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ON CONNECTIONS BETWEEN HYPERGRAPHS AND ALGEBRAS

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Abstract. The aim of the present paper is to translate some algebraic concepts to hypergraphs. Thus we obtain a new language, very useful in the investigation of subalgebra lattices of partial, and also total, algebras. In this paper we solve three such problems on subalgebra lattices, other will be solved in [19]. First, we show that for two arbitrary partial algebras, if their directed hypergraphs are isomorphic, then their weak, relative and strong subalgebra lattices are isomorphic. Secondly, we prove that two partial algebras have isomorphic weak subalgebra lattices iff their hypergraphs are isomorphic. Thirdly, for an arbitrary lattice **L** and a partial algebra **A** we describe (necessary and sufficient conditions) when the weak subalgebra lattice of **A** is isomorphic to **L**.

1.

Investigations of relationships between (total) algebras or varieties of algebras and their lattices of (also total) subalgebras are an important part of universal algebra. For example, the full characterization of the subalgebra lattice of a (total) algebra is given in [5]; moreover, there are many results which characterize subalgebra lattices for algebras which belong to a given variety or a given type (see e.g. [12]). Other papers investigate algebras with special subalgebra lattices (e.g. distributive, modular, etc.) or varieties containing algebras such that their subalgebra lattices satisfy given conditions (see e.g. [7], [13], [15], [21], [22]). Note that several such results concern also classical algebras — Boolean algebras, groups, modules (see e.g. [10], [11], [14], [20]).

The theory of partial algebras provides additional tools for such investigations, because at least four structures may be considered in this case: weak, relative, strong subalgebra and initial segment lattices (see e.g. [3], [6]). It seems that they yield a lot of interesting information on an algebra, also total. Therefore lattices of partial subalgebras may play also an important role in the theory of partial

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(thus also total) algebras. Several results describing connections between a partial algebra and its subalgebra lattices are already known. A characterization of the weak subalgebra lattice is given in [1]. [2] describes monounary partial algebras uniquely determined in the class of all monounary partial algebras of the same type by their weak subalgebra lattices. Moreover, it is shown in [1] that every unary partial algebra can be represented by a graph which uniquely determines its weak subalgebra lattice. Generalizing this idea we introduced in [16] a graph–algebraic language which is very useful in investigations of subalgebra lattices of a unary partial algebra. For instance, in [17] we described all pairs $\langle \mathbf{A}, \mathbf{L} \rangle$, where $\mathbf{A}$ is a unary partial algebra and $\mathbf{L}$ a lattice, such that the strong subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$. Of course, if $\mathbf{A}$ is total, then this lattice is the usual lattice of (total) subalgebras. We also found in [17] necessary and sufficient conditions for two arbitrary unary partial algebras (which can be even of different types) to have isomorphic strong subalgebra lattices. Moreover, applying this language, [18] shows that for total and locally finite unary algebras of finite type, the weak subalgebra lattice uniquely determines the strong subalgebra lattice.

These results mainly concern subalgebra lattices of unary partial algebras. The aim of the paper is to generalize this graph–algebraic language (using experiences and ideas from [16]) onto arbitrary partial algebras. More precisely, we translate some algebraic concepts (e.g. algebra types, four kinds of subalgebras, lattices of subalgebras, etc.) into a hypergraph language. Next, we prove some connections between hypergraphs and partial algebras. This language turned out to be very useful in solutions of some problems on subalgebra lattices of partial algebras. Now we solve three such problems. First, we show that if two partial algebras have isomorphic directed hypergraphs, then their lattices of weak, relative, strong subalgebras and initial segments are isomorphic. Secondly, we prove the inverse result for the weak subalgebra lattice. More precisely, two partial algebras have isomorphic weak subalgebra lattices iff their (undirected) hypergraphs are isomorphic. Of course, the result from [1] is a particular case of this theorem. Thirdly, we solve the following: Let $\mathbf{L}$ be a lattice and $\mathbf{A}$ a partial algebra. When is the weak subalgebra lattice of $\mathbf{A}$ isomorphic to $\mathbf{L}$?

Applying results from this paper we will characterize in a subsequent paper [19] pairs $\langle \mathbf{L}, \langle K, \kappa \rangle \rangle$, where $\mathbf{L}$ is a lattice and $\langle K, \kappa \rangle$ an algebra type, such that there is a partial algebra $\mathbf{A}$ of the type $\langle K, \kappa \rangle$ with the weak subalgebra lattice isomorphic to $\mathbf{L}$. Such a characterization for algebraic lattices and types in the case of total algebras is an important problem of universal algebra (see e.g. [12]) which is not completely solved yet. But for weak subalgebra lattices we will give a complete solution.

The cardinality of a set A is denoted by $|A|$, $\mathbb{N}$ is the set of all non–negative integers, $\mathbf{Card}$ is the class of all cardinal numbers and $\aleph_0 = |\mathbb{N}|$. $\mathbf{Card}^{\mathbb{N}}$ is the class of all infinite sequences $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ of cardinal numbers. $\mathbb{N}^{\mathbb{N}}$ is the set of all infinite sequences of natural numbers and $\mathbb{N}^{\mathbb{N}}_f \subseteq \mathbb{N}^{\mathbb{N}}$ is the set of all infinite sequences in which almost all terms are equal zero. Moreover, $P(A)$ is the family of all subsets of A, $P_k(A) = \{C \in P(A) : \ |C| = k\}$ for $k \in \mathbb{N}$ (of course, $P_0(A) = \{\emptyset\})$, $P_{fin}(A) = \bigcup_{k\in\mathbb{N}} P_k(A)$ and $\prod_{fin}(A) = \bigcup_{k\in\mathbb{N}} A^k$, where $A^k = \underbrace{A \times A \ldots \times A}_{k}$ (note that $A^0 = \{\emptyset\}$).

2.

An (undirected) hypergraph $\mathbf{H} = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{H}}\rangle$ is an ordered triple such that $V^{\mathbf{H}}$ and $E^{\mathbf{H}}$ are arbitrary sets (of vertices and hyperedges respectively), and $I^{\mathbf{H}}$ is a function from $E^{\mathbf{H}}$ into $P_{fin}(V^{\mathbf{H}}) \setminus \{\emptyset\}$. For each $e \in E^{\mathbf{H}}$, the elements of $I^{\mathbf{H}}(e)$ will be called terminal vertices of e. A directed hypergraph $\mathbf{H} = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{H}}\rangle$ is an ordered triple such that $V^{\mathbf{H}}$ and $E^{\mathbf{H}}$ are sets, and $I^{\mathbf{H}} = \langle I_1^{\mathbf{H}}, I_2^{\mathbf{H}}\rangle$ is a function from $E^{\mathbf{H}}$ into $P_{fin}(V^{\mathbf{H}}) \times V^{\mathbf{H}}$. An algebraic hypergraph $\mathbf{H} = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{H}}\rangle$ is an ordered triple such that $V^{\mathbf{H}}$ and $E^{\mathbf{H}}$ are sets, and $I^{\mathbf{H}} = \langle I_1^{\mathbf{H}}, I_2^{\mathbf{H}}\rangle$ is a function from $E^{\mathbf{H}}$ into $\prod_{fin}(V^{\mathbf{H}}) \times V^{\mathbf{H}}$. For each $e \in E^{\mathbf{H}}$ of a directed (algebraic) hypergraph $\mathbf{H}$, $I_1^{\mathbf{H}}(e)$ will be called the initial set (sequence) of e and $I_2^{\mathbf{H}}(e)$ will be called the final vertex of e. Moreover, $e \in E^{\mathbf{H}}$ is a k–edge (where $k \in \mathbb{N}$) iff $I_1^{\mathbf{H}}(e) \in P_k(V^{\mathbf{H}})$ $\big(I_1^{\mathbf{H}}(e) \in (V^{\mathbf{H}})^k\big)$. The initial sets (sequences) of 0–edges are empty, so 0–edges can be identified with their final vertices. The set of all k–edges of $\mathbf{H}$ will be denoted by $E^{\mathbf{H}}(k)$.
The classes of all undirected, directed and algebraic hypergraphs are denoted by $\mathcal{UH}$, $\mathcal{DH}$ and $\mathcal{AH}$.

The above definitions are simple modifications of the concept of hypergraph from [4]. Note that directed graphs are the special case of directed and algebraic hypergraphs simultaneously. Analogously, (undirected) graphs are the special case of hypergraphs. Moreover, since we want to represent partial algebras by hypergraphs, we do not restrict the cardinality of vertex and hyperedge sets.

Observe that with every directed hypergraph we can associate the hypergraph by omitting the orientation of all hyperedges. More formally, ($\{v_1, \ldots, v_0\}$ and $\langle v_1, \ldots, v_0\rangle$ denote the empty set),

Definition 2.1.

(a) Let $\mathbf{H} \in \mathcal{DH}$. Then $\mathbf{H}^*$ is a hypergraph such that $V^{\mathbf{H}^*} = V^{\mathbf{H}}$, $E^{\mathbf{H}^*} = E^{\mathbf{H}}$ and for each $e \in E^{\mathbf{H}^*}$, $I^{\mathbf{H}^*}(e) = I_1^{\mathbf{H}}(e) \cup \{I_2^{\mathbf{H}}(e)\}$.

(b) Let $\mathbf{H} \in \mathcal{AH}$. Then $\mathbf{H}^*$ is the directed hypergraph such that $V^{\mathbf{H}^*} = V^{\mathbf{H}}$, $E^{\mathbf{H}^*} = E^{\mathbf{H}}$ and for each $e \in E^{\mathbf{H}^*}$, $I^{\mathbf{H}^*}(e) = \big\langle \{v_1, \ldots, v_n\}, I_2^{\mathbf{H}}(e)\big\rangle$, where $I_1^{\mathbf{H}}(e) = \langle v_1, \ldots, v_n\rangle$.

(c) Let $\mathbf{H} \in \mathcal{AH}$. Then $\mathbf{H}^{**} = (\mathbf{H}^*)^*$

The axiom of choice easily implies that for each $\mathbf{H} \in \mathcal{UH}$, there is $\mathbf{D} \in \mathcal{DH}$ ($\mathbf{A} \in \mathcal{AH}$) such that $\mathbf{D}^* = \mathbf{H}$ ($\mathbf{A}^{**} = \mathbf{H}$). Analogously for $\mathbf{D} \in \mathcal{DH}$, there is $\mathbf{A} \in \mathcal{AH}$ such that $\mathbf{A}^* = \mathbf{D}$.

For each algebraic (directed) hypergraph $\mathbf{H}$ and $\mathbf{v} \in \prod_{fin}(V^{\mathbf{H}})$ $\big(\mathbf{v} \in P_{fin}(V^{\mathbf{H}})\big)$ we can define the set $E_s^{\mathbf{H}}(\mathbf{v}) = \{e \in E^{\mathbf{H}} \colon I_1^{\mathbf{H}}(e) = \mathbf{v}\}$ and the cardinal number $s^{\mathbf{H}}(\mathbf{v}) = |E_s^{\mathbf{H}}(\mathbf{v})|$. Note that $s^{\mathbf{H}}(\mathbf{v})$ may be an arbitrary cardinal number, because we consider, in general, infinite hypergraphs. Observe also $E_s^{\mathbf{H}}(\emptyset) = E^{\mathbf{H}}(0)$, so $s^{\mathbf{H}}(\emptyset) = |E^{\mathbf{H}}(0)|$.

Having the above definition we can generalize the concept of algebra types onto hypergraphs.

Definition 2.2. Let $\mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{H} \in \mathcal{DH}\big)$ and $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \ldots) \in \mathbf{Card}^{\mathbb{N}}$. Then $\mathbf{H}$ is of type $\underline{\tau}$ iff

$$s^{\mathbf{H}}(\mathbf{v}) \leq \tau_k \quad \text{for all} \quad \mathbf{v} = \langle v_1, \ldots, v_k\rangle \in \textstyle\prod_{fin}(V^{\mathbf{H}})$$
$$\big(\mathbf{v} = \{v_1, \ldots, v_k\} \in P_{fin}(V^{\mathbf{H}})\big).$$

The class of all algebraic (directed) hypergraphs of type $\underline{\tau}$ will be denoted by $\mathcal{AH}(\underline{\tau})$ $\big(\mathcal{DH}(\underline{\tau})\big)$.
$\underline{\tau}$ is a totally finite type iff $\underline{\tau} \in \mathbb{N}^{\mathbb{N}}_f$; it is finite iff $\underline{\tau} \in \mathbb{N}^{\mathbb{N}}$; and it is infinite iff $\underline{\tau} \in \mathbf{Card}^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$.

Note first, that $\mathcal{AH}(\underline{0})$ and $\mathcal{DH}(\underline{0})$, where $\underline{0} = (0, 0, 0, \ldots)$, are the class of all discrete hypergraphs (i.e. hypergraphs without hyperedges). Secondly, for $\underline{\tau}_1 = (\tau^1_0, \tau^1_1, \ldots), \underline{\tau}_2 = (\tau^2_0, \tau^2_1, \ldots) \in \mathbf{Card}^{\mathbb{N}}$, if $\tau^1_i \leq \tau^2_i$ for all $i \in \mathbb{N}$, then $\mathcal{AH}(\underline{\tau}_1) \subseteq \mathcal{AH}(\underline{\tau}_2)$ and $\mathcal{DH}(\underline{\tau}_1) \subseteq \mathcal{DH}(\underline{\tau}_2)$. Thirdly, for every $\mathbf{H} \in \mathcal{AH}(\underline{\tau})$ $\big(\mathbf{H} \in \mathcal{DH}(\underline{\tau})\big)$, $|E^{\mathbf{H}}(0)| = s^{\mathbf{H}}(\emptyset) \leq \tau_0$. Moreover, the following two facts hold:

$\mathbf{H} \in \mathcal{DH}$ of finite type $\underline{\tau} \in \mathbb{N}^{\mathbb{N}}$ is totally finite (i.e. $V^{\mathbf{H}}, E^{\mathbf{H}}$ are finite) iff $\mathbf{H}$ is finite (i.e. $V^{\mathbf{H}}$ is finite).

$\Rightarrow$ is trivial. $\Leftarrow$: Let $N = |V^{\mathbf{H}}| < \aleph_0$. Then $P_k(V^{\mathbf{H}}) = \emptyset$ for $k \geq N+1$ and $|P_k(V^{\mathbf{H}})| < \aleph_0$ for $k \leq N$. Hence, $|P_{fin}(V^{\mathbf{H}})| < \aleph_0$. Since $\underline{\tau} \in \mathbb{N}^{\mathbb{N}}$, $s^{\mathbf{H}}(V) < \aleph_0$ for $V \in P_{fin}(V^{\mathbf{H}})$. Moreover, it is easy to see $E^{\mathbf{H}} = \bigcup_{V \in P_{fin}(V^{\mathbf{H}})} E^{\mathbf{H}}_s(V)$. Thus $E^{\mathbf{H}}$ is a finite sum of finite sets, so $E^{\mathbf{H}}$ is also finite.

an algebraic hypergraph $\mathbf{H}$ of totally finite type $\underline{\tau} \in \mathbb{N}^{\mathbb{N}}_f$ is totally finite iff $\mathbf{H}$ is finite.

$\Rightarrow$ is trivial. $\Leftarrow$: Since $\underline{\tau} \in \mathbb{N}^{\mathbb{N}}_f$, there is $N \in \mathbb{N}$ such that $\tau_k = 0$ for $k \geq N+1$. Then $s^{\mathbf{H}}(\mathbf{v}) = 0$ for $\mathbf{v} \in \bigcup_{m \geq N+1} (V^{\mathbf{H}})^m$ and $s^{\mathbf{H}}(\mathbf{v}) < \aleph_0$ for $\mathbf{v} \in \bigcup_{m=0}^{m=N} (V^{\mathbf{H}})^m$, so $E^{\mathbf{H}} = \bigcup_{v \in \prod_{fin}(V^{\mathbf{H}})} E^{\mathbf{H}}_s(\mathbf{v}) = \bigcup_{v \in \bigcup_{m=0}^{m=N} (V^{\mathbf{H}})^m} E^{\mathbf{H}}_s(\mathbf{v})$. $|\bigcup_{m=0}^{m=N} (V^{\mathbf{H}})^m| < \aleph_0$, because $V^{\mathbf{H}} < \aleph_0$. Thus $E^{\mathbf{H}}$ is a finite sum of finite sets,

We want to represent partial algebras by hypergraphs, so we define various kinds of subhypergraphs.

Definition 2.3. Let $\mathbf{G}, \mathbf{H} \in \mathcal{UH}$. Then

(a) We say that $\mathbf{G}$ is a weak subhypergraph of $\mathbf{H}$ $\big(\mathbf{G} \leq_w \mathbf{H}\big)$ iff $V^{\mathbf{G}} \subseteq V^{\mathbf{H}}$, $E^{\mathbf{G}} \subseteq E^{\mathbf{H}}$, $I^{\mathbf{G}} \subseteq I^{\mathbf{H}}$.
(b) We say that $\mathbf{G}$ is a relative subhypergraph of $\mathbf{H}$ $\big(\mathbf{G} \leq_r \mathbf{H}\big)$ iff $\mathbf{G} \leq_w \mathbf{H}$ and for each $e \in E^{\mathbf{H}}$, if $I^{\mathbf{H}}(e) \subseteq V^{\mathbf{G}}$, then $e \in E^{\mathbf{G}}$.

For each $\mathbf{H} \in \mathcal{UH}$, $S_w(\mathbf{H})$ $\big(S_r(\mathbf{H})\big)$ is the family of all weak (relative) subhypergraphs of $\mathbf{H}$. Note also that the empty hypergraph is simultaneously a weak and relative subhypergraph of $\mathbf{H}$.

Definition 2.4. Let $\mathbf{G}, \mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{G}, \mathbf{H} \in \mathcal{DH}\big)$. Then

(a) We say that $\mathbf{G}$ is a weak subhypergraph of $\mathbf{H}$ $\big(\mathbf{G} \leq_w \mathbf{H}\big)$ iff $V^{\mathbf{G}} \subseteq V^{\mathbf{H}}$, $E^{\mathbf{G}} \subseteq E^{\mathbf{H}}$, $I^{\mathbf{G}} = I^{\mathbf{H}}|_{E^{\mathbf{G}}}$.

(b) We say that $\mathbf{G}$ is a relative subhypergraph of $\mathbf{H}$ $\big(\mathbf{G} \leq_r \mathbf{H}\big)$ iff $\mathbf{G} \leq_w \mathbf{H}$ and for each $e \in E^{\mathbf{H}}$, if $I^{\mathbf{H}}(e) \in \prod_{fin}(V^{\mathbf{G}}) \times V^{\mathbf{G}}$ $\big(I^{\mathbf{H}}(e) \in P_{fin}(V^{\mathbf{G}}) \times V^{\mathbf{G}}\big)$, then $e \in E^{\mathbf{G}}$

(c) We say that $\mathbf{G}$ is a strong subhypergraph of $\mathbf{H}$ $\big(\mathbf{G} \leq_s \mathbf{H}\big)$ iff $\mathbf{G} \leq_r \mathbf{H}$ and for each $e \in E^{\mathbf{H}}$, if $I_1^{\mathbf{H}}(e) \in \prod_{fin}(V^{\mathbf{G}})$ $\big(I_1^{\mathbf{H}}(e) \in P_{fin}(V^{\mathbf{G}})\big)$, then $I_2^{\mathbf{H}}(e) \in V^{\mathbf{G}}$.

(d) We say that $\mathbf{G}$ is a dually strong subhypergraph of $\mathbf{H}$ $\big(\mathbf{G} \leq_d \mathbf{H}\big)$ iff $\mathbf{G} \leq_r \mathbf{H}$ and for each $e \in E^{\mathbf{H}}$, if $I_2^{\mathbf{H}}(e) \in V^{\mathbf{G}}$, then $I_1^{\mathbf{H}}(e) \in \prod_{fin}(V^{\mathbf{G}})$ $\big(I_1^{\mathbf{H}}(e) \in P_{fin}(V^{\mathbf{G}})\big)$.

For each $\mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{H} \in \mathcal{DH}\big)$, $S_w(\mathbf{H})$, $S_r(\mathbf{H})$, $S_s(\mathbf{H})$ and $S_d(\mathbf{H})$ are the families of all weak, relative, strong and dually strong subhypergraphs of $\mathbf{H}$ respectively. Note that the empty hypergraph is simultaneously a weak, relative and dually strong subhypergraph of $\mathbf{H}$. Moreover, if $E^{\mathbf{H}}(0) = \emptyset$, then the empty hypergraph is also a strong subhypergraph of $\mathbf{H}$. In general, for every $\mathbf{G} \leq_s \mathbf{H}$, $I_2^{\mathbf{H}}\big(E^{\mathbf{H}}(0)\big) \subseteq V^{\mathbf{G}}$ and $E^{\mathbf{H}}(0) \subseteq E^{\mathbf{G}}$, because $I_1^{\mathbf{H}}(e) = \emptyset \in \prod_{fin}(V^{\mathbf{G}})$ for each $e \in E^{\mathbf{H}}(0)$.

Now we give a few simple facts (easy proofs are omitted).

Proposition 2.5. *Let $\mathbf{H} \in \mathcal{UH}$ or $\mathbf{H} \in \mathcal{DH}$ or $\mathbf{H} \in \mathcal{AH}$. Then*

(a) *If $\mathbf{G} \leq_w \mathbf{H}$ $\big(\mathbf{G} \leq_r \mathbf{H}\big)$, then $S_w(\mathbf{G}) \subseteq S_w(\mathbf{H})$ $\big(S_r(\mathbf{G}) \subseteq S_r(\mathbf{H})\big)$.*

(b) *If $\mathbf{G} \leq_s \mathbf{H}$ $\big(\mathbf{G} \leq_d \mathbf{H}\big)$, then $S_s(\mathbf{G}) \subseteq S_s(\mathbf{H})$ $\big(S_d(\mathbf{G}) \subseteq S_d(\mathbf{H})\big)$.*

(c) *For $\mathbf{G}_1, \mathbf{G}_2 \leq_w \mathbf{H}$,*
$\mathbf{G}_1 \leq_w \mathbf{G}_2$ $\big(\mathbf{G}_1 = \mathbf{G}_2\big)$ iff $V^{\mathbf{G}_1} \subseteq V^{\mathbf{G}_2}$, $E^{\mathbf{G}_1} \subseteq E^{\mathbf{G}_2}$ $\big(V^{\mathbf{G}_1} = V^{\mathbf{G}_2}$, $E^{\mathbf{G}_1} = E^{\mathbf{G}_2}\big)$.

(d) *For $\mathbf{G}_1, \mathbf{G}_2 \leq_r \mathbf{H}$, $\mathbf{G}_1 \leq_r \mathbf{G}_2$ $\big(\mathbf{G}_1 = \mathbf{G}_2\big)$ iff $V^{\mathbf{G}_1} \subseteq V^{\mathbf{G}_2}$ $\big(V^{\mathbf{G}_1} = V^{\mathbf{G}_2}\big)$.*

(e) *For $\mathbf{G}_1, \mathbf{G}_2 \leq_s \mathbf{H}$ $\big(\mathbf{G}_1, \mathbf{G}_2 \leq_d \mathbf{H}\big)$, $\mathbf{G}_1 \leq_s \mathbf{G}_2$ $\big(\mathbf{G}_1 \leq_d \mathbf{G}_2\big)$ iff $V^{\mathbf{G}_1} \subseteq V^{\mathbf{G}_2}$ $\big(V^{\mathbf{G}_1} \subseteq V^{\mathbf{G}_2}\big)$.*

Of course, for hypergraphs only points (a), (c) and (d) hold.

By the above facts we obtain that for any algebraic or directed (undirected) hypergraph $\mathbf{H}$, the relations $\leq_w$, $\leq_r$, $\leq_s$ and $\leq_d$ ($\leq_w$ and $\leq_r$) are partial orders. Now we show that the families of all subhypergraphs with these orders form complete (and algebraic) lattices.

Proposition 2.6. *Let $\mathbf{H}$ be an undirected or directed or algebraic hypergraph. Then*

(a) *$\mathbf{S}_w(\mathbf{H}) = \langle S_w(\mathbf{H}), \leq_w\rangle$ is a complete (and algebraic) lattice, where the operations of infimum $\bigwedge$ and supremum $\bigvee$ are defined as follows: $\bigwedge_{i\in I} \mathbf{H}_i = \langle \bigcap_{i\in I} V^{\mathbf{H}_i}, \bigcap_{i\in I} E^{\mathbf{H}_i}, \bigcap_{i\in I} I^{\mathbf{H}_i}\rangle$, $\bigvee_{i\in I} \mathbf{H}_i = \langle \bigcup_{i\in I} V^{\mathbf{H}_i}, \bigcup_{i\in I} E^{\mathbf{H}_i}, \bigcup_{i\in I} I^{\mathbf{H}_i}\rangle$, for each non–empty $\{\mathbf{H}_i\}_{i\in I} \subseteq S_w(\mathbf{H})$.*

(b) *$\mathbf{S}_r(\mathbf{H}) = \langle S_r(\mathbf{H}), \leq_r\rangle$ is a complete lattice, where the operation of infimum $\bigwedge$ is defined as above and $\bigvee_{i\in I} \mathbf{H}_i$ is the exactly one relative subhypergraph of $\mathbf{H}$ such that $V^{\bigvee_{i\in I} \mathbf{H}_i} = \bigcup_{i\in I} V^{\mathbf{H}_i}$ for each non–empty family $\{\mathbf{H}_i\}_{i\in I} \subseteq S_r(\mathbf{H})$.*

Moreover, $\mathbf{S}_r(\mathbf{H})$ *is isomorphic to the lattice of all subsets of* $V^{\mathbf{H}}$*, so it is also algebraic.*

Proof. (a): For a non–empty family $\{\mathbf{H}_i\}_{i\in I} \subseteq S_w(\mathbf{H})$ it can be easily verified that $\bigwedge_{i\in I}\mathbf{H}_i$ and $\bigvee_{i\in I}\mathbf{H}_i$ are indeed hypergraphs and weak subhypergraphs of $\mathbf{H}$, and moreover, that they are the infimum and the supremum of $\{\mathbf{H}_i\}_{i\in I}$ in $\langle S_w(\mathbf{H}),\leq_w\rangle$ respectively. Hence, $\langle S_w(\mathbf{H}),\leq_w\rangle$ is a complete lattice. These facts also imply that the family $\mathcal{F}$ of subsets of $V^{\mathbf{H}}\times E^{\mathbf{H}}$ which correspond to weak subhypergraphs of $\mathbf{H}$ (i.e. $W\times F\in\mathcal{F}$ iff the triple $\langle W,F,I|_F\rangle$ is a hypergraph and a weak subhypergraph of $\mathbf{H}$) forms an algebraic closure system, because $\mathcal{F}$ is even closed under arbitrary unions. Moreover, $\langle S_w(\mathbf{H}),\leq_w\rangle$ is isomorphic to $\langle\mathcal{F},\subseteq\rangle$ by Proposition 2.5(c). Thus $\mathbf{S}_w(\mathbf{H})$ is algebraic.

(b): First, we know by (a) that for every non–empty family $\{\mathbf{H}_i\}_{i\in I}\subseteq\mathbf{S}_r(\mathbf{H})$, $\bigwedge_{i\in I}\mathbf{H}_i$ is a well–defined weak subhypergraph of $\mathbf{H}$. Secondly, it can be easily verified that $\bigwedge_{i\in I}\mathbf{H}_i$ is also a relative subhypergraph of $\mathbf{H}$, and moreover, that it is the infimum of this family in the partially ordered set $\langle S_r(\mathbf{H}),\leq_r\rangle$. Thus $\mathbf{S}_r(\mathbf{H})$ is a complete lattice, because $\mathbf{H}\in S_r(\mathbf{H})$ is its greatest element.

Secondly, by Proposition 2.5(d) each subset $W\subseteq V^{\mathbf{H}}$ induces a unique relative subhypergraph of $\mathbf{H}$, i.e. there is exactly one $\mathbf{G}\leq_r\mathbf{H}$ such that $V^{\mathbf{G}}=W$; then $E^{\mathbf{G}}=\{e\in E^{\mathbf{H}}\colon I^{\mathbf{H}}(e)\in W\}$. Hence and by Proposition 2.5(d) we infer that $\mathbf{S}_r(\mathbf{H})$ is isomorphic to the lattice of all subsets of $V^{\mathbf{H}}$. □

Proposition 2.7. *Let* $\mathbf{H}\in\mathcal{AH}$ *or* $\mathbf{H}\in\mathcal{DH}$*. Then*

(a) $\mathbf{S}_s(\mathbf{H})=\langle S_s(\mathbf{H}),\leq_s\rangle$ *is an algebraic lattice, where* $\bigwedge$ *is defined as in Proposition 2.6(a).*

(b) $\mathbf{S}_d(\mathbf{H})=\langle S_d(\mathbf{H}),\leq_d\rangle$ *is an algebraic lattice, where* $\bigwedge$ *and* $\bigvee$ *are defined as in Proposition 2.6(b). Moreover,* $\mathbf{S}_d(\mathbf{H})$ *is a complete sublattice of* $\mathbf{S}_r(\mathbf{H})$.

Remark. Let $\mathbf{H}\in\mathcal{AH}$ or $\mathbf{H}\in\mathcal{DH}$. For each $W\subseteq V^{\mathbf{H}}$, the least (with respect to $\leq_s$) strong subhypergraph of $\mathbf{H}$ containing W will be here denoted by $[W]^s_{\mathbf{H}}$.

Proof. First, by Proposition 2.6(a) the intersection $\bigwedge_{i\in I}\mathbf{H}_i$ of any non–empty family $\{\mathbf{H}_i\}_{i\in I}$ of strong (dually strong) subhypergraphs is a well–defined weak subhypergraph. Secondly, it is easy to see that it is also a strong (dually strong) subhypergraph. Hence, $\bigwedge_{i\in I}\mathbf{H}_i$ is the infimum of $\{\mathbf{H}_i\}_{i\in I}$ in $\langle S_s(\mathbf{H}),\leq_s\rangle$ $\big(\langle S_d(\mathbf{H}),\leq_d\rangle\big)$, because by Proposition 2.6(a) it is the infimum in $\mathbf{S}_w(\mathbf{H})$ and $\leq_s=\leq_w|_{S_s(\mathbf{H})}$ $\big(\leq_d=\leq_w|_{S_d(\mathbf{H})}\big)$. These facts imply that $\mathbf{S}_s(\mathbf{H})$ $\big(\mathbf{S}_d(\mathbf{H})\big)$ is a complete lattice.

Now take an arbitrary non–empty directed family $\{\mathbf{H}_i\}_{i\in I}$ of strong subhypergraphs of $\mathbf{H}$ (i.e. for each $i_1,i_2\in I$, there is $i_3\in I$ such that $\mathbf{H}_{i_1},\mathbf{H}_{i_2}\leq_s\mathbf{H}_{i_3}$). Then it is easily shown that the ordered triple $\big\langle\bigcup_{i\in I}V^{\mathbf{H}_i},\bigcup_{i\in I}E^{\mathbf{H}_i},\bigcup_{i\in I}I^{\mathbf{H}_i}\big\rangle$ is a strong subhypergraph of $\mathbf{H}$. Hence, the family $\mathcal{F}$ of subsets of $V^{\mathbf{H}}$ corresponding to strong subhypergraphs (i.e. $W\in\mathcal{F}$ iff the relative subhypergraph induced by W is also a strong subhypergraph) is closed under arbitrary intersections and unions of directed families, so it is an algebraic closure system. Moreover, Proposition 2.5(d),(e) implies that $\langle S_s(\mathbf{H}),\leq_s\rangle$ is isomorphic to $\langle\mathcal{F},\subseteq\rangle$. Thus $\mathbf{S}_s(\mathbf{H})$ is algebraic.

Finally, take a non–empty family $\{\mathbf{H}_i\}_{i\in I}$ of dually strong subhypergraphs of $\mathbf{H}$. Then it is not difficult to see that the relative subhypergraph induced by $\bigcup_{i\in I} V^{\mathbf{H}_i}$ is also a dually strong subhypergraph. Hence, the family $\mathcal{F}$ of subsets of $V^{\mathbf{H}}$ corresponding to dually strong subhypergraphs of $\mathbf{H}$ (i.e. $W \in \mathcal{F}$ iff the relative subhypergraph induced by W is also a dually strong subhypergraph) is closed under arbitrary intersections and unions, so it is an algebraic closure system. Moreover, Proposition 2.5(d),(e) implies that $\langle S_d(\mathbf{H}), \leq_d\rangle$ is isomorphic to $\langle \mathcal{F}, \subseteq\rangle$. Thus $\mathbf{S}_d(\mathbf{H})$ is algebraic. □

Remark. Propositions 2.5, 2.6 and 2.7 easily imply that for $\mathbf{H} \in \mathcal{UH}$ and its weak (relative) subhypergraph $\mathbf{G}$, $\mathbf{S}_w(\mathbf{G})$ $\big(\mathbf{S}_r(\mathbf{G})\big)$ is a complete sublattice of $\mathbf{S}_w(\mathbf{H})$ $\big(\mathbf{S}_r(\mathbf{H})\big)$. Analogously, for each directed (algebraic) hypergraph $\mathbf{H}$ and its weak, relative, strong or dually strong subhypergraph $\mathbf{G}$, $\mathbf{S}_w(\mathbf{G})$, $\mathbf{S}_r(\mathbf{G})$, $\mathbf{S}_s(\mathbf{G})$ and $\mathbf{S}_d(\mathbf{G})$ are complete sublattices of $\mathbf{S}_w(\mathbf{H})$, $\mathbf{S}_r(\mathbf{H})$, $\mathbf{S}_s(\mathbf{H})$ and $\mathbf{S}_d(\mathbf{H})$ respectively.

Now we prove two less trivial results which will be needed in the next section. We start with a hypergraph generalization of the classical result on the generation of (strong) subalgebras.

Proposition 2.8. *Let* $\mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{H} \in \mathcal{DH}\big)$ *and* $W \subseteq V^{\mathbf{H}}$. *Then* $V^{[W]^s_{\mathbf{H}}} = \bigcup_{n\in\mathbb{N}} X_n$, *where* $X_0 = W$ *and for each* $n \in \mathbb{N}$, $X_{n+1} = \{v \in V^{\mathbf{H}} : \exists_{e\in E^{\mathbf{H}}}\ I_1^{\mathbf{H}}(e) \in \prod_{fin}(X_n)$ *and* $I_2^{\mathbf{H}}(e) = v\} \cup X_n$ $\big(X_{n+1} = \{v \in V^{\mathbf{H}} : \exists_{e\in E^{\mathbf{H}}}\ I_1^{\mathbf{H}}(e) \in P_{fin}(X_n)$ *and* $I_2^{\mathbf{H}}(e) = v\} \cup X_n\big)$.

Proof. Let $\mathbf{G}$ be the relative subhypergraph of $\mathbf{H}$ with $V^{\mathbf{G}} = \bigcup_{n\in\mathbb{N}} X_n$ and take a hyperedge $e \in E^{\mathbf{G}}$ such that $I_1^{\mathbf{H}}(e) \in \prod_{fin}(V^{\mathbf{G}})$ $\big(I_1^{\mathbf{H}}(e) \in P_{fin}(V^{\mathbf{G}})\big)$. Then $I_1^{\mathbf{H}}(e) \in \prod_{fin}(X_m)$ $\big(I_1^{\mathbf{H}}(e) \in P_{fin}(X_m)\big)$ for some $m \in \mathbb{N}$, because $I_1^{\mathbf{H}}(e)$ is a finite sequence (set) and moreover, $X_n \subseteq X_{n+1}$ for each $n \in \mathbb{N}$. Hence, $I_2^{\mathbf{H}}(e) \in X_{m+1} \subseteq V^{\mathbf{G}}$. Thus $\mathbf{G}$ is a strong subhypergraph of $\mathbf{H}$ containing W, because $\mathbf{G} \leq_r \mathbf{H}$ and $X_0 = W$. This fact and the definition of $[W]^s_{\mathbf{H}}$ imply $V^{[W]^s_{\mathbf{H}}} \subseteq V^{\mathbf{G}}$. On the other hand, by a simple induction on $n \in \mathbb{N}$ and by the definition of strong subhypergraphs we obtain $X_n \subseteq V^{[W]^s_{\mathbf{H}}}$ for each $n \in \mathbb{N}$ (because $W = X_0 \subseteq V^{\mathbf{G}}$). Thus $V^{\mathbf{G}} \subseteq V^{[W]^s_{\mathbf{H}}}$, which completes the proof. □

Corollary 2.9. *For each* $\mathbf{H} \in \mathcal{AH}$ *and* $W \subseteq V^{\mathbf{H}}$, $\big([W]^s_{\mathbf{H}}\big)^* = [W]^s_{\mathbf{H}^*}$.

Proof. It is not difficult to show, using Proposition 2.8, that $V^{([W]^s_{\mathbf{H}})^*} = V^{[W]^s_{\mathbf{H}^*}}$. Hence and by Proposition 2.5(d) we obtain the equality of hypergraphs. □

Now we show that for each algebraic (directed) hypergraph $\mathbf{H}$ the function * induces isomorphisms between lattices of subhypergraphs of a given kind of $\mathbf{H}$ and $\mathbf{H}^*$.

Theorem 2.10. *Let* $\mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{H} \in \mathcal{DH}\big)$. *Then* $\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{S}_w(\mathbf{H}^*) \simeq \mathbf{S}_w(\mathbf{H}^{**})$, $\mathbf{S}_r(\mathbf{H}) \simeq \mathbf{S}_r(\mathbf{H}^*) \simeq \mathbf{S}_r(\mathbf{H}^{**})$, $\mathbf{S}_s(\mathbf{H}) \simeq \mathbf{S}_s(\mathbf{H}^*)$, $\mathbf{S}_d(\mathbf{H}) \simeq \mathbf{S}_d(\mathbf{H}^*)$ $\big(\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{S}_w(\mathbf{H}^*)$, $\mathbf{S}_r(\mathbf{H}) \simeq \mathbf{S}_r(\mathbf{H}^*)\big)$.

Proof. Let $\varphi\colon S_w(\mathbf{H}) \longrightarrow S_w(\mathbf{H}^*)$ be a function such that $\varphi(\mathbf{G}) = \mathbf{G}^*$ for each $\mathbf{G} \leq_w \mathbf{H}$. (We assume that $\mathbf{H}$ is algebraic or directed). First, φ is well-defined, because $\mathbf{G}^*$ is, of course, a weak subhypergraph of $\mathbf{H}^*$. Secondly, φ is injective, because if $\varphi(\mathbf{G}_1) = \varphi(\mathbf{G}_2)$, then $V^{\mathbf{G}_1} = V^{\mathbf{G}_2}$ and $E^{\mathbf{G}_1} = E^{\mathbf{G}_2}$, so $\mathbf{G}_1 = \mathbf{G}_2$ by Proposition 2.5(c). Thirdly, let $\mathbf{K} \leq_w \mathbf{H}^*$ be a weak subhypergraph and take the triple $\mathbf{G} = \langle V^{\mathbf{K}}, E^{\mathbf{K}}, I^{\mathbf{H}}|_{E^{\mathbf{K}}}\rangle$. Then obviously $\mathbf{G}$ is a weak subhypergraph of $\mathbf{H}$ and $\varphi(\mathbf{H}) = \mathbf{K}$. Hence, φ is a bijection. Thus, since $\mathbf{S}_w(\mathbf{H})$ and $\mathbf{S}_w(\mathbf{H}^*)$ are total algebras, it is sufficient to show

$$(1) \quad (\mathbf{H}_1 \wedge \mathbf{H}_2)^* = \mathbf{H}_1^* \wedge \mathbf{H}_2^* \text{ and } (\mathbf{H}_1 \vee \mathbf{H}_2)^* = \mathbf{H}_1^* \vee \mathbf{H}_2^* \qquad \text{for each } \mathbf{H}_1, \mathbf{H}_2 \in \mathbf{S}_w(\mathbf{H})$$

By the definition of the operations $\wedge$ and $\vee$ we have the following equalities:

$$V^{(\mathbf{H}_1\wedge\mathbf{H}_2)^*} = V^{\mathbf{H}_1\wedge\mathbf{H}_2} = V^{\mathbf{H}_1} \cap V^{\mathbf{H}_2} = V^{\mathbf{H}_1^*} \cap V^{\mathbf{H}_2^*} = V^{\mathbf{H}_1^*\wedge\mathbf{H}_2^*},$$
$$E^{(\mathbf{H}_1\wedge\mathbf{H}_2)^*} = E^{\mathbf{H}_1\wedge\mathbf{H}_2} = E^{\mathbf{H}_1} \cap E^{\mathbf{H}_2} = E^{\mathbf{H}_1^*} \cap E^{\mathbf{H}_2^*} = E^{\mathbf{H}_1^*\wedge\mathbf{H}_2^*},$$
$$V^{(\mathbf{H}_1\vee\mathbf{H}_2)^*} = V^{\mathbf{H}_1\vee\mathbf{H}_2} = V^{\mathbf{H}_1} \cup V^{\mathbf{H}_2} = V^{\mathbf{H}_1^*} \cup V^{\mathbf{H}_2^*} = V^{\mathbf{H}_1^*\vee\mathbf{H}_2^*},$$
$$E^{(\mathbf{H}_1\vee\mathbf{H}_2)^*} = E^{\mathbf{H}_1\vee\mathbf{H}_2} = E^{\mathbf{H}_1} \cup E^{\mathbf{H}_2} = E^{\mathbf{H}_1^*} \cup E^{\mathbf{H}_2^*} = E^{\mathbf{H}_1^*\vee\mathbf{H}_2^*}.$$

Thus by Proposition2.5(c), since $(\mathbf{H}_1 \wedge \mathbf{H}_2)^*, (\mathbf{H}_1 \vee \mathbf{H}_2)^*, \mathbf{H}_1^* \wedge \mathbf{H}_2^*, \mathbf{H}_1^* \vee \mathbf{H}_2^* \in \mathbf{S}_w(\mathbf{H}^*)$, we obtain (1).

Now take $\mathbf{G} \leq_s \mathbf{H}$ (we assume that $\mathbf{H} \in \mathcal{AH}$). Then by a simple verification we obtain that $\mathbf{G}^*$ is a strong subhypergraph of $\mathbf{H}^*$. Hence, $\varphi|_{\mathbf{S}_s(\mathbf{H})}$ is an injection of $\mathbf{S}_s(\mathbf{H})$ into $\mathbf{S}_s(\mathbf{H}^*)$. It is also not difficult to see that for any $\mathbf{K} \leq_s \mathbf{H}^*$, the weak subhypergraph $\mathbf{G} = \langle V^{\mathbf{K}}, E^{\mathbf{K}}, I^{\mathbf{H}}|_{E^{\mathbf{K}}}\rangle$ of $\mathbf{H}$ is a strong subhypergraph. Thus $\varphi|_{\mathbf{S}_s(\mathbf{H})}$ is bijective. Moreover, it satisfies the first equality of (1) (for $\mathbf{H}_1, \mathbf{H}_2 \leq_s \mathbf{H}$), because the lattices of weak and strong subhypergraphs have the same operation $\wedge$. Thus we must only show the second. By Corollary 2.9 we infer

$$V^{(\mathbf{H}_1\vee\mathbf{H}_2)^*} = V^{\mathbf{H}_1\vee\mathbf{H}_2} = V^{[V^{\mathbf{H}_1}\cup V^{\mathbf{H}_2}]^s_{\mathbf{H}}} = V^{([V^{\mathbf{H}_1}\cup V^{\mathbf{H}_2}]^s_{\mathbf{H}})^*} = V^{[V^{\mathbf{H}_1}\cup V^{\mathbf{H}_2}]^s_{\mathbf{H}^*}} = V^{\mathbf{H}_1^*\vee\mathbf{H}_2^*}.$$

Hence and by Proposition 2.6(d), since $(\mathbf{H}_1 \vee \mathbf{H}_2)^*, \mathbf{H}_1^* \vee \mathbf{H}_2^* \in \mathbf{S}_s(\mathbf{H})$, we obtain $(\mathbf{H}_1 \vee \mathbf{H}_2)^* = \mathbf{H}_1^* \vee \mathbf{H}_2^*$.

In a similar way we can show that $\varphi|_{\mathbf{S}_r(\mathbf{H})}$ (for algebraic and directed hypergraphs) and $\varphi|_{\mathbf{S}_d(\mathbf{H})}$ (only for algebraic) are isomorphisms of $\mathbf{S}_r(\mathbf{H})$ and $\mathbf{S}_d(\mathbf{H})$ onto $\mathbf{S}_r(\mathbf{H}^*)$ and $\mathbf{S}_d(\mathbf{H}^*)$ respectively. Hence and by the above proof, $\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{S}_w(\mathbf{H}^{**})$ and $\mathbf{S}_r(\mathbf{H}) \simeq \mathbf{S}_r(\mathbf{H}^{**})$ (for algebraic), and these isomorphisms are provided by $\psi : S_w(\mathbf{H}) \longrightarrow S_w(\mathbf{H}^{**})$ such that $\psi(\mathbf{G}) = \mathbf{G}^{**}$ for $\mathbf{G} \leq_w \mathbf{H}$ and $\psi|_{\mathbf{S}_r(\mathbf{H})}$. □

Of course, isomorphic algebraic (directed) hypergraphs have isomorphic lattices of subhypergraphs. Now Theorem 2.10 implies the following stronger results:

Corollary 2.11. (a) *For each* $\mathbf{G}, \mathbf{H} \in \mathcal{DH}$, *if* $\mathbf{G}^* \simeq \mathbf{H}^*$, *then* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H})$ *and* $\mathbf{S}_r(\mathbf{G}) \simeq \mathbf{S}_r(\mathbf{H})$.

(b) *For each* $\mathbf{G}, \mathbf{H} \in \mathcal{AH}$, *if* $\mathbf{G}^* \simeq \mathbf{H}^*$ $\big(\mathbf{G}^{**} \simeq \mathbf{H}^{**}\big)$, *then* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H})$, $\mathbf{S}_r(\mathbf{G}) \simeq \mathbf{S}_r(\mathbf{H})$, $\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{S}_s(\mathbf{H})$ *and* $\mathbf{S}_d(\mathbf{G}) \simeq \mathbf{S}_d(\mathbf{H})$ $\big(\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H})$ *and* $\mathbf{S}_r(\mathbf{G}) \simeq \mathbf{S}_r(\mathbf{H})\big)$.

Observe that in the case of the relative subhypergraph lattice the above results are trivial. Because $\mathbf{S}_r(\mathbf{H})$ is just the powerset lattice of $V^{\mathbf{H}}$ (see Proposition 2.6(b)) and $V^{\mathbf{H}^{**}} = V^{\mathbf{H}}$, $V^{\mathbf{H}^*} = V^{\mathbf{H}}$, and moreover, the assumptions $\mathbf{G}^* \simeq \mathbf{H}^*$ and $\mathbf{G}^{**} \simeq \mathbf{H}^{**}$ imply, of course, that $|V^{\mathbf{G}}| = |V^{\mathbf{H}}|$.

3.

We assume knowledge of basic concepts and facts from the theory of partial and total algebras, and also from lattice theory (see e.g. [3], [6], [8] and [9]). Recall only that the type of algebras is a pair $\langle K, \kappa\rangle$, where K is a set of operation symbols and κ is a map of K into $\mathbb{N}$ (for $k \in K$, $\kappa(k)$ is the arity of k). Note that $\kappa^{-1}(i)$ is the set of all i–ary operation symbols in K for each $i \in \mathbb{N}$. The class of all partial algebras (of type $\langle K, \kappa\rangle$) will be denoted by $\mathcal{PA}lg$ $\big(\mathcal{PA}lg(K,\kappa)\big)$

Let $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle, \mathbf{B} = \langle B, (k^{\mathbf{B}})_{k\in K}\rangle \in \mathcal{PA}lg(K,\kappa)$. Recall that $\mathbf{B}$ is a weak subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_w \mathbf{A}$) iff $B \subseteq A$ and $k^{\mathbf{B}} \subseteq k^{\mathbf{A}}$ for all $k \in K$. $\mathbf{B}$ is a relative subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_r \mathbf{A}$) iff $B \subseteq A$ and $k^{\mathbf{B}} = k^{\mathbf{A}} \cap (B^{\kappa(k)} \times B)$, for all $k \in K$. $\mathbf{B}$ is a strong subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_s \mathbf{A}$) iff $B \subseteq A$ and $k^{\mathbf{B}} = k^{\mathbf{A}} \cap (B^{\kappa(k)} \times A)$, for all $k \in K$. $\mathbf{B}$ is an initial segment of $\mathbf{A}$ ($\mathbf{B} \leq_d \mathbf{A}$) iff $B \subseteq A$ and $k^{\mathbf{B}} = k^{\mathbf{A}} \cap (A^{\kappa(k)} \times B)$, for all $k \in K$. The sets $S_w(\mathbf{A})$, $S_r(\mathbf{A})$, $S_s(\mathbf{A})$ and $S_d(\mathbf{A})$ of all weak, relative, strong subalgebras and initial segments of $\mathbf{A}$ respectively, with the relations $\leq_w$, $\leq_r$, $\leq_s$ and $\leq_d$ form complete lattices $\mathbf{S}_w(\mathbf{A})$, $\mathbf{S}_r(\mathbf{A})$, $\mathbf{S}_s(\mathbf{A})$ and $\mathbf{S}_d(\mathbf{A})$.

Definition 3.1. Let $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle \in \mathcal{PA}lg(K,\kappa)$. Then

(a) $\mathbf{H}(\mathbf{A})$ is the algebraic hypergraph such that $V^{\mathbf{H}(\mathbf{A})} = A$, $E^{\mathbf{H}(\mathbf{A})} = \{\langle \mathbf{a}, k, b\rangle \in \prod_{fin}(A) \times K \times A : \langle \mathbf{a}, b\rangle \in k^{\mathbf{A}}\}$ and $I^{\mathbf{H}(\mathbf{A})}(\langle \mathbf{a}, k, b\rangle) = \langle \mathbf{a}, b\rangle$ for each $\langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{A})}$.

(b) $\mathbf{H}^*(\mathbf{A}) = \big(\mathbf{H}(\mathbf{A})\big)^*$ and $\mathbf{H}^{**}(\mathbf{A}) = \big(\mathbf{H}(\mathbf{A})\big)^{**}$.

Note that 0–ary operations (i.e. constants) of a partial algebra $\mathbf{A}$ are represented by 0–edges in $\mathbf{H}(\mathbf{A})$ (and in $\mathbf{H}^*(\mathbf{A})$). Note also that this representation of partial algebras by hypergraphs is a generalization of the suitable construction from [1] (see also [16]) for unary partial algebras.

Proposition 3.2. *For* $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle \in \mathcal{PA}lg(K,\kappa)$, $\mathbf{H}(\mathbf{A}) \in \mathcal{AH}\big(|\kappa^{-1}(0)|, |\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \ldots\big)$

Proof. Let $\mathbf{v} = \langle v_1, \ldots, v_m\rangle \in \prod_{fin}(V^{\mathbf{H}(\mathbf{A})})$ (if $m = 0$, then $\mathbf{v} = \emptyset$). Then Definition 3.1 easily implies $E_s^{\mathbf{H}(\mathbf{A})}(\mathbf{v}) = \{\langle \mathbf{a}, k, b\rangle \in A^m \times \kappa^{-1}(m) \times A : \mathbf{a} = \mathbf{v}$ and $\langle \mathbf{a}, b\rangle \in k^{\mathbf{A}}\}$, so we can take a function $\Phi : E_s^{\mathbf{H}(\mathbf{A})}(\mathbf{v}) \longrightarrow \kappa^{-1}(m)$ such that $\Phi\big(\langle \mathbf{v}, k, b\rangle\big) = k$ for each $\langle \mathbf{v}, k, b\rangle \in E_s^{\mathbf{H}(\mathbf{A})}(\mathbf{v})$. Φ is an injection, since $k^{\mathbf{A}}$ is a partial function for all $k \in K$. Hence, $s^{\mathbf{H}(\mathbf{A})}(\mathbf{v}) = |E_s^{\mathbf{H}(\mathbf{A})}(\mathbf{v})| \leq |\kappa^{-1}(m)|$. □

The inverse result is also true. More precisely ($\simeq$ denotes here an isomorphism of undirected, directed and algebraic hypergraphs; recall that the isomorphism of hypergraphs is a pair of bijections of vertex and hyperedge sets respectively, which preserves terminal vertices of hyperedges),

Theorem 3.3. *Let* $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \dots) \in \mathbf{Card}^{\mathbb{N}}$, $\mathbf{H} \in \mathcal{AH}(\underline{\tau})$ *and* $\langle K, \kappa \rangle$ *be an algebra type such that:*

$(*)$ $|\kappa^{-1}(m)| = \tau_m$ *for each* $m \in \mathbb{N}$.

Then there is $\mathbf{A} \in \mathcal{PA}lg(K, \kappa)$ *such that* $\mathbf{H}(\mathbf{A}) \simeq \mathbf{H}$.

Proof. Let $K_m := \kappa^{-1}(m)$ for each $m \in \mathbb{N}$. Then by $(*)$ there are injections $\Phi(\mathbf{v}) : E_s^{\mathbf{H}}(\mathbf{v}) \longrightarrow K_m$ for all $\mathbf{v} \in \prod_{fin}(V^{\mathbf{H}})$, note that $\Phi(\emptyset)$ (for $\mathbf{v} = \emptyset$) is an injection of $E^{\mathbf{H}}(0)$ into the set of all constant symbols. Now, for each $k \in K$, let $\overline{k} \subseteq (V^{\mathbf{H}})^{\kappa(k)} \times V^{\mathbf{H}}$ be a relation such that for each $\langle \mathbf{a}, b \rangle \in (V^{\mathbf{H}})^{\kappa(k)} \times V^{\mathbf{H}}$: $\langle \mathbf{a}, b \rangle \in \overline{k} \iff \exists_{e \in E_s^{\mathbf{H}}(\mathbf{a})} \Phi(\mathbf{a})(e) = k$ and $I_2^{\mathbf{H}}(e) = b$.

Since $\{\Phi(\mathbf{v})\}_{\mathbf{v} \in \prod_{fin}(V^{\mathbf{H}})}$ are injections, $\overline{k}$ is a partial function of $(V^{\mathbf{H}})^{\kappa(k)}$ into $V^{\mathbf{H}}$ for $k \in K$. Thus a pair $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k \in K} \rangle$, where $A = V^{\mathbf{H}}$ and $k^{\mathbf{A}} = \overline{k}$ for $k \in K$, is a partial algebra of type $\langle K, \kappa \rangle$.

Now we prove $\mathbf{H} \simeq \mathbf{H}(\mathbf{A})$. Let $\varphi_V = id_{V^{\mathbf{H}}}$ (i.e. φ_V is the identity function on $V^{\mathbf{H}}$) and let $\varphi_E : E^{\mathbf{H}} \longrightarrow \prod_{fin}(A) \times K \times A$ be a mapping such that $\varphi_E(e) = \langle I_1^{\mathbf{H}}(e), \Phi\big(I_1^{\mathbf{H}}(e)\big)(e), I_2^{\mathbf{H}}(e) \rangle$ for $e \in E^{\mathbf{H}}$. First, it is easily shown that φ_E is injective, because $\{\Phi(\mathbf{v}) : \mathbf{v} \in \prod_{fin}(V^{\mathbf{H}})\}$ is a family of injections. Secondly, by the definition of $\mathbf{A}$ we obtain $V^{\mathbf{H}} = A = V^{\mathbf{H}(\mathbf{A})}$ and $\varphi_E(E^{\mathbf{H}}) = E^{\mathbf{H}(\mathbf{A})}$ and $I^{\mathbf{H}(\mathbf{A})}\big(\varphi_E(e)\big) = I^{\mathbf{H}}(e)$ for $e \in E^{\mathbf{H}}$. Hence, $\varphi = \langle \varphi_V, \varphi_E \rangle$ is an isomorphism of $\mathbf{H}$ onto $\mathbf{H}(\mathbf{A})$. □

The following result is an immediate consequence of Theorem 3.3 (because for each directed (undirected) hypergraph $\mathbf{H}$, there is $\mathbf{D} \in \mathcal{AH}$ such that $\mathbf{D}^* \simeq \mathbf{H}$ $(\mathbf{D}^{**} \simeq \mathbf{H})$):

Corollary 3.4. *Let* $\mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{H} \in \mathcal{DH}$ *or* $\mathbf{H} \in \mathcal{UH}\big)$. *Then there is a partial algebra* $\mathbf{A}$ *such that* $\mathbf{H}(\mathbf{A}) \simeq \mathbf{H}$ $\big(\mathbf{H}^*(\mathbf{A}) \simeq \mathbf{H}$) *or* $\mathbf{H}^{**}(\mathbf{A}) \simeq \mathbf{H}$ *respectively*$\big)$.

Now we prove that for any partial algebra $\mathbf{A}$, its subalgebra lattices are isomorphic to the subhypergraph lattices of $\mathbf{H}(\mathbf{A})$, and thus also of $\mathbf{H}^*(\mathbf{A})$ by Theorem 2.10. These theorems imply, of course, that for every two partial algebras, if their algebraic or directed hypergraphs are isomorphic, then their lattices of subalgebras are isomorphic. To this purpose we have to prove some auxiliary facts.

Proposition 3.5. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{PA}lg(K, \kappa)$. *Then*

(a) $\mathbf{B} \leq_w \mathbf{A}$ *iff* $\mathbf{H}(\mathbf{B}) \leq_w \mathbf{H}(\mathbf{A})$ *iff* $\mathbf{H}^*(\mathbf{B}) \leq_w \mathbf{H}^*(\mathbf{A})$ *iff* $\mathbf{H}^{**}(\mathbf{B}) \leq_w \mathbf{H}^{**}(\mathbf{A})$.

(b) $\mathbf{B} \leq_r \mathbf{A}$ *iff* $\mathbf{H}(\mathbf{B}) \leq_r \mathbf{H}(\mathbf{A})$ *iff* $\mathbf{H}^*(\mathbf{B}) \leq_r \mathbf{H}^*(\mathbf{A})$ *iff* $\mathbf{H}^{**}(\mathbf{B}) \leq_r \mathbf{H}^{**}(\mathbf{A})$.

(c) $\mathbf{B} \leq_s \mathbf{A}$ *iff* $\mathbf{H}(\mathbf{B}) \leq_s \mathbf{H}(\mathbf{A})$ *iff* $\mathbf{H}^*(\mathbf{B}) \leq_s \mathbf{H}^*(\mathbf{A})$.

(d) $\mathbf{B} \leq_d \mathbf{A}$ *iff* $\mathbf{H}(\mathbf{B}) \leq_d \mathbf{H}(\mathbf{A})$ *iff* $\mathbf{H}^*(\mathbf{B}) \leq_d \mathbf{H}^*(\mathbf{A})$.

(e) $\mathbf{A} = \mathbf{B}$ *iff* $\mathbf{H}(\mathbf{A}) = \mathbf{H}(\mathbf{B})$ *iff* $\mathbf{H}^*(\mathbf{A}) = \mathbf{H}^*(\mathbf{B})$ *iff* $\mathbf{H}^{**}(\mathbf{A}) = \mathbf{H}^{**}(\mathbf{B})$.

Proof. Obviously (e) follows from (a) and Proposition 2.5(c).

(a): Assume that $\mathbf{B} \leq_w \mathbf{A}$. Then $V^{\mathbf{H}(\mathbf{B})} \subseteq V^{\mathbf{H}(\mathbf{A})}$. Moreover, for each $k \in K$, $\mathbf{a} \in B^{\kappa(k)}$ and $b \in B$: $\langle \mathbf{a}, k, b \rangle \in E^{\mathbf{H}(\mathbf{B})} \Rightarrow \langle \mathbf{a}, b \rangle \in k^{\mathbf{B}} \Rightarrow \langle \mathbf{a}, b \rangle \in k^{\mathbf{A}} \Rightarrow \langle \mathbf{a}, k, b \rangle \in E^{\mathbf{H}(\mathbf{A})}$. Now applying the definitions of $I^{\mathbf{H}(\mathbf{B})}$ and $I^{\mathbf{H}(\mathbf{A})}$ we get $\mathbf{H}(\mathbf{B}) \leq_w \mathbf{H}(\mathbf{A})$.

Assume that $\mathbf{H}(\mathbf{B}) \leq_w \mathbf{H}(\mathbf{A})$. Then $B \subseteq A$. Moreover, for each $k \in K$ and $\mathbf{a} \in B^{\kappa(k)}$ and $b \in B$: $\langle \mathbf{a}, b\rangle \in k^{\mathbf{B}} \Rightarrow \langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{B})} \Rightarrow \langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{A})} \Rightarrow \langle \mathbf{a}, b\rangle \in k^{\mathbf{A}}$. Hence, $\mathbf{B} \leq_w \mathbf{A}$.

(b): Let $\mathbf{B} \leq_r \mathbf{A}$ and $\langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{A})}$. Then by Definition 3.1 and by the definition of relative subalgebras $I^{\mathbf{H}(\mathbf{A})}(\langle \mathbf{a}, k, b\rangle) \in \prod_{fin}(V^{\mathbf{H}(\mathbf{B})}) \times V^{\mathbf{H}(\mathbf{B})} \Rightarrow \langle \mathbf{a}, b\rangle \in k^{\mathbf{A}}, \mathbf{a} \in B^{\kappa(k)}, b \in B \Rightarrow \langle \mathbf{a}, b\rangle \in k^{\mathbf{B}} \Rightarrow \langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{B})}$. Hence and by (a), $\mathbf{H}(\mathbf{B}) \leq_r \mathbf{H}(\mathbf{A})$.

Let $\mathbf{H}(\mathbf{B}) \leq_r \mathbf{H}(\mathbf{A})$, $k \in K$. Then by the definition of relative subhypergraphs, for all $\mathbf{a} \in A^{\kappa(k)}$, $b \in A$: $\langle \mathbf{a}, b\rangle \in k^{\mathbf{A}}, \mathbf{a} \in B^{\kappa(k)}, b \in B \Rightarrow \langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{A})}, I^{\mathbf{H}(\mathbf{A})}(\langle \mathbf{a}, k, b\rangle) \in \prod_{fin}(V^{\mathbf{H}(\mathbf{B})}) \times V^{\mathbf{H}(\mathbf{B})} \Rightarrow \langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}(\mathbf{B})} \Rightarrow \langle \mathbf{a}, b\rangle \in k^{\mathbf{B}}$. Hence and by (a), $\mathbf{B} \leq_r \mathbf{A}$.

Of course, the second and the third equivalence (in the above two cases) follows from the proof of Theorem 2.10. The analogous proofs of (c) and (d) are omitted. □

Proposition 3.6. *Let* $\mathbf{A} \in \mathcal{P}\mathcal{A}lg(K, \kappa)$. *Then for each* $\mathbf{H} \leq_w \mathbf{H}(\mathbf{A})$, *there is* $\mathbf{B} \leq_w \mathbf{A}$ *such that* $\mathbf{H} = \mathbf{H}(\mathbf{B})$. *If* $\mathbf{H}$ *is a relative or strong or dually strong subhypergraph, then* $\mathbf{B}$ *is a relative or strong subalgebra or an initial segment respectively.*

Proof. Take $\mathbf{H} \leq_w \mathbf{H}(\mathbf{A})$ and let $\mathbf{B} = \langle B, (k^{\mathbf{B}})_{k\in K}\rangle$ be a pair such that $B = V^{\mathbf{H}}$, and for each $k \in K$, $\mathbf{a} \in B^{\kappa(k)}$, $b \in B$: $\langle \mathbf{a}, b\rangle \in k^{\mathbf{B}} \Leftrightarrow \langle \mathbf{a}, k, b\rangle \in E^{\mathbf{H}}$. Then it is trivial that $B \subseteq A$ and $k^{\mathbf{B}} \subseteq k^{\mathbf{A}}$ for each $k \in K$. Hence, $\mathbf{B}$ is a partial algebra and $\mathbf{B} \leq_w \mathbf{A}$. Applying once more the definition of $\mathbf{B}$, it can be easily shown that $\mathbf{H} = \mathbf{H}(\mathbf{B})$. Of course, the second part follows from Proposition 3.5. □

For every $\mathbf{A} \in \mathcal{P}\mathcal{A}lg$ and $B \subseteq A$, the least strong subalgebra of $\mathbf{A}$ containing B, and also its carrier, will here be denoted by $[B]^s_{\mathbf{A}}$.

Proposition 3.7. *Let* $\mathbf{A} \in \mathcal{P}\mathcal{A}lg$ *and* $W \subseteq A$. *Then* $\mathbf{H}([W]^s_{\mathbf{A}}) = [W]^s_{\mathbf{H}(\mathbf{A})}$ *and* $\mathbf{H}^*([W]^s_{\mathbf{A}}) = [W]^s_{\mathbf{H}^*(\mathbf{A})}$.

Proof. Let $\mathbf{H}_1 = \mathbf{H}([W]^s_{\mathbf{A}})$ and $\mathbf{H}_2 = [W]^s_{\mathbf{H}(\mathbf{A})}$. Then by Proposition 3.5(a), $\mathbf{H}_1 \leq_s \mathbf{H}(\mathbf{A})$, so $\mathbf{H}_2 \leq_s \mathbf{H}_1$, because $W \subseteq V^{\mathbf{H}_1}$. Moreover, by Proposition 3.6, there is $\mathbf{B} \leq_s \mathbf{A}$ such that $\mathbf{H}(\mathbf{B}) = \mathbf{H}_2$. Then $[W]^s_{\mathbf{A}} \leq_s \mathbf{B}$, since $W \subseteq V^{\mathbf{H}_2} = B$. Hence and by Proposition 3.5(a), $\mathbf{H}_1 \leq_s \mathbf{H}(\mathbf{B}) = \mathbf{H}_2$. Thus $\mathbf{H}_1 \leq_s \mathbf{H}_2 \leq_s \mathbf{H}_1$, so $\mathbf{H}_1 = \mathbf{H}_2$. The second equality is obtained from the first and Corollary 2.9. □

Theorem 3.8. *Let* $\mathbf{A} \in \mathcal{P}\mathcal{A}lg$. *Then*

$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{H}(\mathbf{A}))$, $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{H}(\mathbf{A}))$, $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{H}(\mathbf{A}))$, $\mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{H}(\mathbf{A}))$.

Proof. Let $\varphi\colon S_w(\mathbf{A}) \longrightarrow S_w(\mathbf{H}(\mathbf{A}))$ be a function such that $\varphi(\mathbf{B}) = \mathbf{H}(\mathbf{B})$ for each $\mathbf{B} \leq_w \mathbf{A}$. By Propositions 3.5(e) and 3.6, φ is a well–defined bijection. Moreover, by Proposition 3.5(a), φ and its inverse φ^{-1} preserve the lattice orders. Thus for each $\mathbf{B}_1, \mathbf{B}_2 \in S_w(\mathbf{A})$,

$$\mathbf{H}(\mathbf{B}_1 \wedge \mathbf{B}_2) = \mathbf{H}(\mathbf{B}_1) \wedge \mathbf{H}(\mathbf{B}_2) \quad \text{and} \quad \mathbf{H}(\mathbf{B}_1 \vee \mathbf{B}_2) = \mathbf{H}(\mathbf{B}_1) \vee \mathbf{H}(\mathbf{B}_2) \tag{1}$$

By Proposition 3.5(c), $\varphi|_{\mathbf{S}_s(\mathbf{A})}$ is an injection (since φ is injective) of $\mathbf{S}_s(\mathbf{A})$ into $\mathbf{S}_s(\mathbf{H}(\mathbf{A}))$. Hence and by Proposition 3.6, it is bijective. Moreover, by Proposition 3.5(c), $\varphi|_{\mathbf{S}_s(\mathbf{A})}$ and $(\varphi|_{\mathbf{S}_s(\mathbf{A})})^{-1}$ preserve the orders of $\mathbf{S}_s(\mathbf{A})$ and $\mathbf{S}_s(\mathbf{H}(\mathbf{A}))$ respectively. Thus (1) also holds for each $\mathbf{B}_1, \mathbf{B}_2 \in S_s(\mathbf{A})$.

Using Propositions 3.5(b),(d) and 3.6 we can also obtain that $\varphi|_{\mathbf{S}_r(\mathbf{A})}$ and $\varphi|_{\mathbf{S}_d(\mathbf{A})}$ are lattice isomorphisms of $\mathbf{S}_r(\mathbf{A})$ and $\mathbf{S}_d(\mathbf{A})$ onto $\mathbf{S}_r(\mathbf{H}(\mathbf{A}))$ and $\mathbf{S}_d(\mathbf{H}(\mathbf{A}))$ respectively. □

Corollary 3.9. *Let* $\mathbf{A} \in \mathcal{PA}lg$. *Then*

(a) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{H}^*(\mathbf{A}))$, $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{H}^*(\mathbf{A}))$, $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{H}^*(\mathbf{A}))$, $\mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{H}^*(\mathbf{A}))$.

(b) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{H}^{**}(\mathbf{A}))$, $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{H}^{**}(\mathbf{A}))$.

Proof. (a) and (b) follow from Theorems 2.10 and 3.8. Note that these isomorphisms are provided by the functions $\varphi : S_w(\mathbf{A}) \longrightarrow S_w(\mathbf{H}^*(\mathbf{A}))$ and $\psi : S_w(\mathbf{A}) \longrightarrow S_w(\mathbf{H}^{**}(\mathbf{A}))$ such that $\varphi(\mathbf{B}) = \mathbf{H}^*(\mathbf{B})$ and $\psi(\mathbf{B}) = \mathbf{H}^{**}(\mathbf{B})$ for each $\mathbf{B} \leq_w \mathbf{A}$, and also by $\varphi|_{\mathbf{S}_r(\mathbf{A})}$, $\varphi|_{\mathbf{S}_s(\mathbf{A})}$, $\varphi|_{\mathbf{S}_d(\mathbf{A})}$ and $\psi|_{\mathbf{S}_r(\mathbf{A})}$. □

Of course, isomorphic partial algebras have isomorphic lattices of subalgebras. Now Corollary 3.9 implies the following stronger results (for algebras which can even be of different types):

Corollary 3.10. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{PA}lg$ *be algebras such that* $\mathbf{H}^*(\mathbf{A}) \simeq \mathbf{H}^*(\mathbf{B})$ $\big(\mathbf{H}^{**}(\mathbf{A}) \simeq \mathbf{H}^{**}(\mathbf{B})\big)$. *Then*
$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B})$, $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{B})$, $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{B})$, $\mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{B})$ $\big(\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B}),\ \mathbf{S}_r(\mathbf{A}) \simeq \mathbf{S}_r(\mathbf{B})\big)$.

Hence, in particular, if $\mathbf{H}(\mathbf{A}) \simeq \mathbf{H}(\mathbf{B})$, then the subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ are also isomorphic.

Observe that the above facts in the case of the relative subalgebra lattice are trivial. It is implied by the classical fact that $\mathbf{S}_r(\mathbf{A})$ is isomorphic to the powerset lattice of A (which follows from the simple observation that each subset of A uniquely determines a relative subalgebra of $\mathbf{A}$) and Proposition 2.6(b), because $A = V^{\mathbf{H}(\mathbf{A})} = V^{\mathbf{H}^*(\mathbf{A})} = V^{\mathbf{H}^{**}(\mathbf{A})}$, and the assumptions $\mathbf{H}^*(\mathbf{A}) \simeq \mathbf{H}^*(\mathbf{B})$ and $\mathbf{H}^{**}(\mathbf{A}) \simeq \mathbf{H}^{**}(\mathbf{B})$ imply, of course, $|A| = |B|$.

We assume that the reader knows basic notions and facts concerning lattices (see e.g. [9], [12]). For any complete lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$, its partial order is denoted by $\leq_{\mathbf{L}}$, and 0 denotes the least element of $\mathbf{L}$. Recall that $l \in L \setminus \{0\}$ is an atom iff for all $k \in L$, if $0 \leq_{\mathbf{L}} k \leq_{\mathbf{L}} l$, then $k = 0$ or $k = l$. $l \in L$ is join–irreducible iff for all $k_1, k_2 \in L$, $l = k_1 \vee k_2$ implies $l = k_1$ or $l = k_2$.

Definition 3.11. Let $\mathbf{H} \in \mathcal{UH}$, $v \in V^{\mathbf{H}}$ and $e \in E^{\mathbf{H}}$. Then

(a) $\mathbf{H}(v)$ is the weak subhypergraph of $\mathbf{H}$ which has one vertex v only and no hyperedges.

(b) $\mathbf{H}(e)$ is the weak subhypergraph of $\mathbf{H}$ which has one hyperedge e only and its endpoints as the set of vertices.

(c) $N_V^{\mathbf{H}} = \{\mathbf{H}(v) \in S_w(\mathbf{H}) : \ v \in V^{\mathbf{H}}\}$ and $N_E^{\mathbf{H}} = \{\mathbf{H}(e) \in S_w(\mathbf{H}) : \ e \in E^{\mathbf{H}}\}$.

By a simple verification we obtain the following:

Lemma 3.12. *Let* $\mathbf{H}$ *be a hypergraph. Then* $N_V^{\mathbf{H}}$ *is the set of all atoms of* $\mathbf{S}_w(\mathbf{H})$*, and* $N_E^{\mathbf{H}}$ *is the set of all non-zero and non-atomic join-irreducible elements of* $\mathbf{S}_w(\mathbf{H})$*.*

Now we can show that every hypergraph is uniquely determined by its weak subhypergraph lattice.

Theorem 3.13. *Let* $\mathbf{G}, \mathbf{H} \in \mathcal{UH}$*. Then:* $\mathbf{G} \simeq \mathbf{H}$ *iff* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H})$*.*

Proof. The implication $\Rightarrow$ is obvious.

$\Leftarrow$. First, take an isomorphism $\Phi : \mathbf{S}_w(\mathbf{G}) \longrightarrow \mathbf{S}_w(\mathbf{H})$ and let $\Phi_V = \Phi|_{N_V^{\mathbf{G}}}$ and $\Phi_E = \Phi|_{N_E^{\mathbf{G}}}$. Then by Lemma 3.12, Φ_V (Φ_E) is a bijection of $N_V^{\mathbf{G}}$ $\left(N_E^{\mathbf{G}}\right)$ onto $N_V^{\mathbf{H}}$ $\left(N_E^{\mathbf{H}}\right)$. Secondly, let $\psi_{V,\mathbf{G}} : V^{\mathbf{G}} \longrightarrow N_V^{\mathbf{G}}$ and $\psi_{V,\mathbf{H}} : V^{\mathbf{H}} \longrightarrow N_V^{\mathbf{H}}$ be functions such that $\psi_{V,\mathbf{G}}(w) = \mathbf{G}(w)$ and $\psi_{V,\mathbf{H}}(u) = \mathbf{H}(u)$ for each $w \in V^{\mathbf{G}}$ and $u \in V^{\mathbf{H}}$. Thirdly, let $\psi_{E,\mathbf{G}} : E^{\mathbf{G}} \longrightarrow N_E^{\mathbf{G}}$ and $\psi_{E,\mathbf{H}} : E^{\mathbf{H}} \longrightarrow N_E^{\mathbf{H}}$ be functions such that $\psi_{E,\mathbf{G}}(e) = \mathbf{G}(e)$ and $\psi_{E,\mathbf{H}}(h) = \mathbf{H}(h)$ for every $e \in E^{\mathbf{G}}$ and $h \in E^{\mathbf{H}}$. By Definition 3.11, $\psi_{V,\mathbf{G}}$, $\psi_{V,\mathbf{H}}$, $\psi_{E,\mathbf{G}}$, $\psi_{E,\mathbf{H}}$ are bijections, so we can take $\varphi_V = \psi_{V,\mathbf{H}}^{-1} \circ \Phi_V \circ \psi_{V,\mathbf{G}}$ and $\varphi_E = \psi_{E,\mathbf{H}}^{-1} \circ \Phi_E \circ \psi_{E,\mathbf{G}}$.

We prove that $\varphi = \langle \varphi_V, \varphi_E \rangle$ is an isomorphism of $\mathbf{G}$ onto $\mathbf{H}$. Observe that φ_V (φ_E) is a bijection of $V^{\mathbf{G}}$ $\left(E^{\mathbf{G}}\right)$ onto $V^{\mathbf{H}}$ $\left(E^{\mathbf{H}}\right)$. Thus we must only show $I^{\mathbf{H}}\big(\varphi_E(e)\big) = \varphi_V\big(I^{\mathbf{G}}(e)\big)$ for each $e \in E^{\mathbf{G}}$.

Take $e \in E^{\mathbf{G}}$ and let $w_1, w_2 \in V^{\mathbf{G}}$, $u_1, u_2 \in V^{\mathbf{H}}$ be vertices such that $\{w_1, w_2\} = I^{\mathbf{G}}(e)$ and $\{u_1, u_2\} = I^{\mathbf{H}}(\varphi_E(e))$. Then $\mathbf{G}(w_i) \leq_w \mathbf{G}(e)$ and $\mathbf{H}(u_i) \leq_w \mathbf{H}\big(\varphi_E(e)\big)$ for $i = 1, 2$. Since Φ is a lattice isomorphism, Lemma 3.12 and these facts imply $\Phi_V\big(\mathbf{G}(w_i)\big) \leq_w \Phi_E\big(\mathbf{G}(e)\big)$ and $\Phi_V^{-1}\big(\mathbf{H}(u_i)\big) \leq_w \Phi_E^{-1}\big(\mathbf{H}(\varphi_E(e))\big)$ for $i = 1, 2$. Hence and by the definitions of φ_V, φ_E we obtain $\mathbf{H}\big(\varphi_V(w_i)\big) = \psi_{V,\mathbf{H}}\big(\varphi_V(w_i)\big) = \Phi_V\big(\psi_{V,\mathbf{G}}(w_i)\big) \leq_w \Phi_E\big(\varphi_{E,\mathbf{G}}(e)\big) = \psi_{E,\mathbf{H}}\big(\varphi_E(e)\big) = \mathbf{H}\big(\varphi_E(e)\big)$, and analogously, $\mathbf{G}\big(\varphi_V^{-1}(u_i)\big) \leq_w \mathbf{G}(e)$ for $i = 1, 2$. Thus (see Definition 3.11) $\varphi_V(w_i) \in I^{\mathbf{H}}\big(\varphi_E(e)\big)$ and $\varphi_V^{-1}(u_i) \in I^{\mathbf{G}}(e)$ for $i = 1, 2$. Hence, $I^{\mathbf{H}}\big(\varphi_E(e)\big) = \varphi_V\big(I^{\mathbf{G}}(e)\big)$. $\square$

The following algebraic result on the weak subalgebra lattice is an immediate consequence of Corollary 3.9(b) and Theorem 3.13 (note that partial algebras in this result can be of different types):

Corollary 3.14. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{AP}lg$*. Then:* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B})$ *iff* $\mathbf{H}^{**}(\mathbf{A}) \simeq \mathbf{H}^{**}(\mathbf{B})$*.*

Since for any unary partial algebra, its hypergraph is just a graph, the above theorem is a generalization of the result from [1] for unary algebras (another proof, applying the graph language, is given in [16]).

Theorems 2.10 and 3.13 imply an analogous result for algebraic (directed) hypergraphs:

Corollary 3.15. *Let* $\mathbf{G}, \mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{G}, \mathbf{H} \in \mathcal{DH}\big)$*. Then:* $\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{S}_w(\mathbf{H})$ *iff* $\mathbf{G}^{**} \simeq \mathbf{H}^{**}$ $\big(\mathbf{G}^{*} \simeq \mathbf{H}^{*}\big)$*.*

In the last part of this section we show that the complete algebraic characterization theorem of the weak subalgebra lattice from [1] can be generalized onto hypergraphs. Next we show that for any lattice which satisfies conditions of this characterization the exactly one hypergraph which correspond to this lattice can be constructed directly from this lattice. Finally, we obtain that for a lattice $\mathbf{L}$ and partial algebra $\mathbf{A}$, $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ iff the hypergraph of $\mathbf{L}$ is isomorphic to $\mathbf{H}^{**}(\mathbf{A})$.

Theorem 3.16. *Let $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ be a lattice. Then the following conditions are equivalent:*

(a) *There is $\mathbf{A} \in \mathcal{P}Alg$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$.*
(b) *There is $\mathbf{H} \in \mathcal{AH}$ $\big(\mathbf{H} \in \mathcal{DH}\big)$ such that $\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{L}$.*
(c) *There is exactly one (up to isomorphism) $\mathbf{H} \in \mathcal{UH}$ such that $\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{L}$.*
(d) *$\mathbf{L}$ satisfies the following conditions:*
  (d.1) *$\mathbf{L}$ is an algebraic and distributive lattice,*
  (d.2) *every element is a join of join–irreducible elements,*
  (d.3) *every non–zero join–irreducible element contains only finite (and non–empty) set of atoms,*
  (d.4) *the set of all non–zero and non–atomic join–irreducible elements is an antichain with respect to the lattice order $\leq_{\mathbf{L}}$.*

Proof. The proof of $(a) \Leftrightarrow (d)$ is given in [1]. The equivalence $(a) \Leftrightarrow (b)$ is obtained from Corollary 3.4 and Theorem 3.8 (Corollary 3.9). $(b) \Leftrightarrow (c)$ is obtained from Theorems 2.10, 3.13, since for a directed (undirected) hypergraph $\mathbf{H}$, there is an algebraic (directed) hypergraph $\mathbf{D}$ such that $\mathbf{D}^* \simeq \mathbf{H}$. □

Definition 3.17. Let $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ satisfy (d.1)—(d.4) of Theorem 3.16. Then $\mathbf{U}(\mathbf{L})$ is a hypergraph such that: $V^{\mathbf{U}(\mathbf{L})}$ is the set of all atoms of $\mathbf{L}$, and $E^{\mathbf{U}(\mathbf{L})}$ is the set of all non–zero and non–atomic join–irreducible elements of $\mathbf{L}$, and $I^{\mathbf{U}(\mathbf{L})}(e) = \{v \in V^{\mathbf{U}(\mathbf{L})} : v \leq_{\mathbf{L}} e\}$ for each $e \in E^{\mathbf{U}(\mathbf{L})}$.

A simple verification of this definition (that $\mathbf{U}(\mathbf{L})$ is indeed a hypergraph) is omitted. It is easily shown that for any lattices $\mathbf{L}$ and $\mathbf{K}$, $\mathbf{L} \simeq \mathbf{K}$ implies $\mathbf{U}(\mathbf{L}) \simeq \mathbf{U}(\mathbf{K})$. The following result is also true:

Theorem 3.18. *Let a lattice $\mathbf{L}$ satisfy (d.1) — (d.4) of Theorem 3.16. Then $\mathbf{S}_w\big(\mathbf{U}(\mathbf{L})\big) \simeq \mathbf{L}$.*

Proof. By Theorem 3.16 there is $\mathbf{H} \in \mathcal{UH}$ such that $\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{L}$, so it is sufficient to show $\mathbf{U}(\mathbf{L}) \simeq \mathbf{H}$.

Take an isomorphism $\Phi : \mathbf{S}_w(\mathbf{H}) \longrightarrow \mathbf{L}$ and let (see Definition 3.11) $\Phi_V = \Phi|_{N_V^{\mathbf{H}}}, \Phi_E = \Phi|_{N_E^{\mathbf{H}}}$. Since Φ is a lattice isomorphism, Lemma 3.12 implies that Φ_V (Φ_E) is a bijection of $N_V^{\mathbf{H}}$ $\big(N_E^{\mathbf{H}}\big)$ onto the set of all atoms, i.e. $V^{\mathbf{U}(\mathbf{L})}$ $\big($the set of all non–zero and non–atomic join–irreducible elements, i.e. $E^{\mathbf{U}(\mathbf{L})}\big)$. Moreover, let $\psi_V : V^{\mathbf{H}} \longrightarrow N_V^{\mathbf{H}}$ and $\psi_E : E^{\mathbf{H}} \longrightarrow N_E^{\mathbf{H}}$ be functions such that $\psi_V(v) = \mathbf{H}(v)$ and $\psi_E(e) = \mathbf{H}(e)$ for each $v \in V^{\mathbf{H}}$ and $e \in E^{\mathbf{H}}$. Then ψ_V $\big(\psi_E\big)$ is a bijection of $V^{\mathbf{H}}$ $\big(E^{\mathbf{H}}\big)$ onto $N_V^{\mathbf{H}}$ $\big(N_E^{\mathbf{H}}\big)$ (see Definition 3.11). Hence we infer that $\varphi_V = \Phi_V \circ \psi_V$ and $\varphi_E = \Phi_E \circ \psi_E$ are bijections of $V^{\mathbf{H}}$ and $E^{\mathbf{H}}$ onto $V^{\mathbf{U}(\mathbf{L})}$ and $E^{\mathbf{U}(\mathbf{L})}$

respectively. We want to show that $\varphi = \langle \varphi_V, \varphi_E \rangle$ is an isomorphism of $\mathbf{H}$ and $\mathbf{U}(\mathbf{L})$. Of course, we must only prove $\varphi_V\big(I^{\mathbf{H}}(e)\big) = I^{\mathbf{U}(\mathbf{L})}\big(\varphi_E(e)\big)$ for each $e \in E^{\mathbf{H}}$.

Take $e \in E^{\mathbf{H}}$. Then Definitions 3.11 and 3.17 imply $v \in I^{\mathbf{H}}(e) \Rightarrow \psi_V(v) \leq_w \psi_E(e) \Rightarrow \Phi_V\big(\psi_V(e)\big) \leq_{\mathbf{L}} \Phi_E\big(\psi_E(e)\big) \Rightarrow \varphi_V(v) \leq_{\mathbf{L}} \varphi_E(e) \Rightarrow \varphi_V(v) \in I^{\mathbf{U}(\mathbf{L})}\big(\varphi_E(e)\big)$. Thus $\varphi_V\big(I^{\mathbf{H}}(e)\big) \subseteq I^{\mathbf{U}(\mathbf{L})}\big(\varphi_E(e)\big)$.

Now let $I^{\mathbf{U}(\mathbf{L})}\big(\varphi_E(e)\big) = \{u_1, \dots, u_m\}$, i.e. $u_1, \dots, u_m \leq_{\mathbf{L}} \varphi_E(e)$. Since Φ is a lattice isomorphism, we obtain $\Phi_V^{-1}(u_i) \leq_w \Phi_E^{-1}\big(\varphi_E(e)\big) = \psi_E(e)$ for $i = 1, 2, \dots, m$. Hence and by Definition 3.11 and by Lemma 3.12, $\psi_V^{-1} \circ \Phi_V^{-1}(u_i) \in I^{\mathbf{H}}(e)$ for $i = 1, 2 \dots, m$. Since $\varphi_V^{-1} = \psi_V^{-1} \circ \Phi_V^{-1}$ and $\varphi_E^{-1} = \psi_E^{-1} \circ \Phi_E^{-1}$, we obtain by this fact that $\varphi_V^{-1}(u_i) \in I^{\mathbf{H}}(e)$ for $i = 1, 2, \dots, m$. Thus $I^{\mathbf{U}(\mathbf{L})}\big(\varphi_E(e)\big) \subseteq \varphi_V\big(I^{\mathbf{H}}(e)\big)$.

These two inclusions imply the desired equality. □

Corollary 3.19. *Let $\mathbf{A} \in \mathcal{PA}lg$ and let $\mathbf{L}$ be a lattice satisfying (d.1)—(d.4) of Theorem 3.16. Then*

(a) $\mathbf{U}\big(\mathbf{S}_w(\mathbf{A})\big) \simeq \mathbf{H}^{**}(\mathbf{A})$.
(b) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ *iff* $\mathbf{H}^{**}(\mathbf{A}) \simeq \mathbf{U}(\mathbf{L})$.

Proof. (a): By Corollary 3.9 and Theorem 3.18, $\mathbf{S}_w\big(\mathbf{H}^{**}(\mathbf{A})\big) \simeq \mathbf{S}_w(\mathbf{A})$ and $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w\big(\mathbf{U}\big(\mathbf{S}_w(\mathbf{A})\big)\big)$. Hence and by Theorem 3.13, $\mathbf{U}\big(\mathbf{S}_w(\mathbf{A})\big) \simeq \mathbf{H}^{**}(\mathbf{A})$.

(b): If $\mathbf{S}_w(\mathbf{A})$ and $\mathbf{L}$ are isomorphic, then, of course, $\mathbf{U}\big(\mathbf{S}_w(\mathbf{A})\big) \simeq \mathbf{U}(\mathbf{L})$, so $\mathbf{H}^{**}(\mathbf{A}) \simeq \mathbf{U}(\mathbf{L})$, by (a). If $\mathbf{H}^{**}(\mathbf{A})$ and $\mathbf{U}(\mathbf{L})$ are isomorphic, then their lattices of weak subhypergraphs are also isomorphic. Hence and by Corollary 3.9 and Theorem 3.18 we obtain $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$. □

In a similar way, using Theorem 2.10 we obtain analogous results for hypergraphs:

Corollary 3.20. *Let $\mathbf{H} \in \mathcal{UH}$ $\big(\mathbf{H} \in \mathcal{DH}$ or $\mathbf{H} \in \mathcal{AH}\big)$ and let $\mathbf{L}$ be a lattice satisfying (d.1) — (d.4) of Theorem 3.16. Then*

(a) $\mathbf{U}\big(\mathbf{S}_w(\mathbf{H})\big) \simeq \mathbf{H}$ $\quad\big(\mathbf{U}\big(\mathbf{S}_w(\mathbf{H})\big) \simeq \mathbf{H}^{*}$ *or* $\mathbf{U}\big(\mathbf{S}_w(\mathbf{H})\big) \simeq \mathbf{H}^{**}\big)$.
(b) $\mathbf{S}_w(\mathbf{H}) \simeq \mathbf{L}$ *iff* $\mathbf{H} \simeq \mathbf{U}(\mathbf{L})$ $\big(\mathbf{H}^{*} \simeq \mathbf{U}(\mathbf{L})$, $\mathbf{H}^{**} \simeq \mathbf{U}(\mathbf{L})$ *respectively*$\big)$.

By Proposition 3.2, Theorem 3.3 and Corollary 3.19 we obtain the following simple observation:

Proposition 3.21. *Let $\mathbf{L}$ be a lattice satisfying (d.1)—(d.4) of Theorem 3.16 and $\langle K, \kappa \rangle$ an algebra type. Then there is $\mathbf{A} \in \mathcal{PA}lg(K, \kappa)$ such that $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}$ iff there is $\mathbf{H} \in \mathcal{AH}(\underline{\tau})$ such that $\mathbf{H}^{**} \simeq \mathbf{U}(\mathbf{L})$, where $\tau_k = |\kappa^{-1}(k)|$ for each $k \in \mathbb{N}$.*

Applying this result (and also other results from this paper) we will be able to characterize in the subsequent paper [19] pairs $\big\langle \mathbf{L}, (K, \kappa) \big\rangle$, where $\mathbf{L}$ is a lattice and $\langle K, \kappa \rangle$ is an algebra type, such that there is a partial algebra $\mathbf{A}$ of the type $\langle K, \kappa \rangle$ with $\mathbf{S}_w(\mathbf{A})$ isomorphic to $\mathbf{L}$. Recall that such a characterization for arbitrary algebraic lattices and arbitrary types in the case of total algebras is an important problem of universal algebra (see e.g. [12]) which is not completely solved yet. But for weak subalgebra lattices of partial algebras we will be able to give a complete solution.

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