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(1, 1)-knots via the mapping class group of the twice punctured torus

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Abstract. We develop an algebraic representation for $(1,1)$-knots using the mapping class group of the twice punctured torus $\mathrm{MCG}_2(T)$. We prove that every $(1,1)$-knot in a lens space $L(p,q)$ can be represented by the composition of an element of a certain rank two free subgroup of $\mathrm{MCG}_2(T)$ with a standard element only depending on the ambient space. As notable examples, we obtain a representation of this type for all torus knots and for all two-bridge knots. Moreover, we give explicit cyclic presentations for the fundamental groups of the cyclic branched coverings of torus knots of type $(k, ck+2)$.

Key words. $(1,1)$-knots, Heegaard splittings, mapping class groups, two-bridge knots, torus knots.

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1 Introduction and preliminaries

The topological properties of $(1,1)$-knots, also called genus one 1-bridge knots, have recently been investigated in several papers (see [1], [5], [6], [8], [9], [10], [12], [13], [14], [15], [18], [19], [20], [21], [24], [25], [26]). These knots are very important in the light of some results and conjectures involving Dehn surgery on knots (see in particular [9] and [25]). Moreover, the strict connections between cyclic branched coverings of $(1,1)$-knots and cyclic presentations of groups have been pointed out in [5], [12] and [21].

Roughly speaking, a $(1,1)$-knot is a knot which can be obtained by gluing along the boundary two solid tori with a trivial arc properly embedded. A more formal definition follows. A set of mutually disjoint arcs $\{t_1,\dots,t_b\}$ properly embedded in a handlebody H is *trivial* if there exist b mutually disjoint discs $D_1,\dots,D_b \subset H$ such that $t_i \cap D_i = t_i \cap \partial D_i = t_i$, $t_i \cap D_j = \varnothing$ and $\partial D_i - t_i \subset \partial H$ for all $i, j = 1,\dots,b$ and $i \neq j$. Let $M = H \cup_\varphi H'$ be a genus g Heegaard splitting of a closed orientable 3-manifold M and let $F = \partial H = \partial H'$; a link $L \subset M$ is said to be in *b-bridge position* with respect to F if: (i) L intersects F transversally and (ii) $L \cap H$ and $L \cap H'$ are both

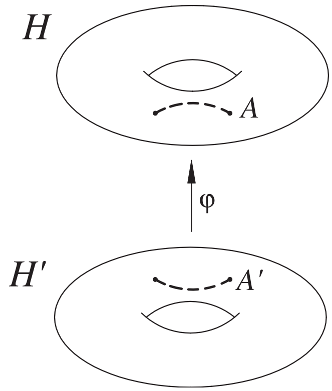

Figure 1. A $(1,1)$-decomposition.

the union of b mutually disjoint properly embedded trivial arcs. The splitting is called a *(g,b)-decomposition of L*. A link L is called a *(g,b)-link* if it admits a (g,b)-decomposition. Note that a $(0,b)$-link is a link in $\mathbf{S}^3$ which admits a b-bridge presentation in the usual sense. So the notion of (g,b)-decomposition of links in 3-manifolds generalizes the classical bridge (or plat) decomposition of links in $\mathbf{S}^3$ (see [7]). Obviously, a $(g,1)$-link is a knot, for every $g \geqslant 0$.

Therefore, a *$(1,1)$-knot* K is a knot in a lens space $L(p,q)$ (possibly in $\mathbf{S}^3$) which admits a $(1,1)$-decomposition

$$(L(p,q),K) = (H,A) \cup_{\varphi} (H',A'),$$

where $\varphi : (\partial H', \partial A') \to (\partial H, \partial A)$ is an (attaching) homeomorphism which reverses the standard orientation on the tori (see Figure 1). It is well known that the family of $(1,1)$-knots contains all torus knots (trivially) and all two-bridge knots (see [16]) in $\mathbf{S}^3$.

In this paper we develop an algebraic representation of $(1,1)$-knots through elements of $\mathrm{MCG}_2(T)$, the mapping class group of the twice punctured torus. In Section 2 we establish the connection between the two objects. In Section 3 we prove that every $(1,1)$-knot in a lens space $L(p,q)$ can be represented by an element of $\mathrm{MCG}_2(T)$ which is the composition of an element of a certain rank two free subgroup and of a standard element only depending on the ambient space $L(p,q)$. This representation will be called "standard". As a notable application, in Sections 4 and 5 we obtain standard representations for the two most important classes of $(1,1)$-knots in $\mathbf{S}^3$: the torus knots and the two-bridge knots. Moreover, applying certain results obtained in [5], we give explicit cyclic presentations for the fundamental groups of all cyclic branched coverings of torus knots of type $(k, ck+2)$, with $c, k > 0$ and k odd.

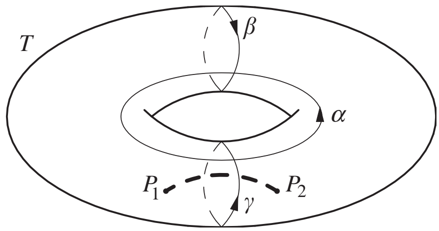

Figure 2. Generators of $MCG_2(T)$.

In what follows, the symbol $L(p,q)$ will denote any lens space, including $\mathbf{S}^3 = L(1,0)$ and $\mathbf{S}^1 \times \mathbf{S}^2 = L(0,1)$. Moreover, homotopy and homology classes will be denoted with the same symbol of the representing loops.

2 (1, 1)-knots and $MCG_2(T)$

Let F_g be a closed orientable surface of genus g and let $\mathscr{P} = \{P_1, \ldots, P_n\}$ be a finite set of distinguished points of F_g, called *punctures*. We denote by $\mathscr{H}(F_g, \mathscr{P})$ the group of orientation-preserving homeomorphisms $h : F_g \to F_g$ such that $h(\mathscr{P}) = \mathscr{P}$. The *punctured mapping class group* of F_g relative to $\mathscr{P}$ is the group of the isotopy classes of elements of $\mathscr{H}(F_g, \mathscr{P})$. Up to isomorphism, the punctured mapping class group of a fixed surface F_g relative to $\mathscr{P}$ only depends on the cardinality n of $\mathscr{P}$. Therefore, we can simply speak of the *n-punctured mapping class group* of F_g, denoting it by $MCG_n(F_g)$. Moreover, for isotopy classes we will use the same symbol of the representing homeomorphisms.

The *n-punctured pure mapping class group* of F_g is the subgroup $PMCG_n(F_g)$ of $MCG_n(F_g)$ consisting of the elements pointwise fixing the punctures. There is a standard exact sequence

$$1 \to PMCG_n(F_g) \to MCG_n(F_g) \to \Sigma_n \to 1,$$

where Σ_n is the symmetric group on n elements. A presentation of all punctured mapping class groups can be found in [11] and in [17].

In this paper we are interested in the two-punctured mapping class group of the torus $MCG_2(T)$. According to previously cited papers, a set of generators for $MCG_2(T)$ is given by a rotation ρ of π radians which exchanges the punctures and the right-handed Dehn twists $t_\alpha, t_\beta, t_\gamma$ around the curves α, β, γ respectively, as depicted in Figure 2. Since ρ commutes with the other generators, we have

$$MCG_2(T) \cong PMCG_2(T) \oplus \mathbb{Z}_2.$$

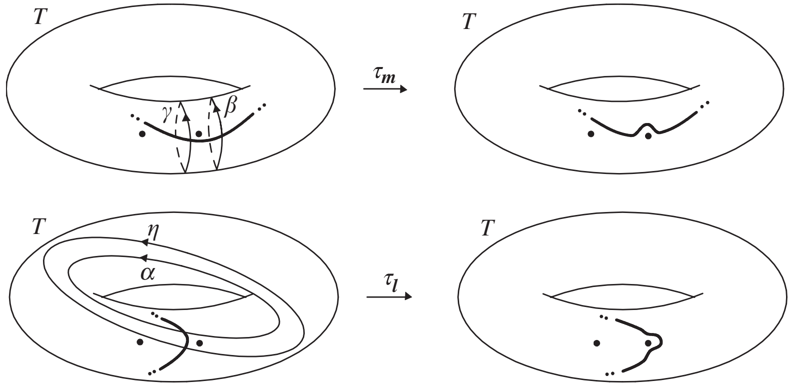

Figure 3. Action of τ_m and τ_l.

The following presentation for $\mathrm{PMCG}_2(T)$ has been obtained in [22]:

$$\langle t_\alpha, t_\beta, t_\gamma \mid t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta, t_\alpha t_\gamma t_\alpha = t_\gamma t_\alpha t_\gamma, t_\beta t_\gamma = t_\gamma t_\beta, (t_\alpha t_\beta t_\gamma)^4 = 1\rangle. \quad (1)$$

The group $\mathrm{PMCG}_2(T)$ (as well as $\mathrm{MCG}_2(T)$) naturally maps by an epimorphism to the mapping class group of the torus $\mathrm{MCG}(T) \cong \mathrm{SL}(2,\mathbf{Z})$, which is generated by t_α and $t_\beta = t_\gamma$. So we have an epimorphism

$$\Omega : \mathrm{PMCG}_2(T) \to \mathrm{SL}(2,\mathbf{Z})$$

defined by $\Omega(t_\alpha) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\Omega(t_\beta) = \Omega(t_\gamma) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

The group $\ker \Omega$ will play a fundamental role in our discussion. In order to investigate its structure, let us consider the two elements $\tau_m = t_\beta t_\gamma^{-1}$ and $\tau_l = t_\eta t_\alpha^{-1}$, where t_η is the right-handed Dehn twist around the curve η depicted in Figure 3. The effect of τ_m and τ_l is to slide one puncture (say P_2) respectively along a meridian and along a longitude of the torus, as shown in Figure 3. Observe that, since $\eta = \tau_m^{-1}(\alpha)$, we have $t_\eta = \tau_m^{-1} t_\alpha \tau_m$.

The following result can be obtained from [3, Theorem 1] and [2, Theorem 5] by classical techniques.

Proposition 1. *The group* $\ker \Omega$ *is freely generated by* $\tau_m = t_\beta t_\gamma^{-1}$ *and* $\tau_l = t_\eta t_\alpha^{-1}$, *where* $t_\eta = \tau_m^{-1} t_\alpha \tau_m$.

Now, let $K \subset L(p,q)$ be a $(1,1)$-knot with $(1,1)$-decomposition $(L(p,q),K) = (H,A) \cup_\varphi (H',A')$ and let $\mu : (H,A) \to (H',A')$ be a fixed orientation-reversing homeomorphism, then $\psi = \varphi \mu_{|\partial H}$ is an orientation-preserving homeomorphism of $(\partial H, \partial A) = (T, \{P_1, P_2\})$. Moreover, since two isotopic attaching homeomorphisms produce equivalent $(1,1)$-knots, we have a natural surjective map from the twice punctured mapping class group of the torus $\mathrm{MCG}_2(T)$ to the class $\mathscr{K}_{1,1}$ of all $(1,1)$-knots

$$\Theta : \psi \in \mathrm{MCG}_2(T) \mapsto K_\psi \in \mathscr{K}_{1,1}.$$

If $\Omega(\psi) = \begin{pmatrix} q & s \\ p & r \end{pmatrix}$, then K_ψ is a $(1,1)$-knot in the lens space $L(|p|,|q|)$ [4, p. 186], and therefore it is a knot in $\mathbf{S}^3$ if and only if $p = \pm 1$.

As will be proved in Section 3, we have the following "trivial" examples:

i) if either $\psi = 1$ or $\psi = t_\beta$ or $\psi = t_\gamma$, then K_ψ is the trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2$;

ii) if $\psi = t_\alpha$, then K_ψ is the trivial knot in $\mathbf{S}^3$.

Moreover, it is possible to prove that if $\psi = t_\alpha t_\beta t_\alpha t_\alpha t_\gamma t_\alpha$, then K_ψ is the knot $\mathbf{S}^1 \times \{P\} \subset \mathbf{S}^1 \times \mathbf{S}^2$, where P is any point of $\mathbf{S}^2$. So, in this case, K_ψ is a standard generator for the first homology group of $\mathbf{S}^1 \times \mathbf{S}^2$.

Every element ψ of $\mathrm{MCG}_2(T)$ can be written as $\psi = \psi' \rho^k$, $k \in \{0,1\}$, where $\psi' \in \mathrm{PMCG}_2(T)$. Since ρ can be extended to a homeomorphism of the pair (H, A), the $(1,1)$-knots K_ψ and $K_{\psi'}$ are equivalent. So, for our discussion it is enough to consider the restriction

$$\Theta' = \Theta_{|\mathrm{PMCG}_2(T)} : \psi \in \mathrm{PMCG}_2(T) \mapsto K_\psi \in \mathscr{K}_{1,1}.$$

3 Standard decomposition

In this section we show that every $(1,1)$-knot $K \subset L(p,q)$ admits a representation by the composition of an element in $\ker \Omega$ and an element which only depends on $L(p,q)$. A representation of this type will be called "standard". Note that a similar result, using a rank three free subgroup of $\mathrm{MCG}_2(T)$, has been obtained in [6, Theorem 3].

First of all, we deal with trivial knots in lens spaces. Let $\mathscr{T}$ be the subgroup of $\mathrm{PMCG}_2(T)$ generated by t_α and t_β. There exists a disk $D \subset H$, with $A \cap D = A \cap \partial D = A$ and $\partial D - A \subset T$, such that $D \cap \alpha = D \cap \beta = \emptyset$. So any element of $\mathscr{T}$ produces a trivial knot in a certain lens space. On the other hand, any trivial knot in a lens space admits a representation through an element of $\mathscr{T}$, as will be proved in Proposition 3.

We need a preparatory result.

Lemma 2. *Let K be a $(1,1)$-knot in $L(p,q)$. Then, for each $r, s \in \mathbb{Z}$ such that $qr - ps = 1$ there exists $\psi \in \mathrm{PMCG}_2(T)$, with $\Omega(\psi) = \begin{pmatrix} q & s \\ p & r \end{pmatrix}$, such that $K = K_\psi$.*

Proof. Let $K = K_{\bar{\psi}}$, with $\Omega(\bar{\psi}) = \begin{pmatrix} q & \bar{s} \\ p & \bar{r} \end{pmatrix}$. Since $q\bar{r} - p\bar{s} = 1$, there exist $c \in \mathbb{Z}$ such that $r = \bar{r} + cp$ and $s = \bar{s} + cq$. If $\psi = \bar{\psi} t_\beta^{-c}$, we have $K_\psi = K_{\bar{\psi}}$, since t_β^{-c} can be extended to a homeomorphism of the pair (H, A). Moreover $\Omega(\psi) = \Omega(\bar{\psi})\Omega(t_\beta^{-c}) = \begin{pmatrix} q & \bar{s} \\ p & \bar{r} \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & \bar{s} + cq \\ p & \bar{r} + cp \end{pmatrix}$. □

For integers p, q such that $0 < q < p$ and $\gcd(p, q) = 1$ consider the sequence of equations of the Euclidean algorithm (with $r_0 = p$, $r_1 = q$):

$$\begin{aligned} r_0 &= a_1 r_1 + r_2 \\ r_1 &= a_2 r_2 + r_3 \\ &\vdots \\ r_{m-2} &= a_{m-1} r_{m-1} + r_m \\ r_{m-1} &= a_m r_m, \end{aligned}$$

with $r_1 > r_2 > \cdots > r_{m-1} > r_m = 1$.

The a_i's are the coefficients of the continued fraction

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_m}}}.$$

In the following we will use the notation $p/q = [a_1, a_2, \ldots, a_m]$.

Proposition 3. *The trivial knot in* $\mathbf{S}^3 = L(1, 0)$ *is represented by* $\psi_{1,0} = t_\beta t_\alpha t_\beta$.

The trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2 = L(0, 1)$ *is represented by* $\psi_{0,1} = 1$.

Let p, q *be integers such that* $0 < q < p$ *and* $\gcd(p, q) = 1$. *If* $p/q = [a_1, a_2, \ldots, a_m]$, *then the trivial knot in the lens space* $L(p, q)$ *is represented by*

$$\psi_{p,q} = \begin{cases} t_\alpha^{a_1} t_\beta^{-a_2} \ldots t_\alpha^{a_m} & \text{if } m \text{ is odd}, \\ t_\alpha^{a_1} t_\beta^{-a_2} \ldots t_\beta^{-a_m} t_\beta t_\alpha t_\beta & \text{if } m \text{ is even}. \end{cases}$$

Proof. Since all the involved homeomorphisms belong to $\mathscr{T}$, all the knots are trivial. It is easy to check (see also [4, p. 186]) that, for suitable $r, s \in \mathbb{Z}$, we have:

$$\begin{pmatrix} q & s \\ p & r \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_m & 1 \end{pmatrix} & \text{if } m \text{ is odd}, \\ \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{if } m \text{ is even}. \end{cases}$$

Since $\Omega(t_\alpha^{a_i}) = \begin{pmatrix} 1 & 0 \\ a_i & 1 \end{pmatrix}$, $\Omega(t_\beta^{a_i}) = \begin{pmatrix} 1 & -a_i \\ 0 & 1 \end{pmatrix}$, and $\Omega(t_\beta t_\alpha t_\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the statement is obtained. □

Now we can prove the result announced at the beginning of this section.

Theorem 4. *Let K be a $(1,1)$-knot in $L(p,q)$. Then there exist $\psi', \psi'' \in \ker \Omega$ such that $K = K_\psi$, with $\psi = \psi' \psi_{p,q} = \psi_{p,q} \psi''$.*

Proof. By Lemma 2, there exists ψ, with $\Omega(\psi) = \Omega(\psi_{p,q})$, such that $K = K_\psi$. It suffices to define $\psi' = \psi \psi_{p,q}^{-1}$ and $\psi'' = \psi_{p,q}^{-1} \psi$. □

A representation $\psi \in \mathrm{PMCG}_2(T)$ of a $(1,1)$-knot will be called *standard* if ψ is of the type described in the previous theorem.

We point out that $(1,1)$-knots admit different (usually infinitely many) standard representations. For example, τ_m^c represents the trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2$, for all $c \in \mathbb{Z}$.

4 Representation of torus knots

In this section we give a standard representation for all torus knots in $\mathbf{S}^3$. Let $K = \boldsymbol{t}(k,h)$ be a torus knot of type (k,h). Then $\gcd(k,h) = 1$, and we can assume that K lies on the boundary $T = \partial H$ of a genus one handlebody H canonically embedded in $\mathbf{S}^3$. The homology class of K is $hl + km$, where l and m respectively denote a longitude and a meridian of T. By slightly pushing (the interior of) an arc $A' \subset K$ outside H and $K - A'$ inside H, we obtain an obvious $(1,1)$-decomposition of K. Observe that $0 < |k| < h$ can be assumed without loss of generality (see [4, p. 45]).

In the next statement $\lfloor x \rfloor$ denotes the integral part of x.

Theorem 5. *The torus knot $\boldsymbol{t}(k,h) \subset \mathbf{S}^3$ is the $(1,1)$-knot K_ψ with:*

$$\psi = \prod_{i=1}^{h} (\tau_m^{\lfloor (i-1)k/h \rfloor - \lfloor ik/h \rfloor} \tau_l^{-1}) t_\beta t_\alpha t_\beta,$$

where $\tau_m = t_\beta t_\gamma^{-1}$ and $\tau_l = \tau_m^{-1} t_\alpha \tau_m t_\alpha^{-1}$.

Proof. Up to isotopy, we can suppose that the arc $A = K_\psi - \mathrm{int}(A')$ lies on ∂H, as in Figure 4. The arc A can be transformed into an arc $\tilde{A}$ in such a way that $\tilde{A} \cup A'$ is a trivial knot in $\mathbf{S}^3$, represented by the standard homeomorphism $\psi_{1,0} = t_\beta t_\alpha t_\beta$, via a suitable sequence of homeomorphisms τ_l and τ_m, according to the following algorithm. Consider the sequence of equations:

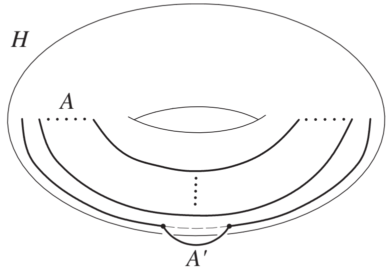

Figure 4.

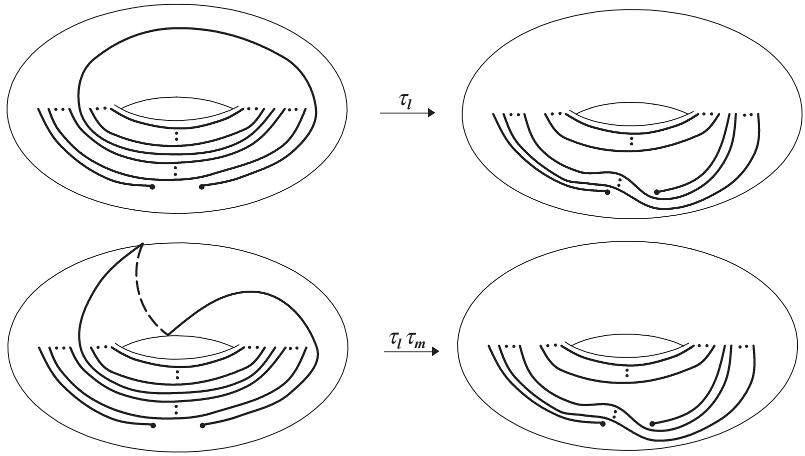

Figure 5.

$$\begin{aligned} k &= q_1 h + r_1, \\ 2k &= q_2 h + r_2, \\ &\vdots \\ hk &= q_h h + r_h, \end{aligned}$$

where $0 \leqslant r_i < h$, for $i = 1, \ldots, h$. Moreover, define $q_0 = 0$. So $q_i = \lfloor ik/h \rfloor$, for $i = 0, 1, \ldots, h$. Now define the homeomorphisms $\psi_i = \tau_l \tau_m^{q_i - q_{i-1}}$, for $i = 1, \ldots, h$. Figure 5 depicts the effect of τ_l and $\tau_l \tau_m$ on A. As a consequence, the homeomorphism $\phi = \psi_h \psi_{h-1} \ldots \psi_1$ transforms the arc A into the arc $\tilde{A}$ (Figure 6 shows the case $\boldsymbol{t}(5,7)$), and therefore we have $\psi_{1,0} = \phi\psi$. So $\phi^{-1}\psi_{1,0}$ represents the torus knot $\boldsymbol{t}(k,h)$. □

For example, $\boldsymbol{t}(5,7) = K_\psi$, with $\psi = \tau_l^{-1}(\tau_m^{-1}\tau_l^{-1})^2 \tau_l^{-1}(\tau_m^{-1}\tau_l^{-1})^3 t_\beta t_\alpha t_\beta$ (see Figure 6).

As a consequence, we obtain a cyclic presentation for the fundamental group for all cyclic branched coverings of a particular class of torus knots.

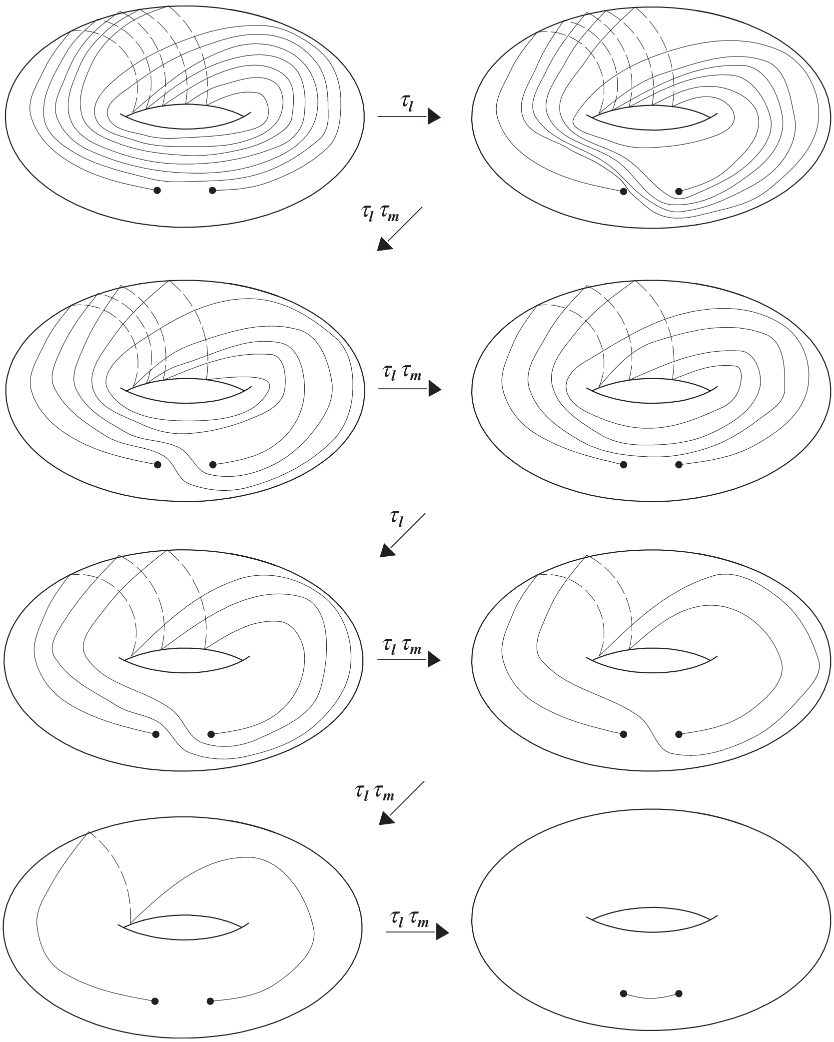

Figure 6. Trivialization of $\boldsymbol{t}(5,7)$.

Proposition 6. *The fundamental group of the n-fold cyclic branched covering of the torus knot* $\boldsymbol{t}(k, ck+2)$, *with* $k > 1$ *odd and* $c > 0$, *admits the cyclic presentation* $G_n(w)$, *where w is equal to*

$$\prod_{i=0}^{(k-3)/2} \left(\prod_{j=0}^{c(k-1)/2} x_{1-i(ck+2)+jk} \prod_{l=0}^{c(k+1)/2} x^{-1}_{ck(k-1)/2-i(ck+2)-lk} \right) \prod_{m=0}^{c(k-1)/2} x_{1-(k-1)(ck+2)/2+mk}$$

(*subscripts are taken modulo n*).

Proof. Let $r = (k-1)/2$. From Theorem 5 we have $\boldsymbol{t}(k, ck+2) = K_\psi$ with $\psi = (\tau_l^{-c}\tau_m^{-1})^r \tau_l^{-1} (\tau_l^{-c}\tau_m^{-1})^r \tau_l^{-c}\tau_m^{-1}\tau_l^{-1} t_\beta t_\alpha t_\beta$. Applying [5, Proposition 1], we obtain $\pi_1(\mathbf{S}^3 - \boldsymbol{t}(k, ck+2)) = \langle \alpha, \gamma \,|\, r(\alpha, \gamma) \rangle$, with $r(\alpha, \gamma) = (\gamma^{-1}\alpha^{cr+1}\gamma^{-1}\alpha^{-c(r+1)-1})^r \gamma^{-1}\alpha^{cr+1}$. Then $H_1(\mathbf{S}^3 - \boldsymbol{t}(k, ck+2)) = \langle \alpha, \gamma \,|\, \alpha - k\gamma \rangle$. Since, up to equivalence, $\omega_f(\gamma) = 1$, we have $\omega_f(\alpha) = k$. We set $\alpha = x\gamma^k$, therefore $\pi_1(\mathbf{S}^3 - \boldsymbol{t}(k, ck+2)) = \langle x, \gamma \,|\, \bar{r}(x, \gamma) \rangle$, with $\bar{r}(x, \gamma) = (\gamma^{-1}(x\gamma^k)^{1+c(k-1)/2}\gamma^{-1}(\gamma^{-k}x^{-1})^{1+c(k+1)/2})^{(k-1)/2}\gamma^{-1}(x\gamma^k)^{1+c(k-1)/2}$. The statement derives from a straightforward application of [5, Theorem 7]. □

For example, the fundamental group of the n-fold cyclic branched covering of $\boldsymbol{t}(5, 7)$ admits the cyclic presentation $G_n(w)$, where

$$w = x_{15}x_{20}x_{25}x_{24}^{-1}x_{19}^{-1}x_{14}^{-1}x_9^{-1}x_8x_{13}x_{18}x_{17}^{-1}x_{12}^{-1}x_7^{-1}x_2^{-1}x_1x_6x_{11}.$$

5 Representation of two-bridge knots

In this section we give a standard representation for all two-bridge knots in $\mathbf{S}^3$. Let $\boldsymbol{b}(a/b)$ be a non-trivial two-bridge knot in $\mathbf{S}^3$ of type (a, b). Then we can assume $\gcd(a, b) = 1$, a odd, b even and $0 < |b| < a$, without loss of generality (see [4, Ch. 12B]). It is known that $\boldsymbol{b}(a/b)$ admits a Conway presentation with an even number of even parameters $[2a_1, 2b_1, \ldots, 2a_n, 2b_n]$ (see Figure 7), satisfying the following relation:

$$\frac{a}{b} = 2a_1 + \cfrac{1}{2b_1 + \cfrac{1}{2a_2 + \cdots + \frac{1}{2b_n}}}.$$

Theorem 7. *The two-bridge knot* $\boldsymbol{b}(a/b) \subset \mathbf{S}^3$ *having Conway parameters* $[2a_1, 2b_1, \ldots, 2a_n, 2b_n]$ *is the* $(1,1)$*-knot* K_ψ *with*:

$$\psi = t_\beta t_\alpha t_\beta \tau_m^{-b_n} t_\varepsilon^{a_n} \ldots \tau_m^{-b_1} t_\varepsilon^{a_1},$$

where $t_\varepsilon = \tau_l^{-1}\tau_m\tau_l\tau_m^{-1}$ *is the right-handed Dehn twist around the curve* ε *depicted in Figure* 8.

Proof. Figure 8 shows the result of the application of $\tau_m^{-b_n} t_\varepsilon^{a_n} \ldots \tau_m^{-b_1} t_\varepsilon^{a_1}$. By applying $\psi_{1,0} = t_\beta t_\alpha t_\beta$ we obtain the two-bridge knot with Conway parameters $[2a_1, 2b_1, \ldots, 2a_n, 2b_n]$.

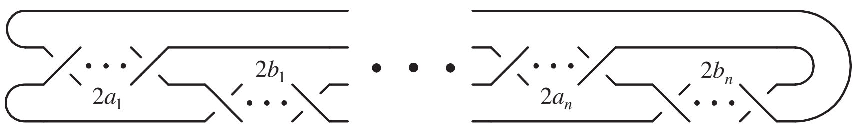

Figure 7. Conway presentation for two-bridge knots.

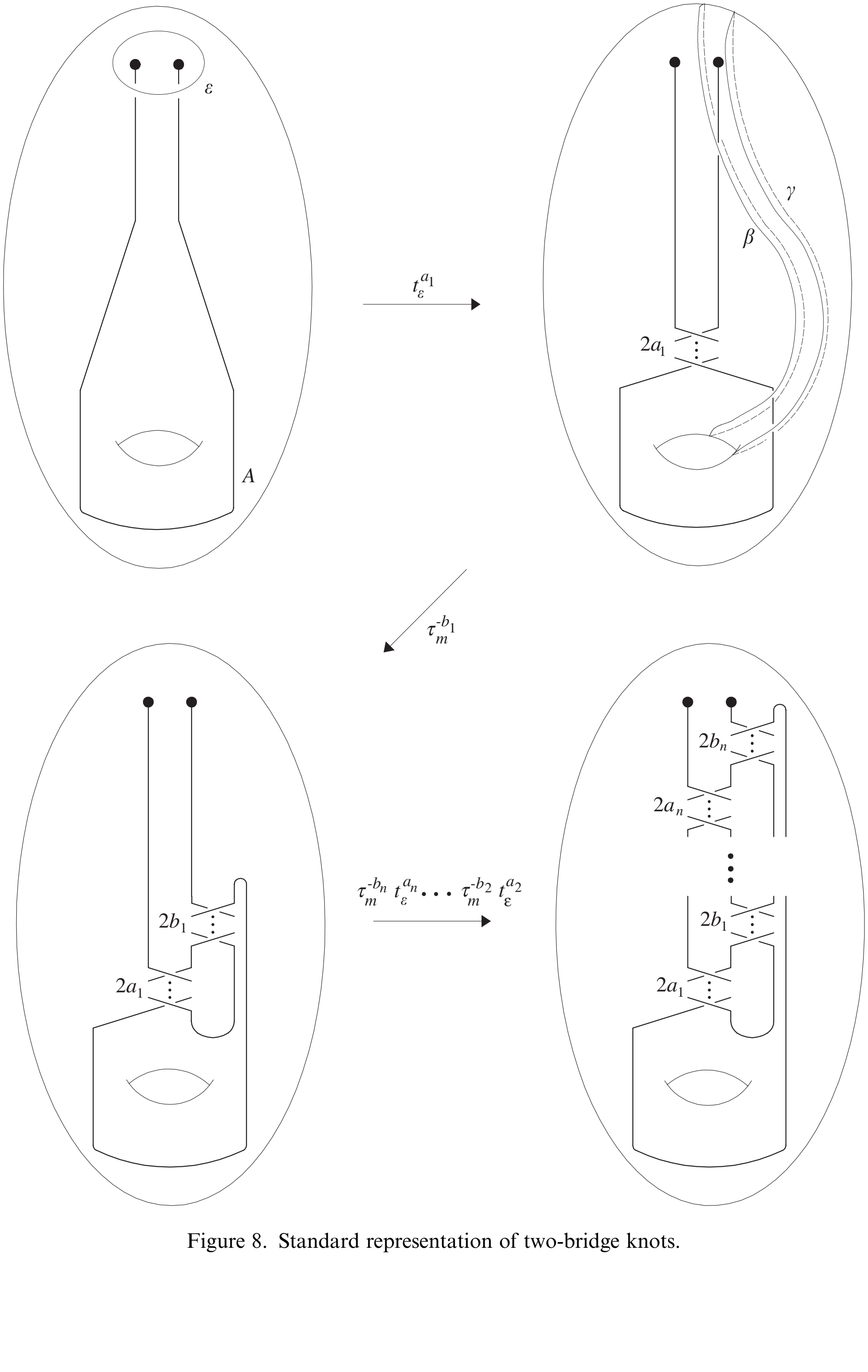

Figure 8. Standard representation of two-bridge knots.

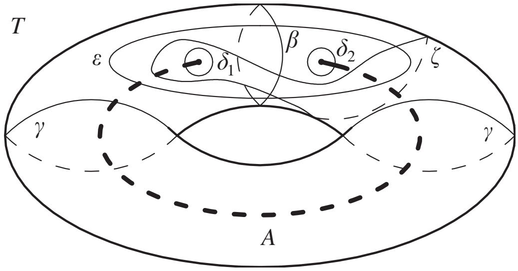

Figure 9.

Now we show that $t_\varepsilon = \tau_l^{-1}\tau_m\tau_l\tau_m^{-1}$ (note that no disk bounded by ε and properly embedded in H is disjoint from A). Referring to Figure 9, the following "lantern" relation $t_\gamma^2 t_{\delta_1} t_{\delta_2} = t_\varepsilon t_\beta t_\zeta$ holds (see [23]). So we obtain $\zeta = t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1}(\gamma)$ and therefore $t_\zeta = t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1}$. Since $t_{\delta_1} = t_{\delta_2} = 1$ we have $t_\varepsilon = t_\gamma^2 t_\zeta^{-1} t_\beta^{-1} = t_\gamma^2 t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} \cdot t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1}$. Now, using the relations of (1) we get

$$\begin{aligned}
t_\varepsilon &= t_\gamma^2 t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} = t_\gamma t_\alpha t_\gamma t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} \\
&= t_\gamma t_\alpha t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\beta t_\gamma^{-1} t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} = t_\gamma t_\alpha t_\beta^{-1} t_\gamma t_\alpha^{-1} t_\gamma^{-1} t_\beta t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} \\
&= t_\gamma t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\gamma^{-1} t_\alpha t_\beta t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} = t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\beta t_\gamma^{-1} t_\alpha t_\beta t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} \\
&= t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\beta t_\gamma^{-1} t_\alpha t_\alpha t_\beta t_\alpha t_\gamma^{-1} t_\alpha^{-1} t_\beta^{-1} = t_\gamma t_\beta^{-1} t_\alpha^{-1} t_\beta t_\gamma^{-1} t_\alpha t_\alpha t_\beta t_\gamma^{-1} t_\alpha^{-1} t_\gamma t_\beta^{-1} \\
&= \tau_m^{-1} t_\alpha^{-1} \tau_m t_\alpha t_\alpha \tau_m t_\alpha^{-1} \tau_m^{-1} = \tau_m^{-1} t_\alpha^{-1} \tau_m t_\alpha \tau_m \tau_m^{-1} t_\alpha \tau_m t_\alpha^{-1} \tau_m^{-1} \\
&= t_\eta^{-1} t_\alpha \tau_m t_\eta t_\alpha^{-1} \tau_m^{-1} = \tau_l^{-1} \tau_m \tau_l \tau_m^{-1}. \qquad \square
\end{aligned}$$

For example, the figure-eight knot $\boldsymbol{b}(5/2)$, which has Conway parameters [2, 2], is the knot K_ψ with $\psi = t_\beta t_\alpha t_\beta \tau_m^{-1} t_\varepsilon$ (see Figure 10).

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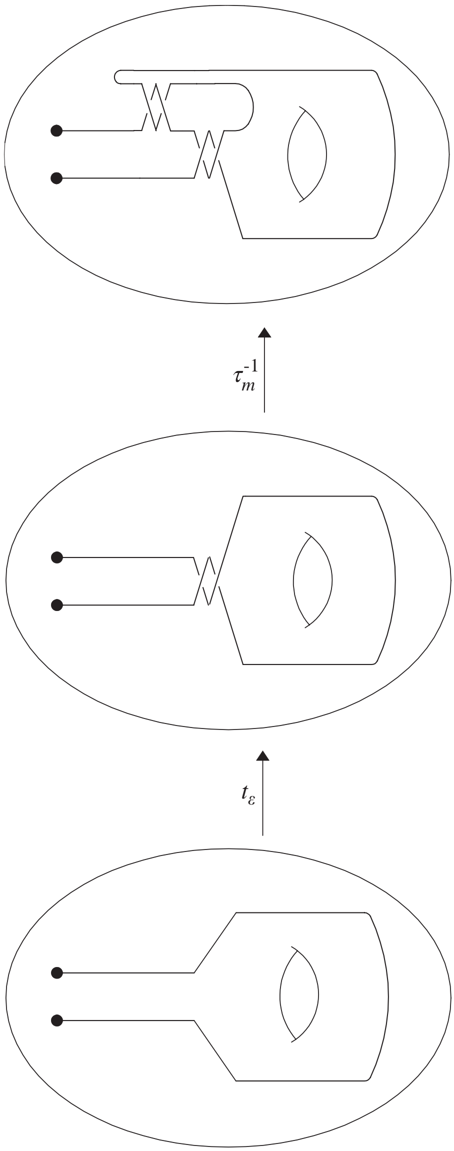

Figure 10. Standard representation of the figure-eight knot.

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