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Prym varieties of pairs of coverings

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Abstract. The Prym variety of a pair of coverings is defined roughly speaking as the complement of the Prym variety of one morphism in the Prym variety of another morphism. We show that this definition is symmetric and give conditions when such a Prym variety is isogenous to an ordinary Prym variety or to another such Prym variety. Moreover in order to show that these varieties actually occur we compute the isogeny decomposition of the Jacobian variety of a curve with an action of the symmetric group S_5.

Key words. Prym variety, group action.

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1 Introduction

Let X be a smooth projective curve over an algebraically closed field k and G a finite group of automorphisms of X. This induces an action of G on the Jacobian JX of X which can be used to decompose JX into a product of smaller dimensional abelian varieties up to isogeny:

$$JX \sim B_1^{d_1} \times \cdots \times B_r^{d_r}$$

The abelian subvarieties B_i correspond one-to-one to the irreducible $\mathbb{Q}$-representations of the group G, which also determine the numbers d_i. One would like to understand the decomposition in terms of the curve and its group action itself. In fact, for many small groups the B_i's can be interpreted as Prym varieties of coverings $X_M \to X_N$, where $M \subset N$ are subgroups of G and X_M and X_N denote the quotients X/M and X/N. This is the case for example for the groups $S_3, S_4, A_4, A_5, D_p, WD_4$ and Q_8 (see [7], [8], [2] and [5]).

For other groups such as S_5 (see Theorem 4.1 below) or the dihedral groups D_n, (see [1] Remark 8.8) not for every B_i there is such a Prym variety. Another type of abelian variety turns up: Let M, N_1 and N_2 be subgroups of G with $M \subset N_1$ and $M \subset N_2$. This gives the following diagram of coverings:

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$$\begin{array}{ccccc} & & X_M & & \\ & f_1 \swarrow & & \searrow f_2 & \\ X_{N_1} & & & & X_{N_2} \\ & g_1 \searrow & & \swarrow g_2 & \\ & & X_N & & \end{array} \tag{1.1}$$

where $N = \langle N_1, N_2 \rangle$, the subgroup generated by N_1 and N_2. Let $P(f_i)$ denote the Prym variety of the covering f_i. Similarly $P(g_i)$ is defined for $i = 1, 2$. Then $f_2^* P(g_2)$ is an abelian subvariety of $P(f_1)$. Since the canonical polarization of JX induces a polarization of $P(f_1)$, the complementary abelian subvariety of $f_2^* P(g_2)$ in $P(f_1)$ is well defined. Similarly the complementary abelian subvariety of $f_1^* P(g_1)$ in $P(f_2)$ is well defined. It turns out that both complementary abelian subvarieties coincide as subvarieties of JX_M. We denote this subvariety by $P(f_1, f_2)$ or $P(X_{N_1} \leftarrow X_M \rightarrow X_{N_2})$ and call it the *Prym variety of the pair of coverings* (f_1, f_2).

In Section 2 we introduce the Prym variety $P(f_1, f_2)$ slightly more generally for any pair of coverings of smooth projective curves $(f_1 : X \rightarrow X_1, f_2 : X \rightarrow X_2)$ and prove its main properties. In Section 3 we prove some auxiliary results on group actions needed in the last section, where we work out the decomposition of JX in the case of an action of the symmetric group S_5 of degree 5.

2 Definition of $P(f_1, f_2)$

Let $f : X \rightarrow Y$ be a morphism of degree n of smooth projective curves over an algebraically closed field k. Denote by $J_X := \mathrm{Pic}^0(X)$ and $J_Y := \mathrm{Pic}^0(Y)$ the Jacobians of X and Y. Pulling back line bundles defines a homomorphism

$$f^* : J_Y \rightarrow J_X.$$

f^* has finite kernel and is an embedding if and only if f does not factor via a cyclic étale cover of degree $\geqslant 2$ (see [4], Proposition 11.4.3). The norm map of line bundles (see [3], Section 6.5) defines a homomorphism

$$N_f : J_X \rightarrow J_Y.$$

The *Prym variety* $P(f)$ *of the morphism* f is defined to be the abelian subvariety

$$P(f) := \ker(N_f)^0$$

of J_X where the 0 means the connected component containing 0. Note that N_f is not necessarily a Prym variety in the classical sense, i.e. the canonical polarization of J_X does not necessarily induce a multiple of a principal polarization on $P(f)$. Suppose $g : Y \rightarrow Z$ is a second morphism of smooth projective curves, say of degree m. The Prym varieties of f, g and gf are related as follows:

Proposition 2.1. *$P(f)$ and $f^*P(g)$ are abelian subvarieties of $P(gf)$ and the addition map gives an isogeny*

$$P(f) \times f^*P(g) \to P(gf).$$

Proof. The addition map yields an isogeny

$$P(gf) \times (gf)^* J_Z \to J_X.$$

Combing the analogous isogenies $P(f) \times f^* J_Y \to J_X$ and $f^*P(g) \times (gf)^* J_Z \to f^* J_Y$ we obtain that the addition map gives an isogeny

$$P(f) \times f^*P(g) \times (gf)^* J_Z \to J_X.$$

Since $P(f)$ and $f^*P(g)$ are obviously abelian subvarieties of $P(gf)$, this implies the assertion. □

Now suppose that we are given a commutative diagram of finite morphisms of smooth projective curves:

$$\begin{array}{ccccc} & & X & & \\ & f_1 \swarrow & & \searrow f_2 & \\ X_1 & & & & X_2 \\ & g_1 \searrow & & \swarrow g_2 & \\ & & Y & & \end{array} \tag{2.1}$$

Then we have

Proposition 2.2. *Suppose g_1 and g_2 do not both factorize via the same morphism $Y' \to Y$ of degree $\geqslant 2$. Then the Prym variety $f_2^*P(g_2)$ is an abelian subvariety of the Prym variety $P(f_1)$.*

Proof. First assume that f_1 and f_2 do not both factorize via a morphism $f : X \to X'$. The universal property of the fibre product over Y yields a diagram

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ & & X_1 \times_Y X_2 & & \\ & p_1 \swarrow & & \searrow p_2 & \\ X_1 & & & & X_2 \\ & g_1 \searrow & & \swarrow g_2 & \\ & & Y & & \end{array}$$

where $n : X \to X_1 \times_Y X_2$ denotes the normalization map and $p_i : X_1 \times_Y X_2 \to X_i$ the projection maps and $f_i = p_i n$ for $i = 1$ and 2. According to [3], Proposition 6.5.8 we have

$$N_{p_1} p_2^*(L) = g_1^* N_{g_2}(L)$$

for any line bundle L on X_2. But the norm map of line bundles is also defined for the map n (see [3] Section 6.5, condition II is satisfied) and we have

$$N_{f_1}(f_2^*(L)) = N_{p_1}N_n(n^*p_2^*(L)) = N_{p_1}p_2^*(L).$$

Both equations together imply the assertion is this case.

In the general case suppose f_i factorizes as $f_i = f_i'f$ with some morphism of smooth projective curves $f : X \to X'$ and $f_i' : X' \to X_i$ for $i = 1$ and 2. By what we have just shown, $f_2'^*P(g_2)$ is an abelian subvariety of $P(f_1')$. So $f_2^*P(g_2)$ is an abelian subvariety of $f^*P(f_1')$ which is an abelian subvariety of $P(f_1)$ according to Proposition 1.1. □

Remark 2.3. The assumption that g_1 and g_2 do not factorize via the same morphism $Y' \to Y$ is necessary for the validity of Proposition 1.2. To give an example, let $h : Y \to \mathbb{P}_1$ be a finite covering. Replace g_i by hg_i for $i = 1, 2$. Then $P(hg_2) = JX_2$ and and it is easy to give an example of a diagram (1.1) where $f_2^*JX_2$ is not an abelian subvariety of $P(f_1)$.

The canonical principal polarization induces a polarization on $P(f_1)$. Hence the complementary abelian subvariety P_1 of the abelian subvariety $f_2^*P(g_2)$ in $P(f_1)$ is well defined (see [4], Section 5.3). The addition map induces an isogeny of polarized abelian varieties

$$P_1 \times f_2^*P(g_2) \to P(f_1).$$

In the same way the canonical principal polarization of JX induces a polarization on $P(f_2)$. Hence the complementary abelian subvariety P_2 of $f_1^*P(g_1)$ in $P(f_2)$ is well defined and the addition map induces an isogeny of polarized abelian varieties

$$f_1^*P(g_1) \times P_2 \to P(f_2).$$

P_1 and P_2 are both abelian subvarieties of JX with induced polarizations, say H_1 and H_2. We have:

Proposition 2.4. *The polarized abelian subvarieties (P_1, H_1) and (P_2, H_2) of JX coincide.*

Proof. It suffices to show that $P_1 = P_2$ since the polarizations are induced by the canonical principal polarization of JX. By definition of the Prym varieties the addition maps induce isogenies

$$f_1^*g_1^*J_Y \times f_1^*P(g_1) \times f_2^*P(g_2) \times P_1 \to f_1^*J_{X_1} \times P(f_1) \to J_X$$

and

$$f_2^*g_2^*J_Y \times f_2^*P(g_2) \times f_1^*P(g_1) \times P_2 \to f_2^*J_{X_2} \times P(f_2) \to J_X$$

where all abelian varieties are subvarieties of J_X. Obviously we have $f_1^*g_1^*J_Y = f_2^*g_2^*J_Y$. So if Z denotes the image of $f_1^*g_1^*J_Y \times f_1^*P(g_1) \times f_2^*P(g_2)$ in J_X the addition map gives isogenies

$$Z \times P_1 \to J_X \quad \text{and} \quad Z \times P_2 \to J_X.$$

Now the corresponding decompositions of the tangent spaces are orthogonal with respect to the hermitian form associated to the canonical polarization of J_X. This implies that on the one hand P_1 and on the other hand P_2 is the complement of the abelian subvariety Z in J_X. Since the complement is uniquely determined, this implies the assertion. □

We call the abelian variety $P_1 = P_2$ or more precisely the polarized abelian variety $(P_1, H_1) = (P_2, H_2)$ the *Prym variety of the pair of coverings* (f_1, f_2) and denote it by $P(f_1, f_2)$ or $P(X_1 \leftarrow X \rightarrow X_2)$. Note that $P(f_1, f_2)$ is defined for *any* pair $(f_1 : X \to X_1, f_2 : X \to X_2)$ of coverings of smooth projective curves. Given (f_1, f_2), the curve Y in the diagram (2.1) is the smooth projective curve corresponding to the function field $k(X_1) \cap k(X_2)$. If for example $f_1 = f_2$ then we have obviously $P(f_1, f_1) = 0$.

Applying the Hurwitz formula, it is easy to compute the dimension of $P(f_1, f_2)$. We do this only in the most important case where the function fields satisfy $k(X_1)k(X_2) = k(X)$ and $k(X_1) \cap k(X_2) = k(Y)$, i.e. the hypotheses of Proposition 2.2 are satisfied and X is the normalization of $X_1 \times_Y X_2$. Then we have $d_1 := \deg(f_1) = \deg(g_2)$ and $d_2 := \deg(f_2) = \deg(g_1)$. Moreover for any covering f of smooth projective curves let ϑ_f denote the degree of the ramification divisor of f. Then we have

Proposition 2.5.

$$\dim P(f_1, f_2) = (d_1 - 1)(d_2 - 1)(g(Y) - 1) + 1/2[\vartheta_{f_1} + (d_1 - 1)\vartheta_{g_1} - \vartheta_{g_2}].$$

3 Isogenies between Prym varieties and Prym varieties of pairs

Let again G be a finite group acting on a smooth projective curve X. If $M \subset N$ and $M' \subset N'$ are two pairs of subgroups of G, it may happen that the Prym varieties $P(X_M/X_N)$ and $P(X_{M'}/X_{N'})$ are isogenous. Similarly this may happen for Prym varieties of pairs. In this section we give a criterion for this. Since we need this only in the case of the symmetric group S_5, we will assume in this section and without further notice that every irreducible $\mathbb{Q}$-representation of the group G is absolutely irreducible. We will see that then the Prym varieties and Prym varieties of pairs depend only on the induced representations of the trivial representations of the subgroups in question. For a general group we will come to this question in a subsequent paper.

The action of G on the curve X induces an action on its Jacobian JX and thus an algebra homomorphism

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(JX).$$

If e denotes any idempotent of the algebra $\mathbb{Q}[G]$, we define

$$\operatorname{Im}(e) := \operatorname{Im}(\rho(me)) \subseteq JX$$

where m is some positive integer such that $me \in \mathbb{Z}[G]$. $\operatorname{Im}(e)$ is an abelian subvariety of JX, which certainly does not depend on the chosen integer m.

Let $W_1, \dots, W_r$ denote the irreducible $\mathbb{Q}$-representations of G. We assume in the sequel that W_1 is the trivial representation and that $d_i = \dim W_i$ for $i = 1, \dots, r$. If e_i denotes the central idempotent of $\mathbb{Q}[G]$ associated to W_i and $A_i = \operatorname{Im}(e_i)$ the corresponding abelian subvariety of JX for $i = 1, \dots, r$, then the addition map induces an isogeny (see [5], Proposition 2.1)

$$\mu : A_1 \times \cdots \times A_r \to A. \tag{3.1}$$

If $d_i > 1$ the abelian variety A_i can be decomposed further: Since W_i is absolutely irreducible, it admits up to a positive constant a uniquely determined G-invariant scalar product (see [9]). Fix one of these for every i and denote it by $(\, ,)$. Let $\{w_{i,1}, \dots, w_{i,d_i}\}$ be a basis of W_i, orthogonal with respect to $(\, ,)$, and define

$$p_{w_{i,j}} := \frac{d_i}{|G| \cdot \|w_{i,j}\|^2} \sum_{g \in G} (w_{i,j}, g w_{i,j}) g.$$

Schur's character relations (see [9], Chapter 2, Corollary 3 of Proposition 4) can be translated into terms of idempotents as follows (see [5], Proposition 3.3): $p_{w_{i,1}}, \dots, p_{w_{i,d_i}}$ are orthogonal idempotents in $\mathbb{Q}[G]$ satisfying

$$p_{w_{i,1}} + \cdots + p_{w_{i,d_i}} = e_i.$$

This implies that if $B_{i,j} := \operatorname{Im}(p_{w_{i,j}})$, the addition map induces an isogeny

$$\mu_i : B_{i,1} \times \cdots \times B_{i,d_i} \to A_i. \tag{3.2}$$

Moreover, since the minimal left ideals of $\mathbb{Q}[G]$ generated by the idempotents $p_{w_{i,j}}$ are pairwise isomorphic for a fixed i, it follows that the abelian varieties $B_{i,1}, \dots, B_{i,d_i}$ are pairwise isogenous (see [5]). Combining everything we obtain:

There are abelian subvarieties $B_1, \dots, B_r$ and an isogeny

$$JX \sim B_1^{d_1} \times \cdots \times B_r^{d_r}. \tag{3.3}$$

The action of G on JX induces an action on the tangent space $T_0 JX$. Denoting $V_i = W_i \otimes \mathbb{C}$, we obtain a decomposition

$$T_0 JX \simeq V_1^{n_1} \times \cdots \times V_r^{n_r}. \tag{3.4}$$

Comparing this with the decomposition (3.3) implies $T_0(B_i^{d_i}) \simeq V_i^{n_i}$. This gives $d_i \cdot \dim B_i = n_i \cdot \dim V_i$. But $\dim V_i = d_i$ and thus

$$n_i = \dim B_i.$$

Let H denote the canonical polarization of JX. It can be considered as a positive definite hermitian form on T_0JX. Since the group G of automorphisms of JX is induced by the automorphism group G of the curve X, it preserves the polarization H. This implies that we may change the isomorphism (3.4) in such a way that H restricts to the scalar product $(\ , \ )$ on $W_i \subset V_i$ for all $i = 1, \dots, r$. We fix this isomorphism in the sequel. Using this we can show:

Proposition 3.1. *Let $M \subset N$ be subgroups of the group G. Then*

$$P(X_M/X_N) \sim B_2^{s_2} \times \cdots \times B_r^{s_r}$$

with $s_i = \dim W_i^M - \dim W_i^N$ for $i = 2, \dots, r$.

Note that in the special case $M = \{1\}$ and $N = G$ Proposition 3.1 gives the well known fact

$$P(X/Y) \sim B_2^{d_2} \times \cdots \times B_r^{d_r}$$

since $\dim W_i^{\{1\}} - \dim W_i^G = \dim W_i = d_i$ for $i = 2, \dots, r$.

Proof. For $i = 1, \dots, r$ choose an orthogonal basis

$$\{w_{i,1}, \dots, w_{i,t_i}, w_{i,t_i+1}, \dots, w_{i,t_i+s_i}, w_{i,t_i+s_i+1}, \dots, w_{i,d_i}\}$$

of W_i in such a way that

$$W_i^N = \langle w_{i,1}, \dots, w_{i,t_i} \rangle \quad \text{and} \quad W_i^M = \langle w_{i,1}, \dots, w_{i,t_i+s_i} \rangle.$$

Then

$$W_i^M = W_i^N + \langle w_{i,t_i+1}, \dots, w_{i,t_i+s_i} \rangle,$$

the sum being orthogonal.

It is easy to see that $p_{w_{i,j}}$ is the projection of W_i onto the 1-dimensional subspace spanned by $w_{i,j}$ (see [6], Remarque (2), page 53). It follows that

$$W_i^M = \sum_{j=1}^{t_i+s_i} p_{w_{i,j}}(W_i) \quad \text{and} \quad W_i^N = \sum_{j=1}^{t_i} p_{w_{i,j}}(W_i).$$

Since the sums are orthogonal, this implies

$$W_i^M = W_i^N + \sum_{j=t_i+1}^{t_i+s_i} p_{w_{i,j}}(W_i).$$

This equation immediately yields, if we again denote $V_i = W_i \otimes \mathbb{C}$:

$$V_i^M = V_i^N + \sum_{j=t_i+1}^{t_i+s_i} p_{w_{i,j}}(V_i)$$

the sums being orthogonal.

On the other hand the tangent map at the origin to $p_{w_{i,j}} : JX \to JX$ is $p_{w_{i,j}} : T_0JX \to T_0JX$. So the tangent space at the origin of the subvariety $\sum_{i=1}^{r} \sum_{j=1}^{t_i+s_i} \operatorname{Im} p_{w_{i,j}} \subset JX$ is $\sum_{i=1}^{r} \sum_{j=1}^{t_i+s_i} p_{w_{i,j}}(T_0JX)$. But

$$\sum_{i=1}^{r} \sum_{j=1}^{t_i+s_i} p_{w_{i,j}}(T_0JX) = (T_0JX)^M = T_0JX_M.$$

It follows that

$$JX_M \sim \sum_{i=1}^{r} \sum_{j=1}^{t_i+s_i} \operatorname{Im} p_{w_{i,j}}$$

(which is the image of the sum map $\times_{i=1}^{r} \times_{j=1}^{t_i+s_i} B_{i,j} \to JX$). Similarly we have

$$JX_N \sim \sum_{i=1}^{r} \sum_{j=1}^{t_i} \operatorname{Im} p_{w_{i,j}}.$$

Hence, since $t_1 = d_1 (= 1)$ and thus $s_1 = 0$, we obtain the orthogonal decomposition

$$JX_M \sim JX_N \times \sum_{i=2}^{r} \sum_{j=t_i+1}^{t_i+s_i} \operatorname{Im} p_{w_{i,j}}.$$

On the other hand, by definition of the Prym variety we have the orthogonal decomposition

$$JX_M \sim JX_N \times P(X_M \to X_N).$$

Comparing both, orthogonal cancellation gives

$$P(X_M \to X_N) \sim \sum_{i=2}^{r} \sum_{j=t_i+1}^{t_i+s_i} \operatorname{Im} p_{w_{i,j}}. \tag{3.5}$$

This implies the assertion, since B_i is isogenous to $\operatorname{Im} p_{w_{i,j}}$ for all j. □

For any subgroup M of G let $\mathbb{Q}(G/M)$ denote the induced representation of the trivial representation of M in G. Note that for subgroups $M \subset N$ of G,

$\mathbb{Q}(G/N)$ is a subrepresentation of $\mathbb{Q}(G/M)$ so that $\mathbb{Q}(G/M) - \mathbb{Q}(G/N)$ is in fact a $\mathbb{Q}$-representation.

Corollary 3.2. *Suppose* $\mathbb{Q}(G/M) - \mathbb{Q}(G/N) \simeq \bigoplus_{i=2}^{r} W_i^{s_i}$. *Then*

$$P(X_M \to X_N) \sim B_2^{s_2} \times \cdots \times B_r^{s_r}.$$

Proof. For any representation W of G let χ_W denote its character. Since any irreducible representation of G is absolutely irreducible, we may apply Frobenius reciprocity, to give for any $j = 1, \dots, r$

$$\begin{aligned}\dim W_j^M - \dim W_j^N &= (\chi_{\mathbb{Q}(G/M) - \mathbb{Q}(G/N)}, \chi_{W_j}) \\ &= \sum_{i=1}^{r} s_i \cdot (\chi_{W_i}, \chi_{W_j}) = s_j.\end{aligned}$$

Hence Proposition 3.1 gives the assertion. □

Applying Corollary 3.2 twice we obtain

Corollary 3.3. *Let* $M_i \subset N_i$ *be subgroups of* G *such that the representations* $\mathbb{Q}(G/M_i) - \mathbb{Q}(G/N_i)$ *are isomorphic for* $i = 1$ *and* 2. *Then*

$$P(X_{M_1} \to X_{N_1}) \sim P(X_{M_2} \to X_{N_2}).$$

Now suppose we are given the following diagram of subgroups of G

$$\begin{array}{ccccc} & & N_1 & & \\ & \nearrow & & \searrow & \\ M & & & & N = \langle N_1, N_2 \rangle \to G \\ & \searrow & & \nearrow & \\ & & N_2 & & \end{array}$$

where all the maps are the canonical inclusions. This induces the diagram (1.1) of coverings of curves. The equation $N = \langle N_1, N_2 \rangle$ implies that g_1 and g_2 do not both factorize via a morphism $Y' \to X_N$ of degree $\geqslant 2$. With the notation of above we have

Proposition 3.4. $P(f_1, f_2) \sim B_2^{s_2} \times \cdots \times B_r^{s_r}$ *with*

$$s_i = \dim W_i^M + \dim W_i^N - \dim W_i^{N_1} - \dim W_i^{N_2} \quad \textit{for } i = 2, \dots, r.$$

Note that $M \subset N_1$ and $M \subset N_2$ imply $W^{N_1} + W^{N_2} \subset W^M$ and $N = \langle N_1, N_2 \rangle$ implies $W^{N_1} \cap W^{N_2} = W^N$. Hence

$$\dim W^{N_1} + \dim W^{N_2} - \dim W^N \leqslant \dim W^M.$$

So $s_i \geqslant 0$. This in turn implies that $\mathbb{Q}(G/M) + \mathbb{Q}(G/N) - \mathbb{Q}(G/N_1) - \mathbb{Q}(G/N_2)$ is actually a representation.

One can also state the inequality as:

$$\dim W^{N_2} - \dim W^N \leqslant \dim W^M - \dim W^{N_1}$$

which is the reason why we can choose the basis the way we do in the proof below.

Proof. For $i = 1, \dots, r$ we choose an orthogonal basis $\{w_{i,1}, \dots, w_{i,t_i}, w_{i,t_i+1}, \dots, w_{i,t_i+s_i^1}, w_{i,t_i+s_i^1+1}, \dots, w_{i,t_i+s_i^1+s_i^2}, w_{i,t_i+s_i^1+s_i^2+1}, \dots, w_{i,d_i}\}$ of W_i in such a way that

$$W_i^N = \langle w_{i,1}, \dots, w_{i,t_i} \rangle, \quad W_i^{N_1} = \langle w_{i,1}, \dots, w_{i,t_i+s_i^1} \rangle,$$
$$W_i^{N_2} = \langle w_{i,1}, \dots, w_{i,t_i}, w_{i,t_i+s_i^1+1}, \dots, w_{i,t_i+s_i^1+s_i^2} \rangle \quad \text{and}$$
$$W_i^M = \langle w_{i,1}, \dots, w_{i,t_i+s_i^1+s_i^2}, \dots, w_m \rangle.$$

By (3.5) we have

$$f_{N_2}^* P(X_{N_2}/X_N) \sim \sum_{i=2}^{r} \sum_{j=t_i+s_i^1+1}^{t_i+s_i^1+s_2^2} \operatorname{Im} p_{w_{i,j}}$$

all sums being orthogonal with respect to the polarization induced by the canonical polarization H of JX. Since $f_{N_2} = f_2 \cdot f_M$ and f_M^* is an isogeny, this gives

$$f_2^* P(X_{N_2}/X_N) \sim \sum_{i=2}^{r} \sum_{j=t_i+s_i^1+1}^{t_i+s_i^1+s_2^2} \operatorname{Im} p_{w_{i,j}}.$$

In the same way we get

$$P(X_M/X_{N_1}) \sim \sum_{i=2}^{r} \sum_{j=t_i+s_i^1+1}^{m} \operatorname{Im} p_{w_{i,j}}.$$

Since all sums are orthogonal and $P(f_1, f_2)$ is by definition the orthogonal complement of $f_2^* P(X_{N_2}/X_N)$ in $P(X_M/X_{N_1})$, this implies

$$P(f_1, f_2) \sim \sum_{i=2}^{r} \sum_{j=t_i+s_i^1+s_i^2+1}^{m} \operatorname{Im} p_{w_{i,j}}.$$

Since $\operatorname{Im} p_{i,j}$ is isogenous to B_i for all j and moreover (3.1) and (3.2) are isogenies, we obtain

$$P(f_1, f_2) \sim \prod_{i-2}^{r} \prod_{j=t_i+s_i^1+s_i^2+1}^{m} B_i.$$

Now the assertion follows from $m - t_i - s_i^1 - s_i^2 = \dim W_i^M + \dim W_i^N - \dim W_i^{N_1} - \dim W_i^{N_2}$. □

It is easy to see that $\mathbb{Q}(G/M) + \mathbb{Q}(G/N) - \mathbb{Q}(G/N_1) - \mathbb{Q}(G/N_2)$ is actually a representation. Hence in the same way that Corollary 3.2 follows from Proposition 3.1, Proposition 3.4 implies:

Corollary 3.5. *Let $(M \subset N_1, M \subset N_2)$ be a triple of subgroups of G and $N = \langle N_1, N_2 \rangle$. If*

$$\mathbb{Q}(G/M) + \mathbb{Q}(G/N) - \mathbb{Q}(G/N_1) - \mathbb{Q}(G/N_2) \simeq \bigoplus_{i=2}^{r} W_i^{s_i}$$

then

$$P(f_1, f_2) \sim B_2^{s_2} \times \cdots \times B_r^{s_r}$$

Finally Corollaries 3.2 and 3.5 imply

Corollary 3.6. (a) *If $M' \subset N'$ is another pair of subgroups of G such that the representation $\mathbb{Q}(G/M) + \mathbb{Q}(G/N) - \mathbb{Q}(G/N_1) - \mathbb{Q}(G/N_2)$ is isomorphic to the representation $\mathbb{Q}(G/M') - \mathbb{Q}(G/N')$, then*

$$P(f_1, f_2) \sim P(X_{M'} \to X_{N'}).$$

(b) *If $(M' \subset N_1', M' \subset N_2')$ is another triple of subgroups of G and $N' = \langle N_1', N_2' \rangle$ such that the representations $\mathbb{Q}(G/M) + \mathbb{Q}(G/N) - \mathbb{Q}(G/N_1) - \mathbb{Q}(G/N_2)$ and $\mathbb{Q}(G/M') + \mathbb{Q}(G/N') - \mathbb{Q}(G/N_1') - \mathbb{Q}(G/N_2')$ are isomorphic and $f_1' : X_{M'} \to X_{N_1'}$ and $f_2' : X_{M'} \to X_{N_2'}$ denote the corresponding coverings, then*

$$P(f_1, f_2) \sim P(f_1', f_2').$$

In somewhat vague terms Corollaries 3.3 and 3.6 can be expressed by saying: The induced representations of the trivial representations determine the isogeny decomposition.

4 Example: The symmetric group of degree 5

Let X be smooth projective curve with an action of the symmetric group S_5 of degree 5. The group action induces the decomposition (3.3) of the Jacobian JX. Note that if we assume that $g(X/S_5) \geqslant 2$, then every abelian subvariety B_i occurring in (3.3) is positive dimensional according to [5] Theorem 4.1. In this section we apply the results of Section 3 in order to express the abelian subvarieties B_i of decomposition (3.3) in terms of Prym varieties of subgroups and pairs of subgroups of S_5.

We consider S_5 as the group of permutations of the set of integers $\{1, \ldots, 5\}$. In order to state the result consider the following subgroups of S_5:

$A_5 := \langle (1,2,3,4,5), (3,4,5) \rangle$ of order 60,
$S_4 := \langle (2,3), (2,4), (2,5) \rangle$ of order 24,
$A_4 := \langle (2,3)(4,5), (2,4)(3,5), (3,4,5) \rangle$ of order 12,
$D_5 := \langle (1,2,3,4,5), (2,5)(3,4) \rangle$ of order 10,
$D_4 := \langle (2,3), (2,4,3,5) \rangle$ of order 8,
$K := \langle (2,3), (4,5) \rangle$ of order 4,
$L := \langle (2,3), (4,5), (1,2,3) \rangle$ of order 12 and
$M := \langle (1,2,3,4,5), (2,5)(3,4), (2,4,5,3) \rangle$ of order 20.

For any subgroup M of S_5 let $X_M := X/M$ denote the quotient curve of X by the action of M and denote $Y := X/S_5$. If $M \subset N$ is any pair of subgroups of S_5, we denote by $P(X_M \to X_N)$ the Prym variety of the associated covering $X_M \to X_N$. Similarly for any triple of subgroups $(M \subset N_1, M \subset N_2)$ let $P(X_{N_1} \leftarrow X_M \to X_{N_2})$ denote the Prym variety of the pair of morphisms $(X_M \to X_{N_1}, X_M \to X_{N_2})$. With this notation we have:

Theorem 4.1.

$$JX \sim JY \times P(X_{A_5} \to Y) \times P(X_{S_4} \to Y)^4 \times P(X_{A_5} \leftarrow X_{A_4} \to X_{S_4})^4$$
$$\times P(X_M \leftarrow X_{D_5} \to X_{A_5})^5 \times P(X_M \to Y)^5 \times P(X_{D_4} \leftarrow X_K \to X_L)^6$$

There is no pair of subgroups $M \subset N$ of S_5 whose associated Prym variety $P(X_M \to X_N)$ is isogenous to a Prym variety of a pair of morphisms occurring in this decomposition.

Proof. Let $U, U', V, V', W, W', \bigwedge^2 V$ denote the irreducible $\mathbb{C}$-representations of S_5. They are determined by the following character table:

	1	(12)	(12)(34)	(123)	(12)(345)	(1234)	(12345)
#	1	10	15	20	20	30	24
U	1	1	1	1	1	1	1
U'	1	−1	1	1	−1	−1	1
V	4	2	0	1	−1	0	−1
V'	4	−2	0	1	1	0	−1
W	5	1	1	−1	1	−1	0
W'	5	−1	1	−1	−1	1	0
$\bigwedge^2 V$	6	0	−2	0	0	0	1

Observe that all irreducible representations are defined over $\mathbb{Q}$. Hence according to

Equation (3.3) there are abelian subvarieties $B_{U'}, B_V, B_{V'}, B_W, B_{W'}$ and $B_{\wedge^2 V}$ of JX, uniquely determined up to isogeny, such that

$$JX \sim JY \times B_{U'} \times B_V^4 \times B_{V'}^4 \times B_W^5 \times B_{W'}^5 \times B_{\wedge^2 V}^6.$$

We have to identify $B_{U'}, \dots, B_{\wedge^2 V}$ in terms of Prym varieties.

(a) $B_{U'}$: Using the above character table one easily checks:

$$\mathbb{Q}(S_5/A_5) - \mathbb{Q}(S_5/S_5) \simeq U'.$$

So Corollary 3.2 implies $B_{U'} \sim P(X_{A_5} \to Y)$.

(b) B_V: One checks $\mathbb{Q}(S_5/S_4) - \mathbb{Q}(S_5/S_5) \simeq V$. So Corollary 3.2 implies $B_V \sim P(X_{S_4} \to Y)$.

(c) $B_{W'}$: $\mathbb{Q}(S_5/M) - \mathbb{Q}(S_5/S_5) \simeq W'$. Hence Corollary 3.2 implies $B_{W'} \sim P(X_M \to Y)$.

(d) $B_{V'}$: Consider the diagram

$$\begin{array}{ccc} & X_{A_4} & \\ \swarrow & & \searrow \\ X_{A_5} & & X_{S_4} \\ \searrow & & \swarrow \\ & Y & \end{array}$$

One checks $\langle S_4, A_5 \rangle = S_5$ and $\mathbb{Q}(S_5/A_4) + \mathbb{Q}(S_5/S_5) - \mathbb{Q}(S_5/S_4) - \mathbb{Q}(S_5/A_5) \simeq V'$. So Corollary 3.5 yields $B_{V'} \sim P(X_{A_5} \leftarrow X_{A_4} \to X_{S_4})$.

(e) B_W: Consider the diagram

$$\begin{array}{ccc} & X_{D_5} & \\ \swarrow & & \searrow \\ X_M & & X_{A_5} \\ \searrow & & \swarrow \\ & Y & \end{array}$$

One checks $\langle M, A_5 \rangle = S_5$ and $\mathbb{Q}(S_5/D_5) + \mathbb{Q}(S_5/S_5) - \mathbb{Q}(S_5/M) - \mathbb{Q}(S_5/A_5) \simeq W$. So Corollary 3.5 implies $B_W \sim P(X_M \leftarrow X_{D_5} \to X_{A_5})$.

(f) $B_{\wedge^2 V}$: Consider the diagram

$$\begin{array}{ccc} & X_K & \\ \swarrow & & \searrow \\ X_{D_4} & & X_L \\ \searrow & & \swarrow \\ & Y & \end{array}$$

One checks $\langle D_4, L \rangle = S_5$ and $\mathbb{Q}(S_5/K) + \mathbb{Q}(S_5/S_5) - \mathbb{Q}(S_5/D_4) - \mathbb{Q}(S_5/L) \simeq \wedge^2 V$. So Corollary 3.5 gives $B_{\wedge^2 V} \sim P(X_{D_4} \leftarrow X_K \to X_L)$.

It remains to show that $B_{V'}$, B_W and $B_{\wedge^2 V}$ are not isogenous to a Prym variety of a covering associated to a pair of subgroups $M \subset N$ of S_5. For this we computed the Prym varieties of all conjugacy classes of pairs of such subgroups. The computations are a little too long to repeat them here. □

Finally let us give some examples of isogenous Prym varieties as well as Prym varieties of pairs which are isogenous to Prym varieties of coverings. For this consider the following subgroups of S_5:

$C_2 := \langle (2,5)(3,4) \rangle$ of order 2,

$C_3 := \langle (3,4,5) \rangle$ of order 3,

$C_5 := \langle (1,2,3,4,5) \rangle$ of order 5,

$N := \langle (3,4,5), (1,2)(4,5) \rangle$ of order 6,

$S_3 := \langle (4,5), (3,4,5) \rangle$ of order 6 and

$K_1 := \langle (2,3)(4,5), (2,4,3,5) \rangle$ of order 4.

Examples 4.2. (a) $P(X_{D_5} \to X_{A_5}) \sim P(X_{D_4} \to X_{S_4})$.
(b) $P(X_{C_5} \leftarrow X \to X_{C_2}) \sim P(X_{C_2} \to X_{D_5})$.
(c) $P(X_N \leftarrow X_{C_3} \to X_{A_4}) \sim P(X_{C_5} \to X_{D_5})$.
(d) $P(X_{A_4} \leftarrow X_{C_3} \to X_{S_3}) \sim P(X_{K_1} \to X_{D_4})$.

Note that in (a) $X_{D_5} \to X_{A_5}$ is of degree 6 whereas $X_{D_4} \to X_{S_4}$ is of degree 3.

Proof. We have $\mathbb{Q}(S_5/D_5) - \mathbb{C}(S_5/A_5) \simeq W \oplus W' \simeq \mathbb{Q}(S_5/D_4) - \mathbb{Q}(S_5/S_4)$. So Corollary 3.3 implies (a). As for (b), note first that $\langle C_5, C_2 \rangle = D_5$. Then we have $\mathbb{Q}(S_5/(1)) + \mathbb{Q}(S_5/D_5) - \mathbb{Q}(S_5/C_2) - \mathbb{Q}(S_5/C_5) \simeq V + V' + W + W' + (\wedge^2 V)^2 \simeq \mathbb{Q}(S_5/C_2) - \mathbb{Q}(S_5/D_5)$. So Corollary 3.6 (a) implies the assertion. The proof of (c) and (d) is similar. □

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