q'} \simeq \Gamma_p \simeq \mathcal{H}^*$. The connectedness of Γ implies now that the residue at each point of Γ is isomorphic to \mathcal{H}^* . □

Lemma 5.6. *The generalized hexagon \mathcal{H}^* can uniquely be recovered from its distance 1-or-2 graph.*

Proof. Let H^* be the distance 1-or-2 graph of \mathcal{H}^* . Let x and y be two vertices of H^* . If x and y are collinear in \mathcal{H}^* , then $(x^\perp \cap y^\perp)^\perp$ consists of the 3 vertices x, y and z forming a line of \mathcal{H}^* . If x and y are non-collinear, then $(x^\perp \cap y^\perp)^\perp$ consists of just x and y , see [33, 3.5.7]. So the lines of \mathcal{H}^* can uniquely be recovered from the graph H^* . This proves the lemma. □

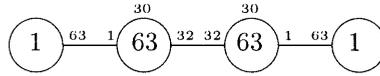
Lemma 5.7. *Let p be a point of Γ . Then Γ_p is isomorphic to \mathcal{H} .*

Proof. Let p be a point of Γ and q a point at distance 2 from p . Suppose Γ_p is isomorphic to \mathcal{H}^* . Then, by Lemma 5.5 we find that Γ_q is also isomorphic to \mathcal{H}^* .

The results of [23] and Lemma 5.4 imply that the μ -graph $p^\perp \cap q^\perp$ consists of all the points of Γ_p at distance 3 (inside Γ_p) from some point $r \in \Gamma_p$. Thus there are $63 \cdot 32/32 = 63$ points at distance 2 from p (and similarly from q in \mathcal{G}). We claim that the point r is at distance 3 from q in \mathcal{G} . Suppose not, then q and r are at distance 2 and $r^\perp \cap q^\perp$ consists of the 32 points of Γ_r opposite to p . As $r^\perp \cap q^\perp \cap p^\perp$ is empty, we have found $64 = 32 + 32$ points in Γ_q . A contradiction. Hence, r is indeed at distance 3 from q .

Next we show that r is the unique point of \mathcal{G} at distance 3 from q . Suppose u is a common neighbor of p and r . Then u is adjacent to some point in $p^\perp \cap q^\perp$. Thus u is at distance 2 from q . So, we can apply the above to u instead of p . The connectedness of Γ_r implies that all the points of Γ_r are at distance 2 from q . In particular, Γ_r consists of all the 63 points of \mathcal{G} that are at distance 2 from q . Moreover, r is the unique neighbor of each of these 63 points which is at distance 3 from q . This im-

plies that Γ contains $1 + 63 + 63 + 1$ points; its distance distribution diagram reads as follows:



In particular, for each point $x \in \Gamma$ there is a unique point \bar{x} at distance 3 from x in \mathcal{G} .

Suppose $x \in \Gamma_p$. Then \bar{x} is at distance 2 from p . Moreover, $\bar{x}^\perp \cap p^\perp$ consists of all the points of Γ_p which are in Γ_p opposite to x .

Suppose that x and y are two points in Γ_p . Then consider the points \bar{x} and \bar{y} . If x and y are opposite in Γ_p , then x and \bar{y} are adjacent as well as y and \bar{x} . Since x and \bar{x} are at distance 3, we find that also \bar{x} and \bar{y} are not adjacent. But that implies that \bar{x} and \bar{y} are adjacent if and only if x and y are adjacent. Thus we can describe all adjacencies in \mathcal{G} in terms of Γ_p . The adjacencies within p^\perp are clear. The point \bar{p} is adjacent to all points \bar{x} , with $x \in \Gamma_p$; the point \bar{x} , $x \in \Gamma_p$, is adjacent to \bar{p} , and all points y of Γ_p at distance 3 from x in Γ_p , and finally \bar{x} and \bar{y} , where x, y are points of Γ_p , are adjacent if and only if x and y are adjacent.

This description shows that the automorphism group $G = \text{Aut}(\mathcal{G})$ of \mathcal{G} contains a subgroup stabilizing p and \bar{p} and inducing $G_2(2)$ on Γ_p and $\Gamma_{\bar{p}}$. As the point p is arbitrary, we see that the group G is transitive and of permutation rank 4 on \mathcal{G} .

Now consider the stabilizer G_p of p inside G . As the generalized hexagon \mathcal{H}^* can be recovered from its distance 1-or-2 graph, see Lemma 5.6 we find that $G_p \simeq G_2(2)$. Let $g \in G_p$ be an element of type $3A$ in the notation of the Atlas [16]. Then g fixes 9 points of the local hexagon Γ_p , and thus 20 points of Γ . Moreover, every element of order 3 in G fixing exactly 20 points of Γ has to be conjugate to g . Hence, as G_p contains 28 conjugates of $\langle g \rangle$, the number of conjugates of $\langle g \rangle$ in G equals $128 \cdot 28/20$, which is impossible. This final contradiction shows that for all p in Γ , the local generalized hexagon Γ_p is isomorphic to \mathcal{H} . \square

So, for the remainder of this subsection we may and do assume that for all points p of Γ , the local generalized hexagon Γ_p is isomorphic to \mathcal{H} . This implies that the graph \mathcal{G} is locally isomorphic to the distance 1-or-2 graph of \mathcal{H} . This graph is isomorphic to the point graph of the symplectic polar space related to $\text{PSp}_6(2)$. From this observation we deduce:

Proposition 5.8. *The graph \mathcal{G} is isomorphic to the collinearity graph of an affine polar space obtained from the $\text{PSp}_8(2)$ polar space by removing a hyperplane.*

Proof. This follows by the results of [14, 24] or [22]. \square

Let \mathcal{S} be the $\text{PSp}_8(2)$ polar space. Up to isomorphism, there are three types of hyperplanes in the $\text{Sp}_8(2)$ polar space, the degenerate ones and the $\text{PO}_8^+(2)$ and $\text{PO}_8^-(2)$ hyperplanes, see [14]. Thus \mathcal{G} is isomorphic to the collinearity graph of an affine polar space \mathcal{AS} obtained from the $\text{Sp}_8(2)$ polar space \mathcal{S} by removing a degenerate hyperplane, a $\text{PO}_8^+(2)$ hyperplane, or a $\text{PO}_8^-(2)$ hyperplane.

By the above proposition, it remains to check which of the three graphs does contain a set of distinguished 4-cliques making it into the point graph of an extended hexagon. Let \mathcal{A} be the (unique) set of affine planes that make \mathcal{G} into the affine polar space \mathcal{AS} . Then, as has been shown in [14, 22] and [18], there is an equivalence relation, called parallelism on the set of \mathcal{A} of affine planes, whose equivalence classes form (part of) the line set of the polar space at infinity \mathcal{AS}_∞ of \mathcal{AS} . The set \mathcal{C} of circles of Γ is a subset of \mathcal{A} . In the lemma below we will show that it consists of a union of parallel classes of \mathcal{A} .

As a consequence of this observation we see that also for Γ we can define a space at infinity, called Γ_∞ , embedded into the polar space at infinity \mathcal{AS}_∞ of \mathcal{AS} . The space Γ_∞ consists of all the points of \mathcal{AS}_∞ , except for the radical of this polar space, and all the lines at infinity corresponding to those parallel classes of affine planes that contain circles of \mathcal{C} .

Lemma 5.9. *If π is a circle in \mathcal{C} , then the whole parallel class of π in \mathcal{A} is in \mathcal{C} .*

Proof. Let π_0 be a plane of \mathcal{C} . Let p be a point of π_0 and q a point in $\pi_0^\perp \setminus \pi_0$. By m we denote the line through p and q . In the hexagon Γ_p we see that m is at distance 1 from some line n on p inside π_0 . Thus there is a plane $\pi_1 \in \mathcal{C}$ on m meeting π_0 in n . Let π_2 be the affine plane of \mathcal{AS} on q which is parallel to π_0 . Fix a line g on q in π_2 , but outside π_1 . In Γ_q we see that g is at distance 1 from some element of π_1 , (indeed, π_0 and g are in a singular 3-space of \mathcal{S}) so there is a unique plane $\pi_3 \in \mathcal{C}$ on g meeting π_1 in a line, k say. Suppose k meets n at the point r of Γ . Then inside Γ_r we encounter the three planes π_0, π_1 and π_3 . Since the planes are inside an affine 3-space, we find that π_0 and π_3 intersect in a line on r distinct from k and n . In particular, the three planes form a triangle in Γ_r . This contradicts $Res(r)$ to be a generalized hexagon. Thus k and n meet at infinity. In particular, $\pi_2 = \pi_3 \in \mathcal{C}$. We can conclude that the unique plane on q which is parallel with π_0 is also in \mathcal{C} .

Let $\pi_0 \in \mathcal{C}$ and $\pi_1 \in \mathcal{A}$ be two parallel planes in \mathcal{AS} . Let l_∞ be the line at infinity in \mathcal{AS}_∞ of π_0 and π_1 . If π_1 contains a point r with $r \in \pi_0^\perp$ then, by the above, π_1 is also in \mathcal{C} . By connectedness of the affine polar space on the singular planes of \mathcal{S} meeting \mathcal{AS}_∞ just in l_∞ and transitivity of being parallel we find that in any case π_1 is in \mathcal{C} . □

We consider the three possibilities for \mathcal{G} separately. First assume \mathcal{G} to be isomorphic to the graph on 136 points obtained by removing a $PO_8^-(2)$ hyperplane from the polar space \mathcal{S} . Then each parallel class of circles contains 10 circles. As there are $136 \cdot 63/4$ circles in Γ , we find $34 \cdot 63/10$ parallel classes of circles of Γ , which is impossible.

Now assume that \mathcal{G} is isomorphic to the graph on 120 points obtained by removing a $PO_8^+(2)$ hyperplane. The geometry at infinity of \mathcal{AS} is now the $PO_8^+(2)$ polar space with 135 points. As each parallel class of circles in \mathcal{AS} contains 6 circles, we find that the space at infinity of Γ consists of all the points of $PO_8^+(2)$ polar space at infinity, and of $120 \cdot 63/(4 \cdot 6) = 315$ lines of this polar space. Each of the 120 $PO_7(2)$ hyperplanes of the polar space \mathcal{AS}_∞ meets the space Γ_∞ at infinity of Γ in a subspace

isomorphic with \mathcal{H} . We will show that Γ_∞ is isomorphic to the near hexagon associated to the $\text{PSP}_6(2)$ dual polar space.

Suppose p is a point and l a line in Γ_∞ , the point p not on l . Then there exist $\text{PO}_7(2)$ hyperplanes of \mathcal{AS}_∞ containing both p and l . Such hyperplanes meet Γ in a subspace isomorphic to \mathcal{H} . So, inside such hyperplanes we see that l contains at least one point at distance ≤ 2 from p (measured in Γ_∞). If p is collinear (inside Γ_∞) with two points on l , then inside \mathcal{AS}_∞ we find p and l to be in a singular plane. However, the intersection of a $\text{PO}_7(2)$ hyperplane containing p and l with Γ_∞ has no plane. This contradiction implies that p is collinear to at most one point on l . Suppose p is, inside Γ_∞ , at distance 2 from two points, say q_1 and q_2 , of l . Let r_1 and r_2 , respectively, be common neighbors of p and q_1 and q_2 , respectively. There is a $\text{PO}_7(2)$ hyperplane H of \mathcal{AS}_∞ containing p, q_1, q_2, r_1 and r_2 . Intersecting Γ_∞ with H we obtain a generalized hexagon containing p, q_1, q_2, r_1 and r_2 . Inside this hexagon we find that p is collinear to a point on l . Thus Γ_∞ is indeed a near hexagon.

Next we show that any two points of Γ_∞ at distance 2 are in a quad. Suppose p and r are two such points. Then they are collinear in the polar space \mathcal{AS}_∞ . But that implies that in any hexagon obtained by intersecting Γ_∞ with a $\text{PO}_7(2)$ hyperplane they stay at distance two and have a common neighbor. By varying these hyperplanes we easily find that p and r have at least two common neighbors, from which we deduce with Yanuska's Lemma ([32]), that they are in a quad.

The above shows that Γ_∞ is a near hexagon with quads and, by [6], it is isomorphic to the $\text{PSP}_6(2)$ near hexagon. But then Γ can be described as the tangent space of the embedded $\text{PSP}_6(2)$ near hexagon Γ_∞ into the $\text{PO}_8^+(2)$ polar space at infinity of \mathcal{AS} . Indeed, Γ consists of the points and lines of the affine polar space \mathcal{AS} ; its circles are those affine planes of \mathcal{AS} whose line at infinity is a line Γ_∞ , which is embedded in the $\text{PO}_8^+(2)$ hyperplane of \mathcal{S} at infinity of \mathcal{AS} . As such an embedding of the $\text{PSP}_6(2)$ near hexagon Γ_∞ is unique up to isomorphism (it is the spin representation of $\text{PSP}_6(2)$), we find that Γ is unique up to isomorphism. In particular, Γ is isomorphic to the extended generalized hexagon on 120 points related to $\text{PSP}_6(2)$. (See also [5].)

Finally assume \mathcal{G} to be isomorphic to the graph on 128 points obtained by removing a degenerate hyperplane ∞^\perp for some point ∞ of the $\text{PSP}_8(2)$ polar space. If we fix a point p of Γ , then the parallel classes of edges and circles of \mathcal{G} on p form the points and lines of a generalized hexagon H_p isomorphic to \mathcal{H} embedded into $p^\perp \cap \infty^\perp$. The lines and singular planes on ∞ meeting the generalized hexagon H_p in a point respectively line we denote by \bar{H}_p . This yields an embedding of \mathcal{H} into the $\text{PSP}_6(2)$ polar space on lines through ∞ .

For p and q adjacent points of Γ we find that \bar{H}_p and \bar{H}_q meet in at least 15 lines, as H_p and H_q meet in 15 lines. For antipodal points p and q in Γ (points at distance 3) we find $\bar{H}_p = \bar{H}_q$. But this implies that for any two points p and q of Γ the intersection of \bar{H}_p and \bar{H}_q contains at least 15 lines.

In total, there are 120 different embeddings of \mathcal{H} into a $\text{PSP}_6(2)$ polar space. They form an orbit under $\text{PSP}_6(2)$. Two distinct embeddings either share 15 lines containing some point at distance ≤ 1 from a fixed point or 9 non-intersecting lines. The graph on these 120 distinct embeddings, with two embeddings adjacent if and only if they intersect in 15 lines is the graph on 120 points of the extended generalized

hexagon related to $\text{PSp}_6(2)$ consider above. As this graph does not contain 64-cliques, we find that there are points p and q in Γ with $\bar{H}_p = \bar{H}_q$. By switching to an antipodal of q , if necessary, we even may assume that these points p and q are adjacent in \mathcal{G} .

Let r be a point different from but cocircular with p and q . Let r_p and r_q , respectively, be the point of the $\text{PSp}_6(2)$ polar space formed by the line through ∞ and the point at infinity of the line through r and p or q , respectively. Then $\bar{H}_r \cap \bar{H}_p$ share at least the 15 lines containing a point at distance ≤ 1 from r_p , and $\bar{H}_r \cap \bar{H}_q$ share at least the 15 lines at distance ≤ 1 from r_q . But since $r_p \neq r_q$ and $\bar{H}_p = \bar{H}_q$, we find that \bar{H}_p and \bar{H}_r share more than 15 lines, and therefore are equal. By connectivity we find that for all points p and q of Γ we have $\bar{H}_p = \bar{H}_q$.

As (up to isomorphism) there is only one embedding of \mathcal{H} into a $\text{PSp}_6(2)$ polar space, we find that (up to isomorphism) the space Γ_∞ at infinity of Γ is unique. But then, as Γ consists of the points and lines of \mathcal{AS} and those affine planes whose line at infinity is in Γ_∞ , uniqueness of Γ follows easily. In fact, we find Γ to be isomorphic to the extended generalized hexagon on 128 points related to the group $2^7 : \text{G}_2(2)$.

The results of this subsection lead to the following.

Proposition 5.10. *Let Γ be locally a generalized hexagon of order $(2, t)$. Then Γ is isomorphic to the extended generalized hexagon on 128 points related to $2^7 : \text{G}_2(2)$, or the extended generalized hexagon on 120 points related to $\text{PSp}_6(2)$.*

5.3 Extensions of the near hexagon of Hamming type. Now we consider the case that Γ is locally the near hexagon of Hamming type, which is the unique near hexagon with parameters $(s, t_2, t) = (2, 1, 2)$.

Suppose p is a point of Γ and let q be a point at distance 2 from p in the graph \mathcal{G} . As before, denote by H the geometric hyperplane of Γ_p consisting of the points not in $p^\perp \cap q^\perp$. As follows from Lemma 4.7, Γ contains a family \mathcal{E}_2 of extended quads. As each point quad pair in the near hexagon of Hamming type is classical, we find that q^\perp meets each extended quad on p in 0 or 6 points. Hence H meets each quad of Γ_p in an ovoid of 3 points or contains the quad. By Proposition 5.1, every point in H is on 0 lines inside H , or is deep. Thus if H contains a quad, then all points in this quad are deep. This would imply that all points of the near hexagon are in H , which is impossible. Hence H contains no deep points, it is an ovoid and has cardinality 9. From this we deduce that \mathcal{G} has diameter 2, and is strongly regular with parameters $(v, k, \lambda, \mu) = (40, 27, 18, 18)$.

Suppose r is a point at distance 2 from p with $p^\perp \cap q^\perp = p^\perp \cap r^\perp$, and suppose $u \in p^\perp \cap q^\perp$. Then in the near hexagon Γ_u we see that all points at distance at most 2 from p and q are also at distance at most 2 from r . One easily checks that this implies that $r = q$. So the 12 points at distance 2 from p all determine different ovoids in Γ_p . But Γ_p admits exactly 12 ovoids. Indeed, each grid in Γ_p admits 6 ovoids, and an ovoid of such grid, together with a point outside that grid non-collinear with the 3 points of the ovoid in the grid determines a unique ovoid in Γ_p . Hence Γ_p admits $6 \cdot 12 / (9 - 3) = 12$ ovoids. So, each point at distance 2 from p can uniquely be identified with the ovoid it determines.

Fix an ovoid \mathcal{O} of the Hamming near hexagon. Then, inside the Hamming near hexagon, each point of this ovoid is at distance 2 from 6 points inside the ovoid, and at distance 3 from 2 points of the ovoid. There are 9 ovoids of the Hamming near hexagon meeting \mathcal{O} in 3 points (which form an ovoid in a grid of the Hamming near hexagon), and 2 ovoids disjoint from \mathcal{O} .

The point q has 9 neighbors in \mathcal{G} that are not adjacent to p . These 9 points form an ovoid in Γ_q , and hence each of these points has 6 common neighbors with q which are not in Γ_p . In particular, these 9 points correspond uniquely to the 9 ovoids of Γ_p meeting H in 3 points. The two points that are at distance 2 from both p and q have 9 common neighbors with p and q . So they correspond to the two ovoids in Γ_p that are disjoint from the ovoid determined by q .

It follows from the above, that the graph \mathcal{G} is unique (up to isomorphism). But as the collinearity graph from the near hexagon can be recovered from its distance 1-or-2 graph (collinear points have 13 common neighbors, while points at distance 2 have 11 common neighbors in the distance 1-or-2 graph), Γ is also unique. Hence we have proven:

Proposition 5.11. *Suppose Γ is locally the near hexagon of Hamming type on 27 points. Then Γ is isomorphic to the extended near hexagon on 40 points related to $\text{PSU}_4(2)$.*

5.4 Extensions of the $\text{PSp}_6(2)$ near hexagon. The near hexagon related to the group $\text{PSp}_6(2)$ is the unique near hexagon with the parameters $(s, t_2, t) = (2, 2, 6)$, it has 135 points. Its distance 1-or-2 graph is isomorphic with the graph on the 135 singular points in an orthogonal $\text{PO}_8^+(2)$ geometry. Two points are adjacent, if and only if they are perpendicular. This graph is a polar graph. Hence, if Γ is locally the $\text{PSp}_6(2)$ near hexagon, then its collinearity graph is locally a polar graph. In particular, we have:

Lemma 5.12. *Suppose Γ is locally the $\text{PSp}_6(2)$ near hexagon. Then \mathcal{G} is isomorphic to the collinearity graph of the affine polar space obtained by removing a hyperplane from the non-degenerate polar space related to $\text{PO}_{10}^+(2)$.*

Proof. See [14]. □

The above proposition shows us that Γ can be embedded into an affine polar space \mathcal{AS} , which together with its space at infinity \mathcal{AS}_∞ forms the $\text{PO}_{10}^+(2)$ polar space \mathcal{S} . This polar space \mathcal{S} has two types of hyperplanes; the degenerate hyperplanes with a unique point in the radical, and the $\text{PO}_9(2)$ hyperplanes, see [14]. As in Subsection 5.3 we can construct a space Γ_∞ at infinity for Γ by taking all points of the space at infinity of the affine polar space \mathcal{AS} together with those lines at infinity corresponding to parallel classes containing circles in \mathcal{C} .

First assume that the removed hyperplane is non-degenerate and consider the space Γ_∞ . It contains 255 points. It is embedded in the polar space at infinity \mathcal{AS}_∞ of \mathcal{AS} , which is a $\text{PO}_9(2)$ polar space. Each point of Γ determines a unique $\text{PO}_8^+(2)$ hyperplane of this polar space meeting the space at infinity of Γ in a near hexagon

isomorphic to Γ_p . As in Subsection 5.3, it follows from the fact that any point-line pair of the polar space is contained in such a hyperplane that the space at infinity of Γ is a near hexagon. By the results of [6], however, we find that no such near hexagon exists.

Thus we can assume that the polar space at infinity of \mathcal{AS} is degenerate. Let ∞ be the radical of \mathcal{AS}_∞ . If we fix a point p of Γ , then the edges and circles on p define a near hexagon H_p isomorphic to Γ_p embedded into the subspace $p^\perp \cap \infty^\perp$ of the $\text{PO}_{10}^+(2)$ polar space. The singular lines and planes of this polar space on the point ∞ meeting H_p in a point or line, respectively, form a near hexagon embedded in the residue of ∞ . This embedded near hexagon will be denoted by \bar{H}_p . We will show that \bar{H}_p and \bar{H}_q are equal for any two points p and q of Γ (compare with Subsection 5.2).

There are 240 different embeddings of the $\text{Sp}_6(2)$ near hexagon in the $\text{PO}_8^+(2)$ polar space, forming one orbit under the automorphism group of the polar space. As Γ contains 256 points, there are at least two distinct points of Γ , p and q say, with $\bar{H}_p = \bar{H}_q$. For any point r of Γ collinear with p and q , respectively, we find that H_r and H_p meet in at least all the points of H_p (or H_q , respectively) not opposite to the point pr_∞ (or qr_∞) which is at infinity of the edge through r and p (or q , respectively). But that implies that $\bar{H}_p = \bar{H}_q$ and \bar{H}_r meet in more than the geometric hyperplane of \bar{H}_p consisting of all points at distance ≤ 2 from the point represented by the line through ∞ and pr_∞ . As \bar{H}_p is generated by that hyperplane and any point outside it (the complement of the hyperplane is connected, see [2]) we find $\bar{H}_p = \bar{H}_q$ and \bar{H}_r to be the same. By connectivity of Γ_p we find that $\bar{H}_r = \bar{H}_p$ for all points r of Γ_p . But then connectivity of Γ implies $\bar{H}_r = \bar{H}_p$ for all r of Γ . Finally the uniqueness of Γ follows easily from the fact that, up to isomorphism, there is a unique embedding of Γ_p in the $\text{PO}_8^+(2)$ polar space. This proves:

Proposition 5.13. *If Γ is an extension of the $\text{PSp}_6(2)$ near hexagon, then it is isomorphic to the extension on 256 points related to the group $2^8 : \text{Sp}_6(2)$.*

5.5 Extensions of the $\text{PSU}_6(2)$ near hexagon. In this subsection we consider the case where for all points p of Γ the local near hexagon Γ_p is isomorphic to the $\text{PSU}_6(2)$ near hexagon on 891 points. We prove:

Proposition 5.14. *Suppose Γ is locally the $\text{PSU}_6(2)$ near hexagon, then it is isomorphic to the extension on 2300 points related to Co_2 .*

Suppose Γ satisfies the hypothesis of the above proposition. Denote the $\text{PSU}_6(2)$ near hexagon by \mathcal{H} . Fix a point p of Γ and a point q at distance 2 in the collinearity graph of Γ . By Proposition 5.1 we know that the set $p^\perp \cap q^\perp$ is the complement of a geometric hyperplane of Γ_p , in which a point is deep or on 15 lines inside the hyperplane.

In the following proposition we classify the geometric hyperplanes of \mathcal{H} having this property. The following lemma will turn out to be useful in the proof of the proposition.

Lemma 5.15. *Suppose X is a set of at most $q + 2$ points in a projective plane of order q meeting every line in 0 or at least 2 points. Then X is a hyperoval, i.e., a set of $q + 2$ points meeting every line in 0 or 2 points.*

Proof. Suppose l is a line meeting X in at least 3 points. Fix a point $p \in X \cap l$. Then, as there are at most $q - 1$ points of X not in l , there is at least one line through p meeting X in just the point p , which leads to a contradiction. \square

Proposition 5.16. *Let H be a proper geometric hyperplane of \mathcal{U} , in which every point is deep or on 15 lines in the hyperplane, then H is one of the orbits of length 567 on the points of \mathcal{U} under a subgroup of $\text{Aut}(\mathcal{U})$ isomorphic to ${}^+\Omega_6^-(3)$.*

Proof. Let H be a hyperplane as in the hypothesis. Suppose Q is a quad in \mathcal{U} . Then H contains Q or intersects it in a hyperplane which is either a subquadrangle of order $(2, 2)$ in Q , or a point and all its neighbors in Q .

Suppose p is a point of H which is not deep. Then the lines and quads of \mathcal{U} on p form a projective plane of order 4. Let X_p be the set of six lines on p not in H . Then a quad on p has either no line in X_p , or at least two lines in X_p . But then Lemma 5.15 implies that X_p is a hyperoval in the projective plane of lines and quads on p .

Suppose $q (\neq p)$ is a point in H on a line l through p . Then by the above there exists a quad containing l and meeting H in a proper subquadrangle. In particular, q is not deep. A quad on p and q with all lines through p inside H has (by the above applied to q) at least three lines through q inside H and hence is contained in H . This implies that the point p is on exactly six quads that are fully contained in H . (These six quads form a dual hyperoval in the projective plane of lines and quads on p .) Moreover, every line on p in H is on exactly two quads contained in H .

Fix a quad Q on p inside H . If r is a point in H , but not in Q , then there is unique line on r meeting Q . This line is contained in H . In particular, H is connected. Moreover, by the above, no point of H is deep. Counting pairs of collinear points (x, y) with $x \in H$ and $y \notin H$, we obtain that $12|H| = 21 \cdot (891 - |H|)$. So, $|H| = 567$. Moreover, H contains $567 \cdot 6/27 = 126$ quads.

Let Q be an arbitrary quad of \mathcal{U} . Then through each point of \mathcal{U} outside Q there is a unique line meeting Q . The map t_Q which fixes all points of Q and interchanges the two points on each line meeting Q in a single point is an automorphism of \mathcal{U} . It is an elation of $\text{PSU}_6(2)$. If the quad Q is inside H then t_Q stabilizes H . Indeed, if r is a point of H not in Q , then $\tau_Q(r)$ is on the unique line through r meeting Q which is in H . So, the stabilizer G_H of H in $\text{Aut}(\mathcal{U})$ contains the 126 involutions t_Q where Q runs over the quads in H . Moreover, this group is transitive on the points of H . Indeed, if x and y are collinear points in H , then on the third point of the line through x and y we can find a quad Q in H but not containing x nor y . For this quad we have $t_Q(x) = y$. So, connectedness of H implies G_H to be transitive on H .

It remains to prove that G_H is isomorphic to the group ${}^+\Omega_6^-(3)$. Using the Atlas [16], or with the help of the classification of groups generated by 3-transpositions, see [21], it is easy to see that G_H is isomorphic to ${}^+\Omega_6^-(3)$. This also follows by the results of [6] or [29]. \square

The set of hyperplanes of \mathcal{U} satisfying the hypothesis of the above proposition will be denoted by \mathcal{GH} . The group $\text{PSU}_6(2)$ contains three conjugacy classes of subgroups isomorphic to ${}^+\Omega_6^-(3)$ and generated by elations. This implies that the group $\text{PSU}_6(2)$ has 3 orbits of length 1408 on the set \mathcal{GH} of geometric hyperplanes satisfying the conditions of the previous proposition. The outer automorphism group of $\text{PSU}_6(2)$ induces the full symmetric group S_3 on the three classes of subgroups, see [16]. Each of the subgroups isomorphic to ${}^+\Omega_6^-(3)$ contains a single conjugacy class of 126 elations and acts transitively by conjugation on the remaining $693 - 126 = 567$ elations in $\text{PSU}_6(2)$.

For each $H \in \mathcal{GH}$ the set of lines in H induces a near hexagon on the 567 points of H , the Aschbacher near hexagon, see [1, 6]. Moreover, we observe that every line of the $\text{PSU}_6(2)$ near hexagon is contained in precisely two quads contained in H .

In the following lemma, we see how we can distinguish between pairs of geometric hyperplanes in one or in distinct $\text{PSU}_6(2)$ -orbits on \mathcal{GH} .

Lemma 5.17. *Let H be a geometric hyperplane in \mathcal{GH} . Then there are 567 elements of \mathcal{GH} intersecting H in 375 points, and 840 elements intersecting H in 351 points. Together with H , these $1408 = 1 + 567 + 840$ hyperplanes form the $\text{PSU}_6(2)$ -orbit on \mathcal{GH} containing H .*

The hyperplane H intersects of each other $\text{PSU}_6(2)$ -orbit on \mathcal{GH} 112 elements in 405 points and 1296 in 357 points.

Proof. Let $G_H \simeq {}^+\Omega_6^-(3)$ be the stabilizer of H in $G = \text{PSU}_6(2)$, and let Q be a quad of \mathcal{U} intersecting H in a subquadrangle of order $(2, 2)$. Then the geometric hyperplane H^{t_Q} intersects H in the 375 points of H that are at distance at most 1 from this subquadrangle. Each hyperplane H' intersecting H in the set of points at distance 1 from a subquadrangle of order $(2, 2)$ is obtained as the image under an involution t conjugate under G_H to t_Q . As we can find 567 such involutions in G , this yields a G_H -orbit of length 567. Notice that each point of H is in $567 \cdot 375/567 = 375$ hyperplanes in this G_H -orbit.

As $\text{PSU}_6(2)$ has permutation rank 3 in its action on the cosets of ${}^+\Omega_6^-(3)$, see [16], the remaining 840 geometric hyperplanes form a G_H -orbit. Each point of \mathcal{H} is in $896 = 1408 \cdot 567/891$ geometric hyperplanes in the G -orbit of H . So, since a point of H is in 375 hyperplanes meeting H from the G_H -orbit described above, such a point is in 520 hyperplanes of the orbit of length 840. But that implies that each hyperplane of that orbit intersects H in $567 \cdot 520/840 = 351$ points.

By \mathcal{GH}_1 we denote the G -orbit of H . Let \mathcal{GH}_i with $i = 2, 3$ be the other two G -orbits of length 1408 on \mathcal{GH} . From Lemma 2.3 of [28], we know that the stabilizer G_H of H has two orbits of length 112 and 1296, respectively, on \mathcal{GH}_i . If $H' \in \mathcal{GH}_i$ is an element in the orbit of length 112, then the stabilizer of H' is a group isomorphic to $3^4 : S_6$, see [28]. This group contains 45 elations and $H \cap H'$ is the Hall near hexagon as described in [6]. In particular, $H' \in \mathcal{GH}$ contains 405 points. Notice that each point of H is in $112 \cdot 405/567 = 80$ hyperplanes in the G_H -orbit of H' .

As each point of H is in 896 hyperplanes of \mathcal{GH}_i , it is in 816 hyperplanes in the G_H -orbit of length 1296 on \mathcal{GH}_i . But then by counting incident point-hyperplane

pairs (p, I) with $p \in H$ and I in the G_H -orbit of length 1296 on \mathcal{GH}_i , we find that $816 \cdot 567 = 1296 \cdot |H \cap J|$, where J is an element of the G_H -orbit of length 1296. Hence, each such hyperplane J meets H in 357 points. \square

Lemma 5.18. *The point graph of Γ is strongly regular with parameters*

$$(v, k, \lambda, \mu) = (2300, 891, 378, 324).$$

Proof. Since the hyperplanes in \mathcal{GH} have no deep points, we see that the graph Γ has diameter 2. The rest is now straightforward by the above. \square

Lemma 5.19. *Let p be a point of Γ . Then the subgraph of the point graph of Γ induced on the vertices distinct and non-adjacent to p , is a graph which is locally the distance 1-or-2 graph of the Aschbacher near hexagon.*

Proof. Straightforward by the above. \square

Let p be a vertex of Γ . For each vertex q non-adjacent to p denote by H_q the geometric hyperplane of the near hexagon induced on Γ_p consisting of all vertices adjacent to p , but not to q .

Suppose q and r are two vertices different from p and non-adjacent to p . Since the two hyperplanes H_q and H_r cannot cover the whole Γ_p , see also 5.17, there is a vertex u adjacent to p, q and r . Inside Γ_u we find a vertex $w \in H_q - H_r$. In particular, we find H_q and H_r to be distinct. Furthermore, we see that there is a vertex adjacent to q and r , but not to p . In particular, the subgraph of the point graph of Γ induced on the set of points at distance 2 from p is connected.

If q and r are adjacent, then, as can be checked within the Aschbacher near hexagon, they have 246 common neighbors that are not adjacent to p , and hence $378 - 246 = 132$ common neighbors in Γ_p . Thus $H_q \cap H_r$ contains $891 - 2 \cdot 324 + 132 = 375$ vertices, and H_q and H_r are in the same $\text{PSU}_6(2)$ -orbit on \mathcal{GH} .

Connectedness of the subgraph of the point graph of Γ induced on the set of points at distance 2 from p , implies that also when q and r are not adjacent, we find H_q and H_r in the same $\text{PSU}_6(2)$ -orbit on \mathcal{GH} .

Fix the $\text{PSU}_6(2)$ -orbits on \mathcal{GH} , containing H_q . (Since $\text{Out}(\text{PSU}_6(2))$ is transitive on the three orbits, it is of no concern, which of the three we have to take.) We can identify each of the 1408 points r of Γ that are not collinear to p with the unique geometric hyperplane H_r in this orbit. On the other hand, for each geometric hyperplane H in this orbit there is a unique vertex r of Γ not adjacent to p with $H_r = H$. As follows from Lemma 5.17, two vertices q and r of Γ , not adjacent to p are adjacent, if and only if $|H_q \cap H_r| = 375$. This shows us that the collinearity graph \mathcal{G} of Γ is unique up to isomorphism.

Consider the subgraph of the collinearity graph of Γ induced on Γ_p . Then two adjacent vertices of this graph are at distance 1 or 2 in the near hexagon Γ_p . They are at distance 1 if and only if they have 121 common neighbors. Hence, the set \mathcal{C} of circles

of Γ can uniquely be recovered from the collinearity graph of Γ , which shows that Γ is unique up to isomorphism. We have proved Proposition 5.14.

5.6 Extensions of the $3^6 : 2 \cdot M_{12}$ near hexagon do not exist. The $3^6 : 2 \cdot M_{12}$ near hexagon is the unique example with parameters $(s, t_2, t) = (2, 1, 11)$. We show that there is no extension of the $3^6 : 2 \cdot M_{12}$ near hexagon satisfying condition (*).

Proposition 5.20. Γ is not an extension of the $3^6 : 2 \cdot M_{12}$ near hexagon.

Proof. Suppose Γ is locally the $3^6 : 2 \cdot M_{12}$ near hexagon, and let p and q be two points of Γ at mutual distance 2 in \mathcal{G} . By H we denote the geometric hyperplane of Γ_p consisting of the complement of $p^\perp \cap q^\perp$ in Γ_p .

From Proposition 5.1 we deduce that each point of H is either deep or on 9 lines inside H and 3 lines meeting the complement of H . Counting pairs of collinear points, one in H and the other not, we obtain that $(|H| - |D|)6 = (729 - |H|)12$, where D is the set of deep points in H . Hence $|H| = 486 + |D|/3 \geq 486$.

Now fix a line l of Γ_p not in H . Let r be the unique point in $l \cap H$. We will count the number of points in H with respect to their distance to l .

On l there is a unique point in H , the point r . Of the points collinear to r there are $18 = 9 \cdot 2$ inside H . The other two points of l are both collinear to 12 points in H , one of them being r . So at distance 1 from l there are $22 + 18 = 40$ points in H . Now consider a point u at distance 2 from l . This point is in a unique grid meeting l in a point. Each point of l is on 55 grids not containing l . On r , one of these 55 grids meets H in an ovoid, $2 \cdot 9 = 18$ grids meet H in 5 points of which only 2 are at distance 2 from r , and the remaining 36 grids meet H in 5 points or are completely contained in H . So, in H there are $2 + 18 \cdot 2 + \alpha \cdot 4 = 38 + 4\alpha$ points at distance 2 from r and l , where α equals the number of grids on r completely inside H .

Consider a point x on l but not in H . On x there are 9 lines which together with l generate a grid meeting H in 5 points. Each of these 9 lines is in 2 grids meeting H in 3 points of an ovoid from which only one point is at distance 2 from l . The other two lines on x are on a unique other grid meeting H in an ovoid with just one point at distance 2 from l . Thus there are $(9 \cdot 2 + 2 \cdot 1)/2 = 10$ grids on x containing exactly one point of H at distance 2 from l and $55 - 10 = 45$ grids on x with 3 points at distance 2 from l . So, in H we find $2(10 + 45 \cdot 3) = 290$ points at distance 3 from r . In particular, H contains $1 + 40 + 38 + 4\alpha + 290 = 369 + 4\alpha$ points and $|H| \leq 369 + 36 \cdot 4 = 513$. Since $486 - 369 = 117$ is not divisible by 4, we find that H has at least one deep point, say d . Let m be a line on d . We will now count the points of H with respect to their distance to m . If m contains a second deep point, then all grids on m are inside H and we can conclude that the third point of m is also deep. But then we find $3 \cdot 22$ points at distance 1 from m and $\geq 3 \cdot 22 \cdot 14/2$ points at distance 2 from m . The total number of points in H is then at least $3 + 66 + 462 = 531$ which contradicts $|H| \leq 513$.

So d is the unique deep point on m . This implies that there are $22 + 16 + 16 = 54$ points at distance 1 from m in H . On d there are 3 lines generating a grid with m not inside H . On each of these three lines we find two grids not containing m , nor a point

of H at distance 2 from m and 8 grids not on m but contained in H . On the 8 lines on d which generate a grid with m contained in H we find 3 grids not containing m nor a point of H at distance 2 from m and 7 grids containing 4 points of H at distance 2 from m . Thus inside H we find $(3 \cdot 8 + 8 \cdot 7)/2 = 40$ grids on d each containing 4 points at distance 2 from l and 15 grids not on m having no point of H at distance 2 from m . This accounts for 160 points at distance 2 from l and d .

Now we consider the grids on a point e of m , where $e \neq d$. By arguments similar to the above we find that among the 55 grids on e not containing m , there are 12 not containing a point in H and at distance 2 from m , there are $24 + 3$ grids containing 2 points of H at distance 2 from m and finally 16 completely inside H . This accounts for $2 \cdot (27 \cdot 2 + 16 \cdot 4)$ points in H at distance 3 from d . In particular, H contains $3 + 54 + 160 + 236 = 453$ points, which contradicts $|H| \geq 486$. \square

5.7 Extensions of the M_{24} near hexagon. In this subsection we assume Γ to be an extension of the M_{24} near hexagon. The M_{24} near hexagon is the unique near hexagon with parameters $s = 2 = t_2$ and $t = 14$. This near hexagon can be described in terms of the Steiner system $S(24, 4, 8)$ on a set of 24 symbols. Its points are the octads of this Steiner system, while lines are the partitions of the 24 elements of the Steiner system into three octads. A dodecad is a set of 12 symbols that is the symmetric difference of two octads meeting in 2 symbols. The complement of a dodecad is also a dodecad. To each dodecad we can associate a geometric hyperplane of the near hexagon consisting of those octads that meet the dodecad in precisely 4 symbols. A pair of complementary dodecads is also called a duum. The two dodecads of a duum define the same hyperplane. Using some detailed knowledge of the Steiner system one can verify that two distinct hyperplanes associated to dodecads intersect in 335 or 315 octads, corresponding to the cases where the dodecads intersect in either 4 or 8 points, or 6 points, respectively.

Let p and q be two points at mutual distance 2 in the collinearity graph \mathcal{G} of Γ , and as usual, let H_q denote the geometric hyperplane which is the complement of $p^\perp \cap q^\perp$ in Γ_p . Then each point in H is on 11 lines inside H and 4 lines meeting H in just one point, or the point is deep.

By the classification of hyperplanes of the M_{24} near hexagon, see [8], the hyperplane H_q is equal to a hyperplane determined by a dodecad as described above. In particular, H_q does contain 495 points, none of them being deep. This implies that the collinearity graph of Γ has diameter 2 and is strongly regular with parameters $(v, k, \lambda, \mu) = (2048, 759, 310, 264)$. Thus there are 1288 points at distance 2 from p . Suppose r is a point not collinear with p , but with $p^\perp \cap q^\perp = p^\perp \cap r^\perp$, i.e., $H_q = H_r$. Then for each point u collinear with both p and q we see that in Γ_u the points at distance at most 2 from p and q are also at distance at most 2 from r . Inside Γ_u we see immediately that this implies that q and r are equal. Hence, the map $q \mapsto H_q$ is a bijection between the points at distance 2 from p and the set of 1288 hyperplanes of Γ_p corresponding to dodecads.

As there are 206 points at distance at most 2 from a fixed point in a hyperplane associated to a dodecad, we find that in the subgraph of \mathcal{G} induced on the set of points at distance two from p , two adjacent points have 206 common neighbors.

Since any two adjacent points in \mathcal{G} have 310 common neighbors, the two adjacent points q and r , both not adjacent to p , have $310 - 206 = 104$ common neighbors in p^\perp . In particular, if two points q and r , both not adjacent to p are adjacent, then the hyperplanes H_q and H_r intersect in $335 = 759 - 2 \cdot \mu + 104$ points. As there are exactly 495 dodecads meeting a fixed dodecad in 4 symbols, we see that two points q and r , both not adjacent to p are adjacent, if and only if the dodecads associated to the hyperplanes H_q and H_r meet in 4 or 8 points. This implies the uniqueness of the point graph \mathcal{G} of Γ . But then the uniqueness of Γ follows easily and we have proven:

Proposition 5.21. *Let Γ be an extension of the M_{24} near hexagon, then it is isomorphic to the extension on 2048 points related to $2^{11} : M_{24}$.*

5.8 Proof of Theorem 1.2. We have finished the proof of Theorem 1.2. Indeed, if Γ is an extended near hexagon as in the hypothesis of 1.2, then for each point p of Γ the local near hexagon Γ_p is either a generalized quadrangle, a generalized hexagon, or a near hexagon with quads. If Γ is locally a generalized quadrangle, then 5.2 applies and if it is locally a generalized hexagon, then 5.10 applies. If Γ is locally a near hexagon with quads, then by [6] it is locally a near hexagon of Hamming type, handled by 5.11, the $\text{PSp}_6(2)$ near hexagon, handled by 5.13, the $\text{PSU}_6(2)$ near hexagon, handled by 5.14, or one of the two near hexagons related to the groups $3^6 : 2 \cdot M_{12}$ or M_{24} , which are handled by 5.20 and 5.21.

6 Classification of line systems

In this section we show how Theorem 1.1 follows from Theorem 1.2.

Let \mathcal{L} be a line system as in the hypothesis of Theorem 1.1. Then as already observed in the introduction, this line system carries the structure of an extension of a near hexagon $\Gamma(\mathcal{L})$, the circles of this extended near hexagon being the sets of 4 lines passing through the vertices of a regular tetrahedron centered at the origin. Let Σ be the set of norm 3 vectors on the lines in \mathcal{L} . Then Σ also carries the structure of an extended near hexagon $\Gamma(\Sigma)$, whose circles are now the sets of 4 vertices on the tetrahedra mentioned above. The extended near hexagon $\Gamma(\Sigma)$ is a 2-fold cover of $\Gamma(\mathcal{L})$, which is an extended near hexagon as in the hypothesis of Theorem 1.2.

We will show that there is a unique way to obtain the 2-fold cover $\Gamma(\Sigma)$ for each of the extended near hexagons of the conclusion of Theorem 1.2.

Consider $\Gamma(\mathcal{L})$ whose point set is \mathcal{L} . For each line p of \mathcal{L} we denote by $+p$ and $-p$ the two norm 3 vectors on p . Then $\{\pm p \mid p \in \mathcal{L}\}$ is the point set of $\Gamma(\Sigma)$. Adjacency in the point graph of $\Gamma(\Sigma)$ is the same as having inner product -1 .

Throughout this section we fix a point p of $\Gamma(\mathcal{L})$ and we show that all adjacencies in $\Gamma(\Sigma)$ can be expressed in terms of adjacencies in $\Gamma(\mathcal{L})$ and subconfigurations of the local near hexagon Γ_p . Without loss of generality, we can assume that all the points adjacent to $+p$ are the points $+q$, and all the points adjacent to $-p$ are the points $-q$, where $q \in p^\perp - \{p\}$.

Lemma 6.1. *Let $q, r \in p^\perp - \{p\}$. Then $+q$ and $+r$, respectively, $-q$ and $-r$ are adja-*

cent, if and only if the points p, q and r are in a circle. The points $+q$ and $-r$ are adjacent, if and only if q and r are adjacent, but not in a circle with p .

Proof. Obvious. □

Lemma 6.2. *Suppose q is a point of $\Gamma(\mathcal{L})$, which is at distance 2 from p in the point graph of $\Gamma(\mathcal{L})$. Then $p^\perp \cap q^\perp$ is a disjoint union of the sets $\{r \in \mathcal{L} \mid +r \in +p^\perp \cap +q^\perp\}$ and $\{r \in \mathcal{L} \mid +r \in +p^\perp \cap -q^\perp\}$. Moreover, the sets $+p^\perp \cap +q^\perp$ and $+p^\perp \cap -q^\perp$ are both cocliques in the near hexagon $\Gamma(\Sigma)_{+p}$.*

Proof. The two sets $+q^\perp$ and $-q^\perp$ are disjoint. Thus certainly the two sets $+p^\perp \cap +q^\perp$ and $+p^\perp \cap -q^\perp$ are disjoint. If $r \in p^\perp \cap q^\perp$, then $+r$ is either adjacent to $+q$ or to $-q$ and hence contained in one of the two sets $+p^\perp \cap +q^\perp$ or $+p^\perp \cap -q^\perp$.

Suppose $+r \in +p^\perp \cap +q^\perp$. Then the points $+q$ and $+p$ are points at distance 3 in the local near hexagon $\Gamma(\Sigma)_{+r}$. In particular, there is no point in $+p^\perp \cap +q^\perp$ adjacent to $+r$, and we may conclude that $+p^\perp \cap +q^\perp$ is a coclique. Clearly the same does also hold for $+p^\perp \cap -q^\perp$. □

Lemma 6.3. *Let Γ be an extended near hexagons of the conclusion of Theorem 1.2. Then, for any pair p, q of points at distance two in the point graph of Γ , there is a unique decomposition of the μ -graph $p^\perp \cap q^\perp$ into two sets P_+ and P_- , which are cocliques in both local near hexagons Γ_p and Γ_q .*

Proof. The existence of a decomposition of the μ -graphs into two cocliques follows by the above lemma and the existence of the line systems related to all the extended near hexagons of Theorem 1.2. It remains to show uniqueness.

The μ -graph is the point set of the complement of some hyperplane H of the local near hexagon Γ_p .

Suppose there are two decompositions $P_+ \cup P_-$ and $Q_+ \cup Q_-$ of the μ -graph. Then $(P_+ \cap Q_+) \cup (P_- \cap Q_-)$ and $(P_+ \cap Q_-) \cup (P_- \cap Q_+)$ are joins of different components of the μ -graph. In particular, the μ -graph and hence the complement of the hyperplane H in the local near hexagon is disconnected. We will show that this leads to a contradiction, except in the case where Γ is the extended near hexagon related to $\text{PSp}_6(2)$.

For generalized quadrangles it is well known that the complements of hyperplanes are connected. Thus we can assume that the diameter of the near hexagon is at least 3.

Suppose the near hexagon contains quads. Then any two points at distance two are contained in a quad. Thus, if two points are in distinct connected components of the complement of H , then they are at distance 3. Suppose (a, b, c, d) is a path of length 3 in the collinearity graph of the near hexagon with a and d not in H , and in distinct connected components of the complement of H . Thus $b, c \in H$. Let Q be a quad containing b and c . As all points of Q that are at distance ≤ 2 from a and d are in H , this quad has to meet H in $b^\perp \cap Q$ or in $c^\perp \cap Q$, or is contained in H . Since there are $t_2 + 2$ lines on b (respectively c) that are not in H , there are $(t_2 + 2)/t_2$ quads on the

line through c and b meeting H in $c^\perp \cap Q$ (respectively $b^\perp \cap Q$). But that would imply that the third point of this line is on $2(t_2 + 2)$ lines not contained in H . A contradiction to 5.1.

Now consider the case that Γ is locally a generalized hexagon as in (iv) of Theorem 1.2. In this case the μ -graph is the set of points at distance 3 from a fixed point in the generalized hexagon \mathcal{H} , and as is shown in [15], this graph is connected.

So, for all the extended near hexagons different from the extended generalized hexagon related to $\text{PSP}_6(2)$ we have shown the lemma. It remains to consider the case where Γ is the extended generalized hexagon related to $\text{PSP}_6(2)$. Here the situation is a bit different.

Fix two points p and q as in the hypothesis of the lemma. The μ -graph $M := p^\perp \cap q^\perp$ of the collinearity graph of the extended generalized hexagon on 120 points related to $\text{PSP}_6(2)$ is the complement of a hyperplane H of the generalized hexagon Γ_p consisting of 9 lines pairwise at distance 3 in the dual hexagon. Inside the hexagon Γ_p , the complement of such a hyperplane is not connected but has two connected components of 18 points, see [23]. These connected components are distance regular bipartite graph with intersection array $\{3, 2, 2, 1; 1, 1, 2, 3\}$. Thus we can partition the 36 vertices in the complement of H into 4 cocliques of 9 vertices, called C_i , with $i = 1, \dots, 4$, being the 4 halves of the 2 bipartite subgraphs. Up to permutation of the indices we may assume that the two bipartite graphs decomposing M inside Γ_p have vertex set $C_1 \cup C_2$ and $C_3 \cup C_4$, respectively. Now $C_1 \cup C_3$ together with $C_2 \cup C_4$ as well as $C_1 \cup C_4$ with $C_2 \cup C_3$ yield a partition of the complement of H into two cocliques of Γ_p of size 18. These are the only two ways to obtain such a partition. We will show that only one of these two partitions satisfies the hypothesis of the lemma. Therefore consider the graph on M induced by Γ_q . At least one of the two partitions obtained inside Γ_p has to remain a partition into cocliques inside Γ_q . (Indeed, that such a partition exists follows from the existence of the corresponding line system.) Suppose $C_1 \cup C_3$ and $C_2 \cup C_4$ is such a partition. It now suffices to show that $C_1 \cup C_4$ is not a coclique inside Γ_q . Fix a point c of C_1 . Then $c^\perp \cap C_i$, with $i = 2, 3, 4$ contains 3 points. The three points of $c^\perp \cap C_1$ are collinear with c inside Γ_p , and hence not inside Γ_q . The three points of $c^\perp \cap C_3$ as well as the points in $c^\perp \cap C_1$ are also not collinear to c inside Γ_q as, by assumption, $C_1 \cup C_3$ is a coclique inside Γ_q . Hence, the three points of M collinear to c inside Γ_q are the points of $c^\perp \cap C_4$. In particular, $C_1 \cup C_4$ is not a coclique. This shows that also in this case, there is a unique way to partition M into two sets of size 18 which are both cocliques in Γ_p and Γ_q . \square

We are now able to prove that for each of the extended near hexagon Γ from Theorem 1.2 there exists, up to isomorphism, a unique regular line system \mathcal{L} as in the hypothesis of Theorem 1.1 with $\Gamma(\mathcal{L}) \simeq \Gamma$.

We fix the notation as before. If q is a point of $\Gamma(\mathcal{L})$ at distance 2 from p , then the μ -graph $p^\perp \cap q^\perp$ can be uniquely be partitioned as $P_+ \cup P_-$, as in 6.3. The above lemmas show that without loss of generality we can now assume that the point $+q$ of $\Gamma(\Sigma)$ is the point with μ -graph $+p^\perp \cap +q^\perp = P_+$ and $-q$ is the point with $+p^\perp \cap -q^\perp = P_-$.

In the next lemma we will see when two such points are adjacent.

Lemma 6.4. *Let q and r be two adjacent points at distance 2 from p , and $C = \{q, r, u, v\}$ a circle on q and r meeting p^\perp . Then $+q$ (respectively $+r$) is adjacent to εu and ηv for a unique pair $\pi_q = (\varepsilon, \eta) \in \{+, -\}^2$ (respectively π_r) with $\varepsilon \cdot \eta = -$. Moreover, $+q$ and $+r$ are adjacent if and only if $\pi_q = \pi_r$.*

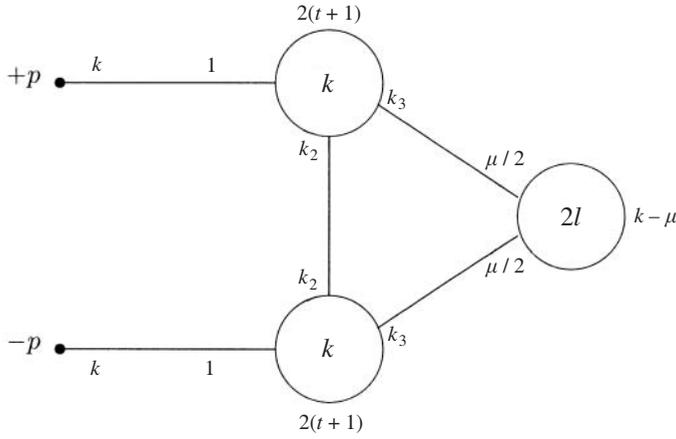
Proof. The point $+q$ is adjacent to εu and ηv for some $\varepsilon, \eta \in \{+, -\}$. Suppose $+q$ is adjacent to $+u$. The points $+u$ and $+v$ are not adjacent in the near hexagon Γ_{+p} . Thus $+u$ is adjacent to $-v$. In the near hexagon Γ_{+q} there is a unique line through $+u$ and one of the points $+v$ or $-v$. Clearly, this can only be $-v$. Hence $+q$ is also adjacent to $-v$. This proves the first part of the lemma.

The second part follows from the observation that the point $+q$ is adjacent to $+r$ if and only if $+r$ is on the unique line in Γ_{+q} through $+u$ and $-v$. Hence if and only if $+r$ is also adjacent to $+u$ and $-v$. □

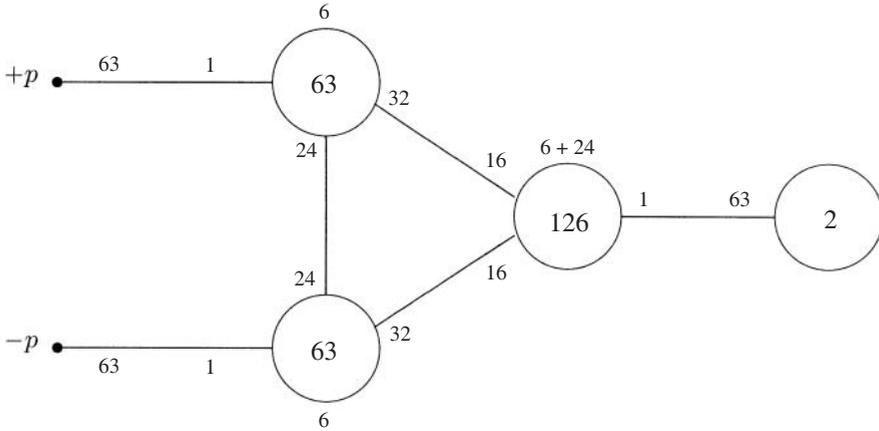
The above lemmas show that the point graph of $\Gamma(\Sigma)$ is uniquely determined by the isomorphism type of $\Gamma(\mathcal{L})$, which clearly implies that $\Gamma(\Sigma)$ is uniquely determined by $\Gamma(\mathcal{L})$, provided that on each edge at distance 2 from p there is a circle meeting p^\perp . In other words, provided that the geometric hyperplane which is the complement of the μ -graph in $\Gamma(\mathcal{L})_p$ has no deep points. This proves that for each of the extended near hexagons $\Gamma(\mathcal{L})$ of 1.2 there is a unique point graph for $\Gamma(\Sigma)$, except possibly, when $\Gamma(\mathcal{L})$ is isomorphic to the extended generalized hexagon related to the group $2^7 : G_2(2)$. So, assume that $\Gamma(\mathcal{L})$ is isomorphic to this extended generalized hexagon related to the group $2^7 : G_2(2)$. Then there is a unique point q at distance 3 from p in the point graph of $\Gamma(\mathcal{L})$. There are 126 points $\pm r$ at distance 2 from $+p$, with r at distance 2 from p in the point graph of $\Gamma(\mathcal{L})$. Of these 126 points 63 are adjacent to $+q$, the other 63 to $-q$. The induced subgraphs on the neighbors of $+q$ (or $-q$) is a generalized hexagon isomorphic to \mathcal{H} . The subgraph induced on the 126 points is isomorphic to the subgraph induced on the 126 points adjacent to $+p$ or $-p$. There is a unique way to partition the 126 points into two sets of 63 points on which the induced subgraph is the point graph of the hexagon \mathcal{H} . (Two adjacent vertices are in the same 63-set if and only if they have 5 common neighbors.) Thus also in this case $\Gamma(\Sigma)$ is uniquely determined by the isomorphism type of $\Gamma(\mathcal{L})$.

It remains to prove that there is a unique presentation of the point graph of $\Gamma(\Sigma)$ as a set of norm 3 vectors in a line system satisfying the hypothesis of Theorem 1.1. But the point graph of $\Gamma(\Sigma)$ uniquely determines the Gram matrix of the set of vectors Σ , and hence the line system \mathcal{L} . This finishes the proof of Theorem 1.1.

We want to finish the paper with some additional remarks. Consider any of the line systems \mathcal{L} from Theorem 1.1 different from the one related to $2^7 : G_2(2)$. Then the point graph of $\Gamma(\mathcal{L})$ is strongly regular. Suppose the parameters of this strongly regular graph are (k, λ, μ) and $l = k(k - \lambda - 1)/\mu$. Then the point graph of $\Gamma(\Sigma)$ has distance distribution diagram as follows. (Here $\lambda = k_1 = 2(t + 1)$, $k_2 = \frac{2^2 t(t+1)}{t_2+1}$ and $k_3 = \frac{2^3 t(t-t_2)}{t_2+1}$, i.e., k_i is the number of points at distance i from a fixed point in the local near hexagon.)



If $\Gamma(\mathcal{L})$ is isomorphic to the $2^7 : G_2(2)$ extended generalized hexagon then the distribution diagram for the point graph of $\Gamma(\Sigma)$ is as follows.



In all cases the line system \mathcal{L} can be realized as follows. Suppose V is the real vector space with basis Σ and equipped with an inner product for which Σ is an orthogonal basis. Let A be the 0, 1-adjacency matrix of the point graph of $\Gamma(\Sigma)$. From the distribution diagrams it is easy to see that the vector

$$(+p) - (-p) - (1/3)(\sum_{q \in p^\perp - \{p\}} ((+q) - (-q)))$$

is an eigenvector with eigenvalue $-k/3$ of A , where k is the number of points in the local near hexagon. Orthogonal projection of the vectors of Σ onto the $-k/3$ -eigenspace of A and rescaling yields a representation of Σ as norm 3 vectors on a tetrahedrally closed regular line system. The dimension of this representation is equal to the multiplicity of the eigenvalue $-k/3$ of A and equals $|\mathcal{L}|/(1 + k/9)$, see [7].

We notice that parts of this section could also be obtained using the theory on covers of diagram geometries, e.g. see [31]. For example, there it is shown that the extended near hexagon on 2300 points related to the group Co_2 has a unique universal cover, which is the extended near hexagon related to the 4600 vectors on the 2300 lines in \mathbb{R}^{23} in the line system of Example 3.1.

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H. Cuypers, Department of Mathematics, Eindhoven University of Technology, P.O. Box 513,
5600 MB Eindhoven, The Netherlands
Email: hansc@win.tue.nl