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FIXED POINT THEOREMS FOR WEAKLY SEQUENTIALLY CLOSED MAPS

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Abstract. A number of fixed point theorems are presented for weakly contractive maps which have weakly sequentially closed graph. Our results automatically lead to new existence theorems for differential inclusions in Banach spaces relative to the weak topology.

1. Introduction

This paper presents some new fixed point theorems for multivalued maps which are weakly contractive and have weakly sequentially closed graph. Our theory will then be used to establish some new existence theorems for differential inclusions in Banach spaces relative to the weak topology. Motivated by a paper for Cichoń [5] we will discuss in detail the inclusion

$$y(t) \in x_0 + \int_0^t G(s, y(s))\, ds \quad \text{on} \quad [0,T]\,; \tag{1.1}$$

here $G : [0,T] \times E \to 2^E$ (here 2^E denotes the family of nonempty subsets of E), $x_0 \in E$ with E a real Banach space. In 1971, Szep [17] discussed the abstract Cauchy problem

$$\begin{cases} y' = f(t,y) \quad \text{on} \quad [0,T] \\ y(0) = x_0 \end{cases} \tag{1.2}$$

with $f : [0,T] \times E \to E$ weakly–weakly continuous and E a reflexive Banach space. Recently [5, 10–13, 16] a more general theory has been presented for (1.2). However the differential inclusion analogue (namely (1.1)) has received very little attention; we refer the reader to Chow and Schuur [3, 4] and O'Regan [14]. In this paper we will discuss in detail differential inclusions in a Banach space relative to the weak topology. Our results improve those in [3, 4, 14].

For the remainder of this section we gather together some notation and preliminary facts. Let Ω_E be the bounded subsets of a Banach space E and let K^w be

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the family of all weakly compact subsets of E. Also let B be the closed unit ball of E. The DeBlasi measure of weak noncompactness is the map $w : \Omega_E \to [0, \infty)$ defined by

$$w(X) = \inf\{t > 0 : \text{ there exists } Y \in K^w \text{ with } X \subseteq Y + t\,B\} \ ; \text{ here } X \in \Omega_E \, .$$

Suppose $F : Z \subseteq E \to 2^E$ maps bounded sets into bounded sets. We call F a α w–contractive map if $0 \leq \alpha < 1$ and $w(F(X)) \leq \alpha\, w(X)$ for all bounded sets $X \subseteq Z$. We next state a theorem of Ambrosetti type (see [10, 16] for a proof).

Theorem 1.1. *Let $H \subseteq C([0,T], E)$ be bounded and equicontinuous. Then*

$$w(H) = \sup_{t \in [0,T]} w(H(t)) = W(H[0,T])$$

where $H(t) = \{\phi(t) : \ \phi \in H\}$ and $H[0,T] = \cup_{t \in [0,T]} \{\phi(t) : \ \phi \in H\}$.

2. Fixed point theory and applications

We begin this section by establishing some fixed point results for multivalued maps. Our results rely on the following fixed point result of Himmelberg [8].

Theorem 2.1. *Let Q be a nonempty, closed, convex subset of a locally convex Hausdorff linear topological space B. Assume $F : Q \to C(Q)$ is upper semicontinuous and $F(Q)$ is relatively compact in B; here $C(Q)$ denotes the family of nonempty, closed, convex subsets of Q. Then F has a fixed point.*

Our first result extends a fixed point theorem of Arino, Gautier and Penot [2].

Theorem 2.2. *Let E be a metrizable locally convex linear topological space and let C be a weakly compact, convex subset of E. Suppose $F : C \to C(C)$ has weakly sequentially closed graph. Then F has a fixed point.*

Proof. Now the graph of F is weakly compact (see [6 pp. 549] or [9 pp.313]). Also [1 pp. 465] implies F is weakly upper semicontinuous. Theorem 2.1 (applied with $B = (E, w)$, the space E endowed with the weak topology) now guarantees that F has a fixed point. □

Remark 2.1. From the proof we see, since the graph of F being weakly compact implies $F(x)$ is weakly closed for each $x \in C$, that it is enough to consider maps $F : C \to K(C)$ where $K(C)$ denotes the family of nonempty, convex subsets of C.

Our next result extends Theorem 2.2 when our space E is a Banach space. This result will be particularly useful when establishing existence results for (1.1).

Theorem 2.3. *Let Q be a nonempty, bounded, convex, closed subset of a Banach space E. Assume $F : Q \to C(Q)$ has weakly sequentially closed graph and also suppose F is α w–contractive (here $0 \leq \alpha < 1$). Then F has a fixed point.*

Proof. Let

$$S_1 = Q \quad \text{and} \quad S_{n+1} = \overline{co}\,(F(S_n)),\ n = 1, 2, \dots \, .$$

It is easy to see that

$$S_{n+1} \subseteq S_n \quad \text{and} \quad w(S_{n+1}) \leq \alpha^n\, w(S_1) \quad \text{for} \quad n = 1, 2, \dots .$$

Since $w(S_n) \to 0$ as $n \to \infty$ we have that $\cap_1^\infty S_n = S_\infty$ is nonempty. In addition S_∞ is weakly closed and convex since each S_n is; in fact S_∞ is weakly compact since $w(S_\infty) = 0$. Also since

$$F(S_n) \subseteq F(S_{n-1}) \subseteq \overline{co}\,(F(S_{n-1})) = S_n \quad \text{for all} \quad n$$

we have $F : S_\infty \to C(S_\infty)$. Theorem 2.2 implies that F has a fixed point in $S_\infty \subseteq Q$. □

We now use Theorem 2.3 to obtain a nonlinear alternative of Leray–Schauder type.

Theorem 2.4. *Let Q and C be closed, bounded, convex subsets of a Banach space E with $Q \subseteq C$. In addition let U be a weakly open subset of Q with $0 \in U$. Assume $\overline{U^w}$ is a weakly compact subset of Q and $F : \overline{U^w} \to CK(C)$ has weakly sequentially closed graph; here $CK(C)$ denotes the family of nonempty, convex , weakly compact subsets of C. Finally suppose $F : \overline{U^w} \to CK(C)$ is a weakly compact map. Then either*
(A1) F has a fixed point;
or
(A2) there is a point $u \in \partial_Q U$ (the weak boundary of U in Q) and $\lambda \in (0,1)$ with $u \in \lambda\, F u$.

Proof. Suppose $(A2)$ does not hold and F does not have a fixed point on $\partial_Q U$. Let

$$H = \left\{ x \in \overline{U^w} :\ x \in \lambda\, F(x) \quad \text{for some} \quad \lambda \in [0,1] \right\} .$$

Now $0 \in H$. Also the graph of F is weakly compact (see [9 pp. 313]; note $\overline{U^w}$ is weakly compact and $F : \overline{U^w} \to CK(C)$ is a weakly compact map). Thus $F : \overline{U^w} \to CK(C)$ is a weakly closed map (i.e. has weakly closed graph). It is now easy to check that H is closed in (E, w) [Let (x_α) be a net in H, so $x_\alpha \in \lambda_\alpha\, F(x_\alpha)$ for some $\lambda_\alpha \in [0,1]$, with $x_\alpha \to x_0 \in \overline{U^w}$ in (E, w) (i.e. x_α converges weakly to x_0). Without loss of generality assume $\lambda_\alpha \to \lambda_0 \in [0,1]$. Let $N(x, \lambda) = \lambda\, F(x)$ and it is easy to check that $N : \overline{U^w} \times [0,1] \to CK(C)$ is a weakly closed map. Consequently $x_0 \in \lambda_0\, F(x_0)$]. In fact H is compact in (E, w) since $H \subseteq co\,(F(H) \cup \{0\})$.

Now $(A2)$ does not hold and F does not have a fixed point on $\partial_Q U$, so $H \cap \partial_Q\, U = \emptyset$. Also (E, w) is Tychonoff so exists a continuous (continuous in (E, w)) $\mu : \overline{U^w} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial_Q\, U) = 0$. Let

$$J(x) = \begin{cases} \mu(x)\, F(x), & x \in \overline{U^w} \\ \{0\}, & x \in C \setminus \overline{U^w}. \end{cases}$$

It is easy to see that $J : C \to CK(C)$ is a weakly closed map. Also $J : C \to CK(C)$ is a weakly compact map since $J(X) \subseteq co\,(F(X \cap \overline{U^w}) \cup \{0\})$ for any subset X of C. Now Theorem 2.3 implies that there exists $x \in C$ with $x \in J(x)$.

Now $x \in U$ since $0 \in U$. Consequently $x \in \lambda\, F(x)$ with $0 \le \lambda = \mu(x) \le 1$. Thus $x \in H$, which implies $\mu(x) = 1$ and so $x \in F(x)$. $\square$

Notice we only need (in Theorem 2.4) $\overline{U^w}$ weakly compact and $F : \overline{U^w} \to CK(E)$ weakly compact to deduce that $F : \overline{U^w} \to CK(E)$ is a weakly closed map. We now state a more general version of Theorem 2.4 (the proof only involves minor adjustments in the previous argument).

Theorem 2.5. *Let Q and C be closed, bounded, convex subsets of a Banach space E with $Q \subseteq C$. In addition let U be a weakly open subset of Q with $0 \in U$ and let $F : \overline{U^w} \to CK(C)$ have weakly closed graph. Also suppose $F : \overline{U^w} \to CK(C)$ is α w–contractive with $0 \le \alpha < 1$. Then either*
(A1) F has a fixed point;
or
(A2) there is a point $u \in \partial_Q U$ and $\lambda \in (0,1)$ with $u \in \lambda\, F u$.

Remark 2.2. For example if $\overline{U^w}$ is weakly compact and $F : \overline{U^w} \to CK(C)$ is weakly sequentially upper semicontinuous (here we do not assume F is a weakly compact map) then $F : \overline{U^w} \to CK(C)$ is weakly upper semicontinuous (see [14]) and consequently $F : \overline{U^w} \to CK(C)$ has weakly closed graph.

Essentially the same reasoning as in Theorem 2.4 of [14] establishes the following result.

Theorem 2.6. *Let $E = (E, \|\,.\,\|)$ be a separable and reflexive Banach space, C and Q are closed, bounded, convex subsets of E with $Q \subseteq C$ and $0 \in Q$. Also assume $F : Q \to CK(C)$ has weakly sequentially closed graph. In addition suppose*

$$(2.1)\qquad \begin{cases} \textit{for any } \ \Omega_\epsilon = \{x \in E : d(x,Q) \le \epsilon\},\ \epsilon > 0, \ \ \textit{if } \ \{(x_j, \lambda_j)\}_{j=1}^{\infty} \\ \textit{is a sequence in } Q \times [0,1] \ \ \textit{with } \ x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q \ \textit{and} \\ \lambda_j \to \lambda \ \ \textit{and if } \ x \in \lambda\, F(x) \ \ \textit{and } \ 0 \le \lambda < 1, \ \textit{then} \\ \{\lambda_j\, F(x_j)\} \subseteq Q \ \ \textit{for } \ j \ \ \textit{sufficiently large; here } \ \partial_{\Omega_\epsilon} Q \\ \textit{is the weak boundary of } \ Q \ \ \textit{relative to } \Omega_\epsilon,\ d(x,y) = |x - y| \\ \textit{and } \ \rightharpoonup \ \ \textit{denotes weak convergence} \end{cases}$$

holds. Then F has a fixed point in Q.

Remark 2.3. Note Q and C are weakly compact subsets of E. Consequently $F : Q \to CK(C)$ is a weakly compact map. Also $F : Q \to CK(C)$ has weakly closed graph.

Remark 2.4. A special case of (2.1) is the following:

$$(2.2)\qquad \begin{cases} \text{if } \ \{(x_j, \lambda_j)\}_{j=1}^{\infty} \ \text{ is a sequence in } \ Q \times [0,1] \ \text{ with } \ x_j \rightharpoonup x \\ \text{and } \ \lambda_j \to \lambda \ \text{ and if } \ x \in \lambda\, F(x) \ \text{ with } \ 0 \le \lambda < 1, \\ \text{then } \ \{\lambda_j\, F(x_j)\} \subseteq Q \text{for } \ j \ \text{ sufficiently large.} \end{cases}$$

We now use Theorem 2.3 to establish some general existence principles for the abstract operator inclusion

$$(2.3)\qquad y(t) \in F\, y(t) \quad \text{on} \quad [0,T]\,.$$

Solutions to (2.3) will be sought in $C([0,T],E)$.

Theorem 2.7. *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,T],E)$. Suppose $F : Q \to C(Q)$ has wk–sequentially closed graph (i.e. for any sequence $(x_n, y_n)_1^\infty$ in $Q \times Q$, $y_n \in F(x_n)$ for $n \in \{1,2,...\}$ with $x_n(t) \to x(t)$ in (E,w) for each $t \in [0,T]$ and $y_n(t) \to y(t)$ in (E,w) for each $t \in [0,T]$, then $y \in F(x)$). Also assume there exists α, $0 \le \alpha < 1$ with $w(F(X)) \le \alpha\, w(X)$ for all subsets X of Q. Then (2.3) has a solution in Q.*

Proof. Let (x_n) be a sequence in Q. Recall [10, 16], since Q is equicontinuous, that $x_n \rightharpoonup x$ iff $x_n(t) \to x(t)$ in (E,w) for each $t \in [0,T]$; here $x_n \rightharpoonup x$ means x_n converges weakly to $x \in C([0,T],E)$. Consequently $F : Q \to C(Q)$ has wk–sequentially closed graph is equivalent to saying $F : Q \to C(Q)$ has weakly sequentially closed graph. To see this suppose $F : Q \to C(Q)$ has wk–sequentially closed graph. Suppose $(x_n, y_n)_1^\infty$ is a sequence in $Q \times Q$, $y_n \in F(x_n)$ for $n \in \{1,2,...\}$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$; here $x, y \in C([0,T],E)$. Then $x_n(t) \to x(t)$ in (E,w) for each $t \in [0,T]$ and $y_n(t) \to y(t)$ in (E,w) for each $t \in [0,T]$. Now since $F : Q \to C(Q)$ has wk–sequentially closed graph then $y \in F(x)$. The other way is similar. Our result now follows from Theorem 2.3. □

Theorem 2.8. *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,T],E)$. Suppose $F : Q \to C(Q)$ has wk–sequentially closed graph and assume*

$$F\,Q\,(t) \text{ is weakly relatively compact in } E \text{ for each } t \in [0,T] \tag{2.4}$$

holds. Then (2.3) has a solution.

Proof. Now $w(F\,Q\,(t)) = 0$ for each $t \in [0,T]$ and so $w(F\,Q) = 0$ by Theorem 1.1 i.e. $F\,Q$ is weakly relatively compact in $C([0,T],E)$. The result now follows from Theorem 2.7. □

We now discuss a special case of (2.1) (which was modelled off a first order differential inclusion [3,4,5]), namely

$$y(t) \in x_0 + \int_0^t G(s, y(s))\, ds \quad \text{for} \quad t \in [0,T]\,. \tag{2.5}$$

here $x_0 \in E$ and $E = (E, |\,.\,|)$ is a real Banach space.

When we are discussing (2.5) we will assume the following conditions hold:

$$G : [0,T] \times E \to C(E) \tag{2.6}$$

$$\begin{cases} \text{for each continuous } y : [0,T] \to E \text{ there exists a scalarly measurable} \\ \text{function } v : [0,T] \to E \text{ with } v(t) \in G(t, y(t)) \text{ a.e. on } [0,T] \\ \text{and } v \text{ is Pettis integrable on } [0,T] \end{cases} \tag{2.7}$$

and

$$(2.8)\qquad \begin{cases} \text{for any } \ r>0 \ \text{ there exists } \ h_r\in L^1[0,T] \ \text{ with } \ |G(t,y)|\le h_r(t) \\ \text{for a.e. } \ t\in[0,T] \ \text{ and all } \ y\in E \ \text{ with } \ |y|\le r; \ \text{ here} \\ |G(t,y)|=\sup\{|w|: \ w\in G(t,y)\}\,. \end{cases}$$

Assign a multivalued operator

$$(2.9)\qquad F: C([0,T],E)\to C(C([0,T],E))$$

by letting

$$(2.10)\qquad F\,y(t)=\{x_0+\textstyle\int_0^t v(s)\,ds: \ v:[0,T]\to E \ \text{ Pettis integrable with } \ v(t)\in G(t,y(t)) \ \text{ a.e. } \ t\in[0,T]\,\}\,.$$

We first show (2.9) makes sense. To see this let $y\in C([0,T],E)$. By (2.7) there exists a Pettis integrable $v:[0,T]\to E$ with $v(t)\in G(t,y(t))$ for a.e. $t\in[0,T]$. Thus F is well defined. Let $u(t)=x_0+\int_0^t v(s)\,ds$. To see that $u\in C([0,T],E)$ first notice that there exists $r>0$ with $|y|_0=\sup_{[0,T]}|y(t)|\le r$. Now (2.8) implies that there exists a constant $h_r\in L^1[0,T]$ with

$$|G(t,y(t))|\le h_r(t) \quad \text{for a.e. } \ t\in[0,T]\,.$$

Let $t,\,x\in[0,T]$ with $t>x$. Without loss of generality assume $u(t)-u(x)\neq 0$. Then there exists (consequence of the Hahn Banach theorem) $\phi\in E^\star$ with $|\phi|=1$ and $|u(t)-u(x)|=\phi(u(t)-u(x))$. Thus

$$|u(t)-u(x)|=\phi\left(\int_x^t v(s)\,ds\right)\le\int_x^t h_r(s)\,ds\,,$$

so $u\in C([0,T],E)$. Consequently

$$F: C([0,T],E)\to 2^{C([0,T],E)}\,.$$

To show (2.9) it just remains for us to show that F has closed (in $C([0,T],E)$) values (note F has automatically convex values since (2.6) is true). Let $y\in C([0,T],E)$. Suppose $w_n\in F\,y$, $n=1,2,\dots$. Then there exists Pettis integrable $v_n:[0,T]\to E$, $n=1,2,\dots$ with $v_n(s)\in G(s,y(s))$ a.e. $s\in[0,T]$. Suppose

$$w_n(t)\to x_0+\int_0^t v(s)\,ds=w(t) \quad \text{in} \quad C([0,T],E)\,.$$

Fix $t\in(0,T]$ and $\phi\in E^\star$. Then $\phi(v_n)\to\phi(v)$ in $L^1[0,t]$ so $\phi(v_n)\to\phi(v)$ in measure. Thus there exists a subsequence S of integers with

$$\phi(v_n(s))\to\phi(v(s)) \quad \text{for a.e. } \ s\in[0,t] \ (\text{as } \ n\to\infty \ \text{ in } \ S\,)\,.$$

Now since $v_n(s)\in G(s,y(s))$ for a.e. $s\in[0,t]$ and since the values of G are closed and convex (so weakly closed) we have $v(s)\in G(s,y(s))$ for a.e. $s\in[0,t]$. Thus $w\in F\,y$ and so F has closed (in $C([0,T],E)$) values

Theorem 2.9. *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,T],E)$. Suppose* (2.6), (2.7) *and* (2.8) *hold. Let F be defined as in* (2.10) *and assume the following conditions hold:*

$$(2.11)\qquad F: Q\to C(C([0,T],E)) \ \textit{ has } \ wk\text{–}\textit{sequentially closed graph}$$

$$F : Q \to C(Q)\,, \tag{2.12}$$

and

$$F\,Q\,(t) \ \text{ is weakly relatively compact in } \ E \ \text{ for each } \ t \in [0,T]\,. \tag{2.13}$$

Then (2.5) *has a solution in* Q.

Remark 2.5. Notice (2.13) can be replaced by any condition that guarantees that F is a α w–contractive map (here $0 \le \alpha < 1$).

Remark 2.6. If E is reflexive then (2.13) is automatically satisfied since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology.

Remark 2.7. Condition (2.11) is our replacement for the condition of Type A in [3, 4]. Certainly if $G(p)$ is point valued for every $p \in [0,T] \times E$ and if for each $t \in [0,T]$, $G_t = G(t,\,.\,)$ is weakly sequentially continuous then (2.11) is satisfied. To see this let $(x_n, y_n)_1^\infty$ be a sequence in $Q \times Q$ with $x_n(t) \to x(t)$ in (E,w) for each $t \in [0,T]$ and $y_n(t) \to y(t)$ in (E,w) for each $t \in [0,T]$ and

$$y_n(t) = F\,x_n(t) = x_0 + \int_0^t G(s, x_n(s))\,ds\,.$$

Fix $t \in (0,T]$. Then since G_t is weakly sequentially continuous the Lebesgue Dominated Convergence Theorem for the Pettis integral [7 Corollary 4] implies for each $\phi \in E^\star$ that $\phi(y_n(t)) = \phi(F\,x_n(t)) \to \phi(F\,x(t))$ i.e. $y_n(t) \to F\,x(t)$ in (E,w). We can do this for each $t \in [0,T]$. Consequently $y(t) = F\,x(t)$.

Proof. The result follows immediately from Theorem 2.8. □

We illustrate the generality of our existence principle by now establishing an existence theorem for (2.3). We only consider the case when E is reflexive (only minor adjustments are needed for the more general case).

Theorem 2.10. *Let E be a reflexive Banach space and suppose* (2.6) *and* (2.7) *hold. Let F be defined as in* (2.10) *and assume* (2.11) *holds. In addition suppose the following conditions hold:*

$$\begin{cases} \text{there exists } \ \alpha \in L^1[0,T] \ \text{ and } \ \psi : (0,\infty) \to (0,\infty) \\ \text{a nondecreasing, continuous function such that} \\ |G(s,u)| \le \alpha(s)\,\psi(|u|) \ \text{ for a.e. } \ s \in [0,T] \ \text{ and all } \ u \in E \end{cases} \tag{2.14}$$

and

$$\int_0^T \alpha(s)\,ds < \int_{|x_0|}^\infty \frac{dx}{\psi(x)}\,. \tag{2.15}$$

Then (2.3) *has a solution in* $C([0,T],E)$.

Proof. Let

$$Q = \{y \in C([0,T],E) : \quad \begin{array}{l} |y(t)| \le b(t) \ \text{ for } \ t \in [0,T] \ \text{ and} \\ |y(t) - y(s)| \le |b(t) - b(s)| \ \text{ for } \ t,\, s \in [0,T]\} \end{array}$$

where

$$b(t) = I^{-1}\left(\int_0^t \alpha(s)\,ds\right) \quad \text{and} \quad I(z) = \int_{|x_0|}^{z} \frac{dx}{\psi(x)}\,.$$

Notice Q is a closed, convex, bounded, equicontinuous subset of $C([0,T],E)$. Let F be as defined in (2.10) and notice (2.13) is satisfied (see Remark 2.6). The result follows immediately from Theorem 2.9 once we show F maps Q into Q. To see this take $u \in F\,Q$. Then there exists $y \in Q$ with $u \in F\,y$ and there exists a Pettis integrable $v : [0,T] \to E$ with $u(t) = x_0 + \int_0^t v(s)\,ds$ and $v(t) \in G(t, y(t))$ for a.e. $t \in [0,T]$. Without loss of generality assume $u(s) \neq 0$ for all $s \in [0,T]$. Then there exists $\phi_s \in E^\star$ with $|\phi_s| = 1$ and $\phi_s(u(s)) = |u(s)|$. Consequently for each fixed $t \in [0,T]$ we have

$$\begin{aligned} |u(t)| &= \phi_t(u(t)) \leq |x_0| + \int_0^t \alpha(s)\,\psi(|y(s)|)\,ds \\ &\leq |x_0| + \int_0^t \alpha(s)\,\psi(b(s))\,ds \leq |x_0| + \int_0^t b'(s)\,ds = b(t) \end{aligned}$$

since

$$\int_{|x_0|}^{b(s)} \frac{dx}{\psi(x)} = \int_0^s \alpha(x)\,dx\,.$$

Next suppose $t,\, s \in [0,T]$ with $t > s$. Without loss of generality assume $u(t) - u(s) \neq 0$. Then there exists $\phi \in E^\star$ with $|\phi| = 1$ and $|u(t) - u(s)| = \phi(u(t) - u(s))$. Consequently

$$|u(t) - u(s)| \leq \int_s^t \alpha(x)\,\psi(|y(x)|)\,ds \leq \int_s^t \alpha(x)\,\psi(b(x))\,ds \leq \int_s^t b'(x)\,dx = b(t) - b(s)\,.$$

Hence $u \in Q$ and we are finished. □

To conclude the paper we again discuss

$$y(t) \in F\,y(t) \quad \text{on} \quad [0,T]\,. \tag{2.16}$$

Associated with (2.16) we consider for each $n \in N^+ = \{1, 2, \ldots\}$ the equations (think of these as corresponding numerical approximations)

$$y(t) \in F_n\,y(t) \quad \text{on} \quad [0,T]\,. \tag*{$(2.17)^n$}$$

Let Q be a subset of $C([0,T],E)$ and let $T_\beta : Q \to 2^{C([0,T],E)}$ for each $\beta \in J$ (some index set). The collection $\{T_\beta : \beta \in J\}$ is weakly collectively α w–contractive (here $0 \leq \alpha < 1$) if

$$w\left(\bigcup_{\beta \in J} T_\beta\ X\right) \leq \alpha\, w(X)$$

for all bounded sets X of Q.

Theorem 2.11. *Let E be a Banach space with Q a nonempty, bounded, closed, convex subset of $C([0,T],E)$. Suppose the following conditions are satisfied:*

(2.18) *for each $n \in N^+$, $F_n : Q \to C(Q)$ has weakly sequentially closed graph*

$$\text{(2.19)} \quad \begin{array}{l} \textit{the collection } \{F_n : n \in N^+\} \textit{ is weakly collectively } \alpha \ w\textit{-contractive} \\ (\textit{here } 0 \le \alpha < 1) \end{array}$$

and

$$\text{(2.20)} \quad \begin{cases} \textit{the sequence of maps } \{F_n\}_1^\infty \textit{ has the following closure property:} \\ \textit{if } \{z_n\}_1^\infty \textit{ is any sequence in } Q \textit{ with } z_n \in F_n\, z_n,\ n = 1, 2, \ldots \\ \textit{and there exists } z_0 \in Q \textit{ with } z_n \rightharpoonup z_0 \textit{ in } C([0,T],E), \\ \textit{then } z_0 \in F\, z_0\,. \end{cases}$$

Then there exists a subsequence S of N^+ and a sequence (x_n) of solutions of $(2.17)^n$, $n \in S$, with $x_n \rightharpoonup x_0$ (as $n \to \infty$ in S) in $C([0,T],E)$ and x_0 is a solution of (2.16).

Proof. Theorem 2.3 implies that there exists $x_n \in Q$ with $x_n \in F_n\, x_n$. Define the map $\mathcal{K} : Q \to 2^Q$ by

$$\mathcal{K}\, y = \bigcup_{n \in N^+} F_n\, y \quad \text{for} \quad y \in Q.$$

Notice for any subset X of Q,

$$w(\mathcal{K}\, X) = w\left(\bigcup_{y \in X} \mathcal{K}\, y\right) = w\left(\bigcup_{y \in X} \bigcup_{n \in N^+} F_n\, y\right) = w\left(\bigcup_{n \in N^+} F_n\, X\right) \le \alpha\, w(X)\,.$$

Now apply Theorem 3.1 of [15] (note $d(x_n, \mathcal{K}\, x_n) = 0$ for each $n \in N^+$) to deduce that (x_n) has a weakly convergent subsequence. Without loss of generality assume its the whole sequence and $x_n \rightharpoonup x_0$ in $C([0,T],E)$. This together with (2.20) implies $x_0 \in F\, x_0$. □

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