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# GLOBAL EXISTENCE AND STABILITY OF SOME SEMILINEAR PROBLEMS

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ABSTRACT. We prove global existence and stability results for a semilinear parabolic equation, a semilinear functional equation and a semilinear integral equation using an inequality which may be viewed as a nonlinear singular version of the well known Gronwall and Bihari inequalities.

## 1. INTRODUCTION

In this paper we shall present a work which improves a recent result of M. Medved [11] as well as the application of the method to other problems such as semilinear functional differential equations and semilinear integral equations. We first report Medved's result from [11]. The author considered the Cauchy problem

$$\left\{ \begin{array}{l} \frac{du}{dt} + Au = f(t,u), \quad u \in X \\ u(0) = u_0 \in X \end{array} \right. \tag{1}$$

where $X$ is an appropriate Banach space and $A : X \to X$ is a sectorial operator. It is known (see [7]) that there is a real number $c$ such that if $\tilde{A} := A + cI$, then $Re\ \sigma(\tilde{A}) > 0$ where $\sigma(\tilde{A})$ is the spectrum of the operator $\tilde{A}$. The fractional power $\tilde{A}^\alpha$ of $\tilde{A}$ is defined in the usual way as the inverse of $\tilde{A}^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\tilde{A}t} dt$ for $\alpha > 0$, where $\Gamma(\alpha)$ is the Eulerian Gamma function. If we denote by $X^\alpha := D(\tilde{A}^\alpha)$ the domain of $\tilde{A}^\alpha$ and $\|x\|_\alpha := \left\| \tilde{A}^\alpha x \right\|$, $x \in X^\alpha$, then $(X^\alpha, \|.\|_\alpha)$ is a Banach space. Furthermore, the operator $-A$ is the infinitesimal generator of an analytic semi-group $\left\{e^{-tA}\right\}_{t \geq 0}$ satisfying for $Re\ \sigma(A) > b > 0$

$$\left\| e^{-tA} x \right\|_\alpha := \left\| \tilde{A}^\alpha e^{-tA} x \right\| \leq dt^{-\alpha} e^{-bt} \|x\|, \quad t > 0, \tag{2}$$

for any $x \in X^\alpha$, where $d > 0$ is a constant.

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If there is an $\alpha \in (0,1)$ such that $f : R \times X^\alpha \to X$, $(t,u) \mapsto f(t,u)$ is locally Hölder in $t$ and locally Lipschitz in $u$, then by a solution of (1) we mean a continuous function $u : (0,T) \to X^\alpha$ with $u(0) = u_0 \in X^\alpha$ such that $f(.,u(.)) : (0,T) \to X$, $t \mapsto f(t,u(t))$ is continuous, $u(t) \in D(A)$, $t \in (0,T)$ and $u$ satisfies (1) on $(0,T)$. A solution $u(t)$ of (1) coincides then (see [13]) with a solution of the integral equation

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s,u(s))ds\,, \quad 0 < t \le T \tag{3}$$

for which $u : (0,T) \to X^\alpha$ is continuous and $f(.,u(.)) : (0,T) \to X$, $t \mapsto f(t,u(t))$ is continuous.

Let $R = (-\infty,\infty)$ and $R^+ = [0,\infty)$. M. Medved [11] proved the following theorem.

**Theorem 1.** *Let $A, f, b$ and $d$ be as above and*

$$\|f(t,u)\| \le t^\kappa \eta(t) \|u\|_\alpha^m\,, \quad m > 1,\ \ \kappa \ge 0$$

*for all $(t,u) \in R^+ \times X^\alpha$, where $\eta : (0,\infty) \to R$ is a continuous, nonnegative function. Then the following assertions hold:*

(1) *Let $0 < \alpha < \min\left\{\frac{1}{2}, \frac{\kappa}{m} + \frac{1}{2pm}\right\}$ for some $p > 1$. Let the function*

$$t \mapsto t^{2q\alpha} \int_0^t \eta(s)^{2q} e^{2q[(1-m)b+m\varepsilon]s} ds$$

*be bounded on the interval $(0,\infty)$ for some $0 < \varepsilon < b$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $u(t)$ be a solution of problem (1) satisfying $u(0) = u_0 \in X^\alpha$, with*

$$(m-1)2^{2q-1}\left(d\|u_0\|\right)^{2q(m-1)} K(\varepsilon)^q L(\varepsilon)^{\frac{q}{p}} (dt^\alpha)^{2q} \int_0^t \eta^{2q} e^{2q[(1-m)b+m\varepsilon]s} ds < 1,$$

*where*

$$K(\varepsilon) = \frac{\Gamma(2\beta-1)}{(2\varepsilon)^{2\beta-1}}, \quad L(\varepsilon) = \frac{\Gamma((2\gamma-2)p+1)}{(2\gamma-2)p+1}, \quad \beta = 1-\alpha.$$

*Then $u(t)$ exists on the interval $(0,\infty)$ and $\lim\limits_{t\to\infty} \|u(t)\|_\alpha = 0$.*

(2) *Let $\frac{1}{2} \le \alpha < min\{1, \frac{\kappa}{m} + \frac{1}{kqm}\}$ for some $k > 1$, $\beta = 1-\alpha = \frac{1}{1+z}$, $z \ge 1$, $q = z+2$. Assume that the function*

$$t \mapsto t^{rq\alpha} \int_0^t \eta(s)^{rq} e^{rq[(1-m)b+m\varepsilon]s} ds$$

*is bounded on the interval $(0,\infty)$ for some $0 < \varepsilon < b$, where $\frac{1}{k} + \frac{1}{r} = 1$. Let $u(t)$ be a solution of problem (1) satisfying $u(0) = u_0$, where*

$$(m-1)2^{rqm}(d\|u_0\|)^{rq(m-1)} P(\varepsilon) t^{rq\alpha} \int_0^t \eta(s)^{rq[(1-m)b+m\varepsilon]s} ds$$

$$\begin{cases} < 1 & \text{for} \quad rq(m-1) \quad \text{even}, \\ \ne 1 & \text{for} \quad rq(m-1) \quad \text{odd}, \end{cases}$$

*where $P(\varepsilon)$ is the expression $(M(\varepsilon)N(\varepsilon))^{rq}$ with $M(\varepsilon) = \left[\frac{\Gamma(1-\alpha p)}{(p\varepsilon)^{1-\alpha p}}\right]^{\frac{1}{p}}$ and $N(\varepsilon) = \left[\frac{\Gamma(kq(\gamma-1)+1)}{(kq\varepsilon)^{kq(\gamma-1)+1}}\right]^{\frac{1}{kq}}$. Then $u(t)$ exists on the interval $(0,\infty)$ and $\lim\limits_{t\to\infty}\|u(t)\|_\alpha = 0$.*

It is our task to weaken the assumptions of this theorem. To this end we use a crucial Lemma which may be found in [12]. This is done in section 2. In section 3 and 4 we discuss applications of this method to semilinear functional differential equations and integral equations respectively.

## 2. A Semilinear Parabolic Equation

For our theorems we need the following lemmas. The first lemma was also used by Medved and can be found in [1]. The second one is crucial to our argument and is reported from [12] with its proof for the sake of completeness. For convenience we shall adopt the same notation as in [11].

**Lemma 2.** *Let $a(t)$, $b(t)$, $k(t)$, $\psi(t)$ be nonnegative, continuous functions on the interval $I = (0,T)$ $(0 < T \le \infty)$, $\omega : (0,\infty) \to R$ be a continuous, nonnegative and nondecreasing function, $\omega(0) = 0$, $\omega(u) > 0$ for $u > 0$ and let $A(t) = \max\limits_{0\le s\le t} a(s)$, $B(t) = \max\limits_{0\le s\le t} b(s)$. Assume that*

$$\psi(t) \le a(t) + b(t)\int_0^t K(s)\omega(\psi(s))ds, \;\; t \in I.$$

*Then*

$$\psi(t) \le \Omega^{-1}\left[\Omega(A(t)) + B(t)\int_0^t K(s)ds\right], \;\; t \in (0,T_1),$$

*where $\Omega(v) = \int\limits_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}$, $v \ge v_0 > 0$, $\Omega^{-1}$ is the inverse of $\Omega$ and $T_1 > 0$ is such that $\Omega(A(t)) + B(t)\int_0^t K(s)ds \in D(\Omega^{-1})$ for all $t \in (0,T_1)$.*

**Lemma 3.** *If $\mu$, $\nu$, $\tau > 0$ and $z > 0$, then*

$$z^{1-\nu}\int_0^z (z-\zeta)^{\nu-1}\zeta^{\mu-1}\exp(-\tau\zeta)d\zeta \le C\tau^{-\mu} \tag{4}$$

*where $C$ is a constant independent of $z$.*

**Proof.** Let $I(z)$ denote the left-hand side of (4). Then by a change of variables

$$I(z) = z^\mu\int_0^1 (1-\xi)^{\nu-1}\xi^{\mu-1}\exp(-\tau z\xi)d\xi.$$

Observing that

$$z^\mu(1-\xi)^{\nu-1}\xi^{\mu-1}\exp(-\tau z\xi) \le \begin{cases} \max(1,2^{1-\nu})z^\mu\xi^{\mu-1}\exp(-\tau z\xi), & 0 \le \xi \le 1/2 \\ 2(1-\xi)^{\nu-1}\Gamma(\mu+1)\tau^{-\mu}, & 1/2 < \xi \le 1, \end{cases}$$

it follows that, $I(z) \le \max(1,2^{1-\nu})\Gamma(\mu)(1+\mu/\nu)\tau^{-\mu}$. □

The next lemma is to be compared with lemma 3 in [11]. In fact it is the counterpart of lemma 3. Note the disappearance of the terms in $\varepsilon$ and the appearance of new terms in $B_1(t)$ and $B_2(t)$.

Besides the use of the previous Lemma, the idea of the proof of this Lemma relies on Medved's method; see Theorem 4 in [10] for the linear version of this result.

**Lemma 4.** *Let $a(t)$, $F(t)$, $\psi(t)$, $b(t)$ be continuous, nonnegative functions on $I=(0,T)$ $(0<T\leq\infty)$, $\beta>0$, $\gamma>0$, $m>1$ and $\psi(t)$ satisfies the inequality*

$$\psi(t)\leq a(t)+b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\psi(s)^m\,ds,\ \ t\in I=(0,T). \tag{5}$$

*Then the following assertions hold:*

(1) *If $\beta>1/2$, $\gamma>1/2$ and $C$ is the constant of lemma 3, then*

$$\psi(t)\leq\Phi(t):=A_1^{1/2}(t)\left[1-(m-1)\Xi_1(t)\right]^{\frac{1}{2(1-m)}}$$

*for all $t\in I=(0,T)$ for which the right-hand side is defined, where*
$\Xi_1(t)=A_1(t)^{m-1}B_1(t)\int_0^t F(s)^2e^{2s}ds$, $A_1(t)=2\max_{0\leq s\leq t}a(s)^2$ *and*
$B_1(t)=C2^{2(1-\gamma)}\max_{0\leq s\leq t}b(s)^2s^{2(\beta-1)}$.

(2) *If $\beta=\frac{1}{1+z}$ for some $z\geq 1$, $\gamma>1-\frac{1}{p}$ and $q=z+2$ then*

$$\psi(t)\leq\Psi(t):=A_2^{1/q}(t)\left[1-(m-1)\Xi_2(t)\right]^{\frac{1}{q(1-m)}}$$

*for all $t\in I=(0,T)$ for which the right-hand side is defined, where for the constant $C$ of lemma 3, $\Xi_2(t)=A_2(t)^{m-1}B_2(t)\int_0^t F(s)^qe^{qs}ds$,*
$A_2(t)=2^{q-1}\max_{0\leq s\leq t}a(s)^q$ *and* $B_2(t)=C^{q/p}p^{q(1-\gamma)-\frac{q}{p}}\max_{0\leq s\leq t}b(s)^qs^{q(\beta-1)}$
*and* $\frac{1}{p}+\frac{1}{q}=1$.

**Proof.** (1) Observe that by the Schwarz inequality

$$\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\psi(s)^m\,ds \leq \left(\int_0^t (t-s)^{2(\beta-1)}s^{2(\gamma-1)}e^{-2s}ds\right)^{\frac{1}{2}}\left(\int_0^t F(s)^2e^{2s}\psi(s)^{2m}ds\right)^{\frac{1}{2}}$$

and by our assumptions it is clear that $2(\beta-1)>-1$ and $2(\gamma-1)>-1$, so that we may apply lemma 3

$$\int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)\psi(s)^m\,ds\leq \left(C2^{1-2\gamma}t^{2(\beta-1)}\right)^{\frac{1}{2}}\left(\int_0^t F(s)^2e^{2s}\psi(s)^{2m}ds\right)^{\frac{1}{2}}. \tag{6}$$

Then (5) and (6) yield

$$\psi(t)\leq a(t)+b(t)\left(C2^{1-2\gamma}t^{2(\beta-1)}\right)^{\frac{1}{2}}\left(\int_0^t F(s)^2e^{2s}\psi(s)^{2m}ds\right)^{\frac{1}{2}},$$

this implies

$$\psi(t)^2 \le 2a(t)^2 + C2^{2-2\gamma}t^{2(\beta-1)}b(t)^2 \int_0^t F(s)^2 e^{2s}\psi(s)^{2m}\,ds\,.$$

Finally, the application of Lemma 2 with $\omega(u) = u^m$,
$\Omega(v) = \frac{1}{1-m}(v^{1-m} - v_0^{1-m})$ and $\Omega^{-1}(z) = \left[(1-m)z + v_0^{1-m}\right]^{\frac{1}{1-m}}$ yields

$$\begin{aligned}\psi(t)^2 &\le \Omega^{-1}\left[\Omega(2\max_{0\le s\le t} a(s)^2) + C2^{2-2\gamma}\max_{0\le s\le t} s^{2(\beta-1)}b(s)^2\int_0^t F(s)^2e^{2s}ds\right]\\ &\le 2\max_{0\le s\le t} a(s)^2\left[1-(m-1)\Xi_1(t)\right]^{\frac{1}{1-m}},\end{aligned}$$

where $\Xi_1(t)$ is as in the Lemma.

(2) For the second part we use Hölder inequality, i.e.

$$\psi(t) \le a(t) + b(t)\left(\int_0^t (t-s)^{(\beta-1)p}s^{(\gamma-1)p}e^{-ps}ds\right)^{\frac{1}{p}}\left(\int_0^t F(s)^q e^{qs}\psi(s)^{qm}ds\right)^{\frac{1}{q}}.$$

Since from the assumptions $(\beta-1)p = \frac{-z(z+2)}{(z+1)^2} > -1$ and $(\gamma-1)p > -1$, lemma 3 implies

$$\psi(t) \le a(t) + b(t)\left(Cp^{(1-\gamma)p-1}t^{(\beta-1)p}\right)^{\frac{1}{p}}\left(\int_0^t F(s)^q e^{qs}\psi(s)^{qm}ds\right)^{\frac{1}{q}}. \tag{7}$$

Applying the inequality

$$(a+b)^r \le 2^{r-1}(a^r + b^r),\quad a\ge 0,\quad b\ge 0,\quad r>1$$

to (7) with $r=q$ we obtain

$$\psi(t)^q \le 2^{q-1}\left\{a(t)^q + C^{\frac{q}{p}}p^{(1-\gamma)q-q}t^{(\beta-1)q}b(t)^q\int_0^t F(s)^q e^{qs}\psi(s)^{qm}ds\right\}. \qquad \square$$

We next apply lemma 2 as in part (1) to get the conclusion.

We are now ready to state our main theorems.

**Theorem 5.** *Let the operator $A$, the function $f$, the numbers $b$ and $d$ be as in the introduction and let*

$$\|f(t,u)\| \le t^\kappa\eta(t)\,\|u\|_\alpha^m\,,\quad m>1,\ \kappa\ge 0 \tag{8}$$

*for all $(t,u)\in R^+\times X^\alpha$, where $\eta:(0,\infty)\to R$ is a continuous, nonnegative function. Then the following assertions hold:*

(1) *If $0<\alpha<\min\left\{\frac{1}{2},\frac{1}{m}\left(\kappa+\frac{1}{2}\right)\right\}$ and the function*

$$t\mapsto\int_0^t \exp\left\{(2(1-m)b+2)s\right\}\eta(s)^2ds$$

*is bounded on the interval $(0,\infty)$, then any solution $u(t)$ of (1) such that $u(0) = u_0\in X^\alpha$ and*

$$(m-1)C2^{2(1-\gamma)+m-1}d^{2m}\,\|u_0\|^{2(m-1)}\int_0^t \exp\left\{(2(1-m)b+2)s\right\}\eta(s)^2ds < 1\,,$$

*where $C$ is the constant of lemma 3, exists globally in time and is such that* $\lim_{t\to\infty} \|u(t)\|_\alpha = 0$.

(2) *If* $\frac{1}{2} \le \beta < \min\left\{1, \frac{1}{m}\left(\kappa + \frac{1}{p}\right)\right\}$, $\beta = 1-\alpha = \frac{1}{1+z}$, $z \ge 1$, $q = z+2$, $\frac{1}{p}+\frac{1}{q} = 1$ *and the function*

$$t \to \int_0^t \exp\{((1-m)b+1)qs\}\,\eta(s)^q ds$$

*is bounded on the interval* $(0,\infty)$, *then any solution* $u(t)$ *of (1) such that* $u(0) = u_0 \in X^\alpha$ *and*

$$(m-1)2^{m(q-1)}d^{mq}C^{\frac{q}{p}}p^{(1-\gamma)q-\frac{q}{p}}\|u_0\|^{(m-1)q}\int_0^t \exp\{((1-m)b+1)qs\}\,\eta(s)^q ds$$

$$\begin{cases} <1 & \text{for} \quad (m-1)q \quad \text{even,} \\ \neq 1 & \text{for} \quad (m-1)q \quad \text{odd,} \end{cases}$$

*exists globally in time and* $\lim_{t\to\infty}\|u(t)\|_\alpha = 0$.

**Proof.** (1) It follows from (3), (2) and (8) that

(9) $\|u(t)\|_\alpha \le$

$$dt^{-\alpha}e^{-bt}\|u_0\| + d\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}s^\kappa\eta(s)\|u(s)\|_\alpha^m\,ds\,.$$

Multiplying both sides of (9) by $e^{bt}t^\alpha$ we obtain

$$e^{bt}t^\alpha\|u(t)\|_\alpha \le d\|u_0\| + dt^\alpha\int_0^t (t-s)^{-\alpha}e^{bs}s^\kappa\eta(s)$$

Let $\psi(t) = e^{bt}t^\alpha\|u(t)\|_\alpha$, then

$$\psi(t) \le d\|u_0\| + dt^\alpha\int_0^t (t-s)^{-\alpha}e^{b(1-m)s}s^{\kappa-m\alpha}\eta(s)\psi(s)^m\,ds\,.$$

It is clear from the assumptions that if $\beta-1=-\alpha$ and $\kappa-m\alpha=\gamma-1$ then $\beta>\frac{1}{2}$ and $\gamma>\frac{1}{2}$. Hence we may apply part (1) of lemma 4 with $a(t) = d\|u_0\|$, $b(t) = dt^\alpha$ and $F(t) = e^{b(1-m)t}\eta(t)$ we obtain $\psi(t)\le\Phi(t)$ i.e.

$$\|u(t)\|_\alpha \le t^{-\alpha}e^{-bt}\Phi(t) = t^{-\alpha}e^{-bt}A_1^{\frac{1}{2}}\left[1-(m-1)\Xi_1(t)\right]^{\frac{1}{2(1-m)}}\,,$$

where $A_1 = 2(d\|u_0\|)^2$, $B_1 = 2^{2-2\gamma}C\max_{0\le s\le t}s^{2(\beta-1)}b(s)^2 = 2^{2-2\gamma}Cd^2$ and $\Xi_1(t) = A_1^{m-1}B_1\int_0^t F(s)^2e^{2s}ds$. As $\Phi(t)$ is bounded on the interval $(0,\infty)$ the conclusion follows.

(2) This part is proved similarly using part (2) of lemma 4. □

**Remark.** Our method is also based on a generalization of Gronwall inequality, namely the nonlinear version stated in lemma 2 and the nonlinear singular version in lemma 4. Note however the important role played by lemma 3. It can be considered as a trick which gets rid of the terms $t^{2q\alpha}$ and $t^{rq\alpha}$ in the assumptions of theorem 1 in [11].

If the constants $m$ and $\kappa$ are such that $\frac{1}{m}\left(\kappa+\frac{1}{2}\right)<\frac{1}{2}$ and/or $\frac{1}{m}\left(\kappa+\frac{1}{p}\right)<1$, then the cases $\frac{1}{m}\left(\kappa+\frac{1}{2}\right)\leq\alpha<\frac{1}{2}$ and /or $\frac{1}{m}\left(\kappa+\frac{1}{p}\right)\leq\alpha<1$ are not covered by the preceding theorem. We next treat these cases. In fact the next theorem represents a stability (not exponentially, however) result for all $0<\alpha<1$. Let us first give a lemma (see [8]) which we shall need in the proof of the theorem.

**Lemma 6.** *If $0\leq\alpha<1$ and $\tau,\mu,\sigma>0$, then*

$$\int_0^t q(t-\tau)^{-\alpha}e^{-\tau(t-s)}(\sigma s+1)^{-\mu}ds\leq C(\alpha,\tau,\mu,\sigma)(\sigma t+1)^{-\mu}$$

*where $C(\alpha,\tau,\mu,\sigma)$ is a constant and $q(t)=\min\{1,t\}$.*

**Theorem 7.** *Assume that the hypothesis of theorem 5 hold and let $u(t)$ be a solution of (1) with $u(0)=u_0\in X^\alpha$. Then the following assertions hold:*

(1) *Let $0<\alpha<\frac{1}{2}$ and the function*

$$t\longmapsto(t+1)^{2(\alpha+\varepsilon\kappa)}\int_0^t s^{2(1-\varepsilon)\kappa-2\alpha m}\eta(s)^2ds \tag{10}$$

*be bounded on $(0,\infty)$ for some $0<\varepsilon<1$. If*

(11)

$$(m-1)(2d^2)^mC\left\|u_0\right\|^{2(m-1)}(t+1)^{2(\alpha+\varepsilon\kappa)}\int_0^t s^{2(1-\varepsilon)\kappa-2\alpha m}\eta(s)^2ds<1\,,$$

*where $C$ is the constant of lemma 6, then $u(t)$ exists on the interval $(0,\infty)$ and $\lim\limits_{t\to\infty}\left\|u(t)\right\|_\alpha=0$.*

(2) *Let $\frac{1}{2}\leq\alpha<1$, $\alpha=\frac{z}{z+1}$, $z\geq1$, $q=z+2$ and the function*

$$t\longmapsto(t+1)^{q(\alpha+\varepsilon\kappa)}\int_0^t s^{q(1-\varepsilon)\kappa-2\alpha m}\eta(s)^qds$$

*is bounded on $(0,\infty)$ for some $0<\varepsilon<1$. If*

$$(m-1)2^{m(q-1)}d^{mq}C^{\frac{q}{p}}\left\|u_0\right\|^{(m-1)q}(t+1)^{(\alpha+\varepsilon\kappa)q}\int_0^t s^{[(1-\varepsilon)\kappa-\alpha m]}\eta(s)^qds$$

$$\begin{cases}<1 & \text{for} \quad (m-1)q \quad \text{even},\\ \neq1 & \text{for} \quad (m-1)q \quad \text{odd},\end{cases}$$

*then $u(t)$ exists on the interval $(0,\infty)$ and $\lim\limits_{t\to\infty}\left\|u(t)\right\|_\alpha=0$.*

**Proof.** (1) It is clear that for $0<\varepsilon<1$

$$\left\|u(t)\right\|_\alpha\leq \tag{12}$$

$$dt^{-\alpha}e^{-bt}\left\|u_0\right\|+d\int_0^t(t-s)^{-\alpha}e^{-b(t-s)}(s+1)^{\varepsilon\kappa}s^{(1-\varepsilon)\kappa}\eta(s)\left\|u(s)\right\|_\alpha^m ds\,.$$

Multiplying both sides of (12) by $t^\alpha(t+1)^\delta$ for some $\delta > 0$ (to be chosen later) and denoting by $\psi(t)$ the expression $t^\alpha(t+1)^\delta \|u(t)\|_\alpha$, we obtain

$$\begin{aligned}\psi(t) &\le d(t+1)^\delta e^{-bt}\|u_0\| + \\ &\quad dt^\alpha(t+1)^\delta \int_0^t (t-s)^{-\alpha} e^{-b(t-s)}(s+1)^{\varepsilon\kappa-\delta m} s^{(1-\varepsilon)\kappa-\alpha m}\eta(s)\psi(s)^m\, ds\,,\end{aligned}$$

and by Schwarz inequality

$$\begin{aligned}\psi(t) &\le d(t+1)^\delta\|u_0\| \\ &\quad + dt^\alpha(t+1)^\delta\left(\int_0^t (t-s)^{-2\alpha}e^{-2b(t-s)}(s+1)^{2(\varepsilon\kappa-\delta m)}ds\right)^{\frac{1}{2}} \\ &\quad \left(\int_0^t s^{2[(1-\varepsilon)\kappa-\alpha m]}\eta(s)^2\psi(s)^{2m}ds\right)^{\frac{1}{2}}.\end{aligned}$$

For a fixed $\varepsilon > 0$ satisfying the hypothesis (10) and (11), if $\delta$ is chosen so that $\varepsilon\kappa - \delta m < 0$ then we may apply lemma 6 obtaining

$$\begin{aligned}\psi(t) &\le d(t+1)^\delta\|u_0\| + dt^\alpha(t+1)^\delta\left(C(t+1)^{2(\varepsilon\kappa-\delta m)}\right)^{\frac{1}{2}} \\ &\qquad \left(\int_0^t s^{2[(1-\varepsilon)\kappa-\alpha m]}\eta(s)^2\psi(s)^{2m}ds\right)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\begin{aligned}\psi(t)^2 &\le 2d^2(t+1)^{2\delta}\|u_0\|^2 + 2d^2Ct^{2\alpha}(t+1)^{2\delta}(t+1)^{2(\varepsilon\kappa-\delta m)} \\ &\qquad \left(\int_0^t s^{2[(1-\varepsilon)\kappa-\alpha m]}\eta(s)^2\psi(s)^{2m}ds\right)^{\frac{1}{2}}.\end{aligned}$$

Let us choose $\delta$ such that $\alpha+\delta+\varepsilon\kappa-\delta m = 0$ i.e $\delta = \frac{\alpha+\varepsilon\kappa}{m-1}$. Observe then that the previous condition $\varepsilon\kappa - \delta m < 0$ is satisfied. Next, applying lemma 4 we get

$$\psi(t) \le \sqrt{2}d(t+1)^\delta\|u_0\|\left\{1-(m-1)\Xi_1(t)\right\}^{\frac{1}{2(m-1)}},$$

with

$$\Xi_1(t) = (2d^2)^m C\|u_0\|^{2(m-1)}(t+1)^{2(\alpha+\varepsilon\kappa)}\int_0^t s^{2(1-\varepsilon)\kappa-2\alpha m}\eta(s)^2 ds\,.$$

We deduce the estimate $\|u(t)\|_\alpha \le \frac{\sqrt{2}d\|u_0\|}{t^\alpha}M$, where $M$ is a bound for the expression $\left\{1-(m-1)\Xi_1(t)\right\}^{\frac{1}{2(m-1)}}$.

(2) This part follows similarly using Hölder inequality as in theorem 5 and lemma 4. □

## 3. A Semilinear Functional Differential Equation

Let $A, X, X^\alpha$ be as in the introduction and $r$ a positive real number. We denote by $\mathcal{C}$ the Banach space $C([-r,0];X)$ of all continuous functions from $[-r,0]$ into $X$. If $z$ is a continuous function defined on $[-r,b]$, then for any $t\in[0,b]$, $z_t$ will denote the function in $\mathcal{C}$ defined by $z_t(\theta) = z(t+\theta)$ for $\theta\in[-r,0]$.

We consider the (nonautonomous) semilinear partial functional differential problem

$$\left\{ \begin{array}{l} \frac{du}{dt} + Au = f(t, u_t), \quad t > 0 \\ u_0 = \phi \in \mathcal{C}\,. \end{array} \right. \tag{13}$$

This problem has been investigated by many authors. Local existence, global existence, asymptotic behavior and regularity results may be found, for instance, in Travis and Webb [16,17], Fitzgibbon [3], Rankin [14], and Redlinger [15].

If $\mathcal{C}_\alpha = C([-r, 0]; X^\alpha)$ and $\|\psi\|_{\mathcal{C}_\alpha} = \sup\limits_{-r\leq\theta\leq 0} \|\psi(\theta)\|_\alpha$, then clearly $(\mathcal{C}_\alpha, \|.\|_{\mathcal{C}_\alpha})$ is a Banach space. The integral version of (13) is the following integral problem

$$\left\{ \begin{array}{l} u(t) = e^{-At}\phi(0) + \displaystyle\int_0^t e^{-A(t-s)} f(s, u_s) ds, \quad t > 0 \\ u_0 = \phi \in \mathcal{C}_\alpha \end{array} \right.$$

where $f(.,.) \in C([0,\infty) \times \mathcal{C}_\alpha; X)$. Similar results to those in section 2 hold for this problem under the same hypothesis on the nonlinearity

$$\|f(t, v)\| \leq t^\kappa \eta(t) \|v\|_\alpha^m\,, \quad \text{for} \quad t \geq 0, \;\; v \in \mathcal{C}_\alpha, \;\; m > 1, \;\; \kappa \geq 0\,.$$

Indeed, it suffices to note that for any solution $u$ in $C([-r, T]; X^\alpha)$, $T > 0$

(14)
$$\begin{aligned} \|u(t+\theta)\|_\alpha \quad &\leq \quad d(t+\theta)^{-\alpha} e^{-b(t+\theta)} \|\phi(0)\| \\ &\qquad + d\int_0^{t+\theta} (t+\theta-s)^{-\alpha} e^{-b(t+\theta-s)} \|f(s, u_s)\|\, ds\,, \end{aligned}$$

for $t + \theta \geq 0$, $-r \leq \theta \leq 0$, $t \leq T$. Multiplying both sides of (14) by $(t+\theta)^\alpha e^{b(t+\theta)}$ we obtain for $t + \theta \geq 0$

$$\begin{array}{l} (t+\theta)^\alpha e^{b(t+\theta)} \|u(t+\theta)\|_\alpha \leq d\|\phi(0)\| + \\ d(t+\theta)^\alpha \displaystyle\int_0^{t+\theta} (t+\theta-s)^{-\alpha} e^{bs} s^\kappa \eta(s) \|u_s\|_{\mathcal{C}_\alpha}^m\, ds \end{array}$$

or

$$\begin{array}{l} (t+\theta)^\alpha e^{b(t+\theta)} \|u(t+\theta)\|_\alpha \leq d\|\phi(0)\| + \\ d(t+\theta)^\alpha \displaystyle\int_0^{t+\theta} (t+\theta-s)^{-\alpha} e^{-b(m-1)s} s^{\kappa-m\alpha} s^{m\alpha} e^{bms} \eta(s) \|u_s\|_{\mathcal{C}_\alpha}^m\, ds\,. \end{array}$$

For the analogue to part (1) of theorem 5 for instance we use first Schwarz inequality

$$\begin{aligned} (t+\theta)^\alpha e^{b(t+\theta)} \|u(t+\theta)\|_\alpha \quad &\leq \quad d\|\phi(0)\| + d(t+\theta)^\alpha \\ &\qquad \left(\int_0^{t+\theta} (t+\theta-s)^{-2\alpha} e^{-2b(m-1)s} s^{2(\kappa-m\alpha)} ds\right)^{\frac{1}{2}} \\ &\qquad \left(\int_0^t s^{2m\alpha} e^{2bms} \eta^2(s) \|u_s\|_{\mathcal{C}_\alpha}^{2m}\, ds\right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 3 now implies

(15) $(t+\theta)^{\alpha} e^{b(t+\theta)} \|u(t+\theta)\|_{\alpha} \leq$

$$d\,\|\phi(0)\| + d\left(C[2b(m-1)]^{2(m\alpha-\kappa)-1}\right)^{\frac{1}{2}} \left(\int_0^t s^{2m\alpha} e^{2bms} \eta^2(s) \|u_s\|_{\mathcal{C}_\alpha}^{2m}\, ds\right)^{\frac{1}{2}}.$$

At this stage we pass to the sup over $-r \leq \theta \leq 0$ in the left-hand side of (15), this yields

$$\psi(t) \leq d\,\|\phi(0)\| + d\left(C[2b(m-1)]^{2(m\alpha-\kappa)-1}\right)^{\frac{1}{2}} \left(\int_0^t \eta^2(s)\psi(s)^{2m}\, ds\right)^{\frac{1}{2}}.$$

The rest is exactly as in the proof of theorem 5.

## 4. A Semilinear Integrodifferential Equation

In this section we shall consider the semilinear integrodifferential problem

$$\left\{\begin{array}{l} \frac{du}{dt} = \int_0^t a(t-s)Au(s)ds + f(t,u), \quad t>0 \\ u(0) = u_0 \in X\,, \end{array}\right. \tag{16}$$

where $X$ is a Banach space with norm $\|.\|$. The autonomous case $f(t,u) = f(u)$ was studied by Hattori and Lightbourne in [6]. A global result was established for small and smooth initial data and a nonlinearity satisfying

$$\|f(u)\| \leq C\,\|u\|_{\alpha}^{m}\,, \quad m>1\,.$$

The problem

$$\left\{\begin{array}{l} \frac{du}{dt} = \displaystyle\int_0^t a(t-s)Au(s)ds, \quad t>0 \\ u(0) = u_0 \end{array}\right. \tag{17}$$

has been investigated by DaPrato and Iannelli [2] (see also references in [2]). In the case $a(t) = \frac{1}{\Gamma(\alpha)} t^{-\eta}$, $\eta \in (0,1)$ and $A = -\triangle$, where $\triangle$ is the Laplacian on $\Omega \subset R$, problem (17) corresponds to the fractional evolution problem

$$\left\{\begin{array}{l} D^{\beta} u(x,t) = \triangle u(x,t), \quad x \in \Omega\,, \quad t>0, \\ u(x,0) = u_0(x), \quad x \in \Omega\,, \end{array}\right. \tag{18}$$

where $\beta = 2-\eta$, $1<\beta<2$. $D^{\beta}$ is the inverse of the Riemann-Liouville integral of order $\beta$

$$I^{\beta} g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)ds\,.$$

Problem (18) is an interpolation of the heat equation and the wave equation.

For the unbounded domain $\Omega = R$, Fujita [5] proved an existence and uniqueness result and gave an explicit representation of the solution by means of a probability density. The authors in [9] proved a stability and blow up result for the same specific linear problem with $\Omega = R$ and a forcing term $f(x,t)$.

We claim that the method developped in section 2 apply to problem (16). To see this it is enough to recall a result from [6]:

Let $A : D(A) \subset X \to X$ be a closed linear operator, densely defined on $X$ such that (a) the resolvent set of $A$ satisfies $\rho(A) \supset \{\lambda \in C : |\arg\lambda| < \phi\} \bigcup V$ where $\frac{\pi}{2} < \phi < \pi$ and $V$ is a neighborhood of zero, (b) there exists $M > 0$ such that, for $\lambda \in \rho(A)$, the resolvent of $A$, $R(\lambda; A) = (\lambda I - A)^{-1}$, satisfies $\|R(\lambda; A)\| \leq M/(1 + |\lambda|)$ and the kernel $a(t)$ is such that (c) there exists $\tilde{\phi} \in (\frac{\pi}{2}, \pi)$ for which $\hat{a}(\lambda)$, the Laplace transform of $a$, is analytic and bounded in $\sum(\tilde{\phi})$, $\hat{a}(\lambda) \neq 0$ for $\lambda \in \sum(\tilde{\phi})$ and $\lambda(\hat{a}(\lambda))^{-1} \in \rho(A)$ for $\lambda \in \sum(\tilde{\phi})$, where $\sum(\tilde{\phi}) = \{\lambda \in C : |\arg\lambda| < \tilde{\phi}\}$. Then, if $|\hat{a}(\lambda)| \leq L|\lambda|^r$, for $\lambda \in \sum(\tilde{\phi})$, there exist positive constants $M$ and $\delta$ such that

$$\|(-A)^\alpha T(t)\| \leq M t^{-\alpha(1+r)} e^{-\delta t}, \quad t > 0 \tag{19}$$

with

$$T(t) = \int_{\gamma(\eta,\varepsilon)} e^{\lambda t}(\lambda - \hat{a}(\lambda)A)^{-1} d\lambda$$

where $\eta \in (\frac{\pi}{2}, \tilde{\phi})$, $\varepsilon > 0$ and $\gamma(\eta, \varepsilon) = \{\lambda = \rho e^{\pm i\eta}, \rho \geq \varepsilon\} \bigcup \{\lambda = \varepsilon e^{i\tau} : \tau \in (-\eta, \eta)\}$.

The problem is then approached, using the estimate (19), via the variation of parameters equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s, u(s)) ds\,.$$

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