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ON THE LIMIT CYCLE OF THE LIÉNARD EQUATION

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ABSTRACT. In the paper, we give an existence theorem of periodic solution for Liénard equation $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$. As a result, we estimate the amplitude $\rho(\mu)$ (maximal x-value) of the limit cycle of the van der Pol equation $\dot{x} = y - \mu(x^3/3 - x)$, $\dot{y} = -x$ from above by $\rho(\mu) < 2.3439$ for every $\mu \neq 0$. The result is an improvement of the author's previous estimation $\rho(\mu) < 2.5425$.

1. Introduction

We are interested in the limit cycle of the Liénard equation:

$$\text{(L)} \qquad \dot{x} = y - F(x), \quad \dot{y} = -g(x)\,.$$

The following is our result.

Theorem. *Suppose that the equation* (L) *satisfies the following: (1) F, g are of class C^1 and odd. (2) $g(x) > 0$ for $x \in (0, \infty)$. (3) F has a zero $\beta > 0$ such that $F(x) < 0$ for $x \in (0, \beta)$ and $F(x) > 0$ for $x \in (\beta, \infty)$. (4) There are two C^1 mappings $\phi, \psi : [0, \beta] \to [\beta, \infty)$ such that*

(i) $\phi'(x)g(\phi(x))F(\phi(x)) = g(x)F(x)$, (ii) $\phi(\beta) = \beta$,

(iii) $\psi'(x)g(\psi(x))F(\psi(x)) \geq -g(x)F(x)$, (iv) $\psi'(x)f(\psi(x)) \geq f(x)$,

(v) $\psi'(x)g(\psi(x)) \leq g(x)$, *where* $f = F'$.

Then it has a periodic orbit in the strip $|x| < \psi(\beta)$.

The proof of the above theorem enables us to estimate the amplitude $\rho(\mu)$ (maximal x-value) of the limit cycle of the van der Pol equation:

$$\text{(vdP)} \qquad \dot{x} = y - \mu(x^3/3 - x), \quad \dot{y} = -x\,.$$

The equation (vdP) has a unique limit cycle for every $\mu > 0$. See [7], for example. Since the replacement $(t, x, y, \mu) \to (-t, x, -y, -\mu)$ preserves the form of (vdP), it is sufficient to consider the case $\mu > 0$. The following is an application of our theorem.

Example. *The amplitude $\rho(\mu)$ of the limit cycle of the equation* (vdP) *is estimated by $\rho(\mu) < 2.3233$ for every $\mu > 0$.*

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The upper bound 2.3233 is better than previous results, namely, 2.8025 of [1], 2.5425 of [7] and 2.3439 of [8]. The values indicate that the result is better than those of [1], [7] and [8].

We shall prove the example by constructing the mappings ϕ, ψ satisfying the condition (4) of our theorem. The construction needs the following conditions in addition to (1), (2), (3) of our theorem:

(a) There is a constant $k < G(\infty) - G(\beta)$ satisfying $F(G^{-1}(G(x)+k)) + F(x) \geq 0$ for $x \in [0, \beta]$, where $G(x) := \int_0^x g(u)du$, and $x = G^{-1}(z)$ is the inverse function of $z = G(x)$ for $x \geq 0$.

(b) $P(x) := f(x)/g(x)$ is strictly increasing on $(0, \infty)$.

(c) $Q(x) := P(x)/F(x)$ is strictly decreasing on $(0, \beta)$ and on (β, ∞).

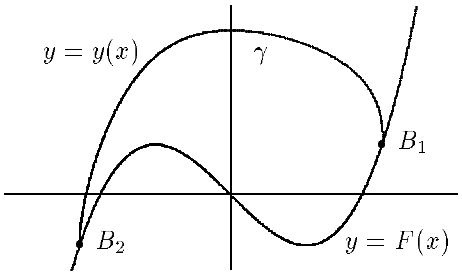
Of course, the equation (vdP) satisfies these conditions. In Section 3, we show the procedure of constructing the mappings. (The procedure can not be repesented as a 'theorem' because its statement can not be short.)

2. Proof of Our Theorem

Since the equation (L) is symmetric with respect to the origin, we consider it only on the region $y \geq F(x)$. We consider an orbit γ which starts from a point B_2 on the left half of $y = F(x)$ and reaches to a point B_1 on the right half of it. We denote the x-coordinates of the points B_1 and B_2 by $b_1 > 0$ and $-b_2 < 0$ respectively. Then we can regard the y-coordinate of γ as a function of $x \in (-b_2, b_1)$. we denote it by $y = y(x)$. By the definition, the function $y(x)$ is strictly increasing on $(-b_2, 0)$ and strictly decreasing on $(0, b_1)$. In the proof of our theorem, we use the following notation:

(2.1) $$v_1(x) = y(x) - F(x) \geq 0, \quad v_2(x) = y(-x) - F(-x) \geq 0\,.$$

To see the above notations visually, for the equation (vdP) with $\mu = 1$, we give below the graphs of $y = F(x)$ and $y = y(x)$.



Lemma. *Suppose that the equation* (L) *satisfies the condtion* (1), (2) *of the theorem. If the orbit* γ *satisfies* $-\int_{-b}^{b} \frac{g(x)F(x)}{y(x) - F(x)}dx < 0$ (> 0) *for* $b = \min\{b_1, b_2\}$, *then* $b_1 < b_2$ $(b_1 > b_2)$.

Proof. We can easily confirm the following:

$$\frac{d}{dx}\Big(\frac{1}{2}y(x)^2+G(x)\Big)=-\frac{g(x)F(x)}{y(x)-F(x)}\,. \tag{2.2}$$

If $b_1 \geq b_2$, then $y(x)$ is defined on $[-b_2, b_2]$. So we calculate as follows:

$$\begin{aligned} &-\int_{-b_2}^{b_2}\frac{g(x)F(x)}{y(x)-F(x)}dx \\ &\qquad=\frac{1}{2}\Big(y(b_2)^2-y(-b_2)^2\Big)=\frac{1}{2}\Big(y(b_2)^2-F(b_2)^2\Big)\geq 0\,. \end{aligned} \tag{2.3}$$

We can prove the other case in a similar way. □

Proof of Theorem. First, we prove that $\psi(\beta)\geq\phi(0)$. Since $\phi'(x)<0$ for $x\in(0,\beta)$, the mapping ϕ reverses the order, that is, $0<x_1<x_2<\beta$ implies $\phi(x_1)>\phi(x_2)$. By integrating (iii) and (i) on $[0,\beta]$, we obtain that

$$\int_{\psi(0)}^{\psi(\beta)}g(u)F(u)du\geq-\int_0^{\beta}g(u)F(u)du=\int_{\phi(\beta)}^{\phi(0)}g(u)F(u)du\,. \tag{2.4}$$

By (ii) and it, we obtain that

$$\int_{\phi(0)}^{\psi(\beta)}g(u)F(u)du\geq\int_{\beta}^{\psi(0)}g(u)F(u)du\geq 0\,. \tag{2.5}$$

Since we have $g(x)F(x)>0$ for $x\in(\beta,\infty)$, we obtain that $\psi(\beta)\geq\phi(0)$.

We shall show that the orbit γ with $b_2=\psi(\beta)$ winds toward inside. To prove it by a contradiction, we assume that $b_1\geq b_2$. For $x\in(0,\beta)$, since we have $\phi'(x)<0$, we calculate as follows:

$$\frac{d}{dx}\Big(y(x)-y(\phi(x))\Big)=-\frac{g(x)}{v_1(x)}+\frac{\phi'(x)g(\phi(x))}{v_1(\phi(x))}<0\,. \tag{2.6}$$

By integrating (2.6) on $[x,\beta]$, we obtain that

$$\begin{aligned} v_1(x)-v_1(\phi(x))&=y(x)-y(\phi(x))+F(\phi(x))-F(x)\\ &\geq y(\beta)-y(\phi(\beta))\geq 0\,. \end{aligned} \tag{2.7}$$

For $x\in(0,\beta)$, by using (iv) and (v), we calculate as follows:

$$\begin{aligned} &\frac{d}{dx}\Big(v_2(x)-v_2(\psi(x))\Big)\\ &=-\frac{g(x)}{v_2(x)}+\frac{\psi'(x)g(\psi(x))}{v_2(\psi(x))}+f(x)-\psi'(x)f(\psi(x))\\ &\leq\frac{g(x)}{v_2(x)v_2(\psi(x))}\Big(v_2(x)-v_2(\psi(x))\Big)\,. \end{aligned} \tag{2.8}$$

By integrating it on $[x,\beta]$, we obtain that

$$v_2(x)-v_2(\psi(x))\geq v_2(\beta)\cdot\exp\Big(-\int_x^{\beta}\frac{g(u)}{v_2(u)v_2(\psi(u))}du\Big)>0\,. \tag{2.9}$$

To apply the lemma, we consider the following integral:

$$-\int_{-\psi(\beta)}^{\psi(\beta)} \frac{g(x)F(x)}{y(x)-F(x)}dx = -\int_0^{\psi(\beta)} \frac{g(x)F(x)}{v_1(x)}dx - \int_0^{\psi(\beta)} \frac{g(x)F(x)}{v_2(x)}dx\,. \tag{2.10}$$

By using (i) and (2.7), we calculate the first term of the right-hand side of (2.10) as follows:

$$\begin{aligned} &\le -\int_0^{\beta} \frac{g(x)F(x)}{v_1(x)}dx - \int_{\phi(\beta)}^{\phi(0)} \frac{g(x)F(x)}{v_1(x)}dx \\ &= -\int_0^{\beta} \frac{g(x)F(x)}{v_1(x)}dx + \int_0^{\beta} \frac{\phi'(x)g(\phi(x))F(\phi(x))}{v_1(\phi(x))}dx < 0\,. \end{aligned} \tag{2.11}$$

By using (iii) and (2.9), we calculate the second term of the right-hand side of (2.10) as follows:

$$\begin{aligned} &\le -\int_0^{\beta} \frac{g(x)F(x)}{v_2(x)}dx - \int_{\psi(0)}^{\psi(\beta)} \frac{g(x)F(x)}{v_2(x)}dx \\ &= -\int_0^{\beta} \frac{g(x)F(x)}{v_2(x)}dx - \int_0^{\beta} \frac{\psi'(x)g(\psi(x))F(\psi(x))}{v_2(\psi(x))}dx < 0\,. \end{aligned} \tag{2.12}$$

By the lemma, we obtain that $b_1 < b_2$. It is a contradiction. So we conclude that $b_1 < b_2$. Thus the orbit γ winds toward inside. On the other hand, since we have $g(x)F(x) < 0$ near the origin, we obtain from the lemma that every orbit near the origin winds toward outside. Hence the equation has a periodic orbit in the strip $|x| < \psi(\beta)$. □

3. Construction of the Mappings

When the equation (L) satisfies the conditions (1), (2), (3) of our theorem and (a), (b), (c) of Section 1, we can construct the mappings ϕ, ψ satisfying the condition (4) of our theorem.

Procedure of Construction. We define $\phi(x)$ as the implicit function of

$$\int_x^{\phi} g(u)F(u)du = 0\,. \tag{3.13}$$

Of course, $\phi(\beta) = \beta$. By differentiating (3.13), we obtain that

$$\phi'(x)g(\phi(x))F(\phi(x)) - g(x)F(x) = 0\,. \tag{3.14}$$

Hence the mapping ϕ satisfies (i) and (ii) of our theorem.

We take the smallest constant $k < G(\infty) - G(\beta)$ satisfying

$$F(G^{-1}(G(x)+k)) + F(x) \ge 0 \tag{3.15}$$

for $x \in [0, \beta]$. Then at least one value of $x \in [0, \beta)$ satisfies the equality of (3.15). We denote by $\theta \in [0, \beta)$ the maximal x satisfying the equality of (3.15). We shall construct the mapping ψ separately on three intervals $[0,\theta]$, $(\theta,\eta]$ and $(\eta,\sqrt{3}]$, where η will be determined later.

For $x\in[0,\theta]$, we define $\psi(x)$ by

$$\psi(x)=G^{-1}(G(x)+k)\,. \tag{3.16}$$

By differentiating it, we obtain that

$$\psi'(x)g(\psi(x))-g(x)=0\,. \tag{3.17}$$

By combining (3.13) and it, we obtain that

$$\psi'(x)g(\psi(x))F(\psi(x))+g(x)F(x)\geq 0\,. \tag{3.18}$$

By substituting $\psi'(x)$ from (3.17), we obtain that

$$\psi'(x)f(\psi(x))-f(x)=g(x)\Big(P(\psi(x))-P(x)\Big)\geq 0\,. \tag{3.19}$$

For $x\in(\theta,\eta]$, we define $\psi(x)$ by the implicit function of

$$\int_{\psi(\theta)}^{\psi}g(u)F(u)du+\int_{\theta}^{x}g(u)F(u)du=0\,. \tag{3.20}$$

Since the function $R(x):=Q(\psi(x))+Q(x)$ is strictly decreasing on (θ,β), and since

$$R(\theta)=-\frac{1}{F(\theta)}\Big(P(\psi(\theta))-P(\theta)\Big)>0\,,\quad R(\beta-0)=-\infty\,, \tag{3.21}$$

it has the unique zero η in (θ,β). By differentiating (3.20), we obtain that

$$\psi'(x)g(\psi(x))F(\psi(x))+g(x)F(x)=0\,. \tag{3.22}$$

By substituting $\psi'(x)$ from (3.22), we obtain that

$$\psi'(x)f(\psi(x))-f(x)=-g(x)F(x)\Big(Q(\psi(x))+Q(x)\Big)\geq 0\,. \tag{3.23}$$

By integrating it on $[\theta,x]$, we obtain that

$$F(\psi(x))+F(x)\geq F(\psi(\theta))+F(\theta)=0\,. \tag{3.24}$$

By substituting $\psi'(x)$ from (3.22), we obtain that

$$g(x)-\psi'(x)g(\psi(x))=\frac{g(x)}{F(\psi(x))}\Big(F(\psi(x))+F(x)\Big)\geq 0\,. \tag{3.25}$$

For $x\in(\eta,\beta]$, we define $\psi(x)$ by the implicit function of

$$\int_{\psi(\eta)}^{\psi}f(u)du-\int_{\eta}^{x}f(u)du=0\,. \tag{3.26}$$

By differentiating it, we obtain that

$$\psi'(x)f(\psi(x))-f(x)=0\,. \tag{3.27}$$

By substituting $\psi'(x)$ from (3.27), we obtain that

$$\begin{aligned}&\psi'(x)g(\psi(x))F(\psi(x))+g(x)F(x)\\&=\frac{g(x)F(x)}{Q(\psi(x))}\Big(Q(\psi(x))+Q(x)\Big)\geq 0\,.\end{aligned} \tag{3.28}$$

By substituting $\psi'(x)$ from (3.27), we obtain that

$$g(x) - \psi'(x)g(\psi(x)) = \frac{g(x)}{P(\psi(x))}\Big(P(\psi(x)) - P(x)\Big) \geq 0\,. \tag{3.29}$$

Hence the mapping ψ satisfies (iii), (iv), (v) of our theorem. □

The equation (vdP) satisfies the conditions (1), (2), (3) of our theorem and (a), (b), (c) of Section 1. So we can immediately apply the above construction to it.

Proof of Example. For the equation (vdP), we can calculate as follows:

$$F(G^{-1}(G(x)+\sqrt{3}))^2 - F(x)^2 = \frac{2}{3}\sqrt{3}\mu^2\Big(x^2 - (2-\sqrt{3})\Big) \geq 0\,. \tag{3.30}$$

So we obtain that $k = \sqrt{3}$, $\theta = \sqrt{2-\sqrt{3}}$ and $\psi(\theta) = \sqrt{2+\sqrt{3}}$. By the definition of η, we obtain that

$$R(\eta) = \frac{\psi(\eta)^2 - 1}{\psi(\eta)^4 - 3\psi(\eta)^2} + \frac{\eta^2 - 1}{\eta^4 - 3\eta^2} = 0\,. \tag{3.31}$$

By putting $x = \eta$ to (3.20), we obtain that

$$\psi(\eta)^5 - 5\psi(\eta)^3 + \eta^5 - 5\eta^3 - 6\sqrt{3} + 4\sqrt{6} = 0. \tag{3.32}$$

By solving the simultaneous equations (3.31), (3.32) by computer, we obtain that $\eta \approx 1.3784$ and $\psi(\eta) \approx 2.2006$. By putting $x = \sqrt{3}$ to (3.26), we obtain that

$$\psi(\sqrt{3})^3 - 3\psi(\sqrt{3}) - \psi(\eta)^3 + 3\psi(\eta) + \eta^3 - 3\eta = 0\,. \tag{3.33}$$

By solving (3.33) by computer, we obtain that $\psi(\sqrt{3}) \approx 2.3233$. Hence the equation has a periodic orbit in the strip $|x| < 2.3233$. □

4. A Conjecture

Since the limit cycle of the van der Pol equation is unique, its amplitude $\rho(\mu)$ is a continuous function of the parameter $\mu \neq 0$. We already know that $\rho(\mu) \to 2$ as $\mu \to 0$ and that $\rho(\mu) \to 2$ as $\mu \to \infty$. See [5], for example. More precisely, we found in [4] that $\rho(\mu) = 2 + (7/96)\mu^2 + O(\mu^3)$ for sufficiently small $\mu > 0$, and in [2] that $\rho(\mu) = 2 + (0.7793\cdots)\mu^{-4/3} + o(\mu^{-4/3})$ for sufficiently large $\mu > 0$.

By a computer experiment, we have the following table:

μ	$\downarrow 0$	0.1	1.0	2.0	3.0	3.2
ρ	$\downarrow 2$	2.00010	2.00862	2.01989	2.02330	2.02341
μ	3.3	3.4	4.0	5.0	10	$\uparrow \infty$
ρ	2.02342	2.02341	2.02296	2.02151	2.01429	$\downarrow 2$

We calculate the amplitude ρ of the above table by using the Runge-Kutta method with a step size 2^{-20}. By the above table, we realize that the example is not sharp. So we want to pose the following:

Conjecture. The amplitude $\rho(\mu)$ of the limit cycle of the van der Pol equation is estimated by $2 < \rho(\mu) < 2.0235$ for every $\mu > 0$.

However, to estimate the amplitude is a very difficult problem. An attempt to estimate the amplitude is done by Giacomini and Neukirch [3].

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