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GENERALIZED KRAWTCHOUK POLYNOMIALS: NEW PROPERTIES

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Abstract. Orthogonality conditions and recurrence relations are presented for generalized Krawtchouk polynomials. Coefficients are evaluated for the expansion of an arbitrary polynomial in terms of these polynomials and certain special values for generalized Krawtchouk polynomials are obtained. Summations of some of these polynomials and of certain products are also considered.

1. Introduction

MacWilliams, Sloane and Goethals (1972) gave some properties of generalized Krawtchouk polynomials. Sookoo [3] presented orthogonality conditions and recurrence relations.

In this paper we study generalized Krawtchouk polynomials further. In Section 2, we define these polynomials and characters of the Galois field with q elements and also present various notations. In Section 3, further orthogonality relations are obtained and these are used in Section 4 to obtain the coefficients for the expansion of a polynomial in terms of the generalized Krawtchouk polynomials. Section 5 is devoted to recurrence relations. In Section 6, we consider certain summations of generalized Krawtchouk polynomials and also products. All generalized Krawtchouk polynomials can be expressed in terms of certain special values of these polynomials of the lowest order. These special values we present in Section 7.

2. Definitions and notations

Generalized Krawtchouk polynomials are defined in terms of characters (c.f. MacWilliams and Sloane (1978)) of a Galois field $GF(q)$.

Notation. Let $q = p^m$, where p is a prime number and m is a natural number. Let α be a primitive elements of $GF(q)$, the Galois field of order q. If $\beta \in GF(q)$, we set

$$\beta = \beta_0 + \beta_1 + \beta_2\alpha^2 + \cdots + \beta_{m-1}\alpha^{m-1} \,,$$

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where $0 \leq \beta_i \leq p-1$ $(i = 0, 1, \ldots, m-1)$, β can also be considered as the tuple $(\beta_0, \beta_1, \ldots, \beta_{m-1})$.

Definition. For any element $\beta = (\beta_0, \beta_1, \ldots, \beta_{m-1})$ of $GF(q)$ the *character* χ_β of $GF(q)$ is defined by

$$\begin{aligned} \chi_\beta(\lambda) &= \xi^{\beta_0\lambda_0 + \beta_1\lambda_1 + \cdots + \beta_{m-1}\lambda_{m-1}} \\ &\forall\ \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}) \in GF(q)\,, \end{aligned}$$

where ξ is the complex number $\mathrm{e}^{2\pi i/p}$.

Notation. For natural numbers n and q, $V(n,q)$ denotes

$$\left\{ (s_0, s_1, \ldots, s_{q-1}) \middle| s_0, s_1, \ldots, s_{q-1} = 0, 1, \ldots, n \quad \text{and} \ \sum_{i=0}^{q-1} s_i = n \right\}.$$

MacWilliams, Sloane and Goethals (1972) defined generalized Krawtchouk polynomials. An equivalent definition, suitable for our purpose, is the following.

Definition. For elements $\overline{p} = (p_0, p_1, \ldots, p_{q-1})$ and

$$\overline{s} = (s_0, s_1, \ldots, s_{q-1}) \quad \text{of} \quad V(n,q)\,,$$

the generalized Krawtchouk polynomial $K(p_0, p_1, \ldots, p_{q-1}; s_0, s_1, \ldots s_{q-1})$ (or $K(\overline{p}; \overline{s})$) is

$$\sum_{r_{ij}} \frac{\overline{s}!}{\overline{r}_0! \overline{r}_1! \ldots \overline{r}_{q-1}!} \prod_{k=1}^{q-1} \left[\chi(\alpha^{k-1})\right]^{R(k)}$$

where the summation is taken over all natural numbers r_{ij} $(i, j = 0, 1, \ldots, q-1)$ satisfying

$$\sum_{j=0}^{q-1} r_{ij} = s_i \qquad (i = 0, 1, \ldots, q-1)$$

and

$$\sum_{i=0}^{q-1} r_{ij} = p_j \qquad (j = 0, 1, \ldots, q-1)$$

and where

$$\begin{aligned} \overline{s}! &= s_0! s_1! \ldots s_{q-1}! \\ \overline{r_i} &= r_{i0}! r_{i1}! \ldots r_{i(q-1)}! \quad (i = 0, 1, \ldots, q-1) \end{aligned}$$

and

$$\begin{aligned} R(k) = {} & r_{1k} + r_{2(k-1)} + \cdots + r_{k1} \\ & + r_{(k+1)(q-1)} + r_{(k+2)(q-2)} + \cdots + r_{(q-1)(k+1)} \end{aligned}$$

for $k = 1, 2, \ldots, q-1$.

3. Orthogonality relations

Sookoo (1998) has obtained orthogonality relations, which are used to obtain those given below.

Theorem 3.1. *Let $\overline{p}, \overline{t} \in V(n,q), \overline{p} = (p_0, p_1, \ldots, p_{q-1})$ and $\overline{t} = (t_0, t_1, \ldots, t_{q-1})$. Then*

$$\sum_{\overline{s} \in V(n,q)} K(\overline{p}; \overline{s}) \overline{K(\overline{s}; \overline{t})} = q^n \prod_{i=0}^{q-1} \delta_{p_i, t_i}$$

Proof. Let $\overline{s} = (s_0, s_1, \ldots, s_{q-1})$. From Theorem 4.2 of [3]

$$\sum_{\overline{s} \in V(n,q)} \frac{1}{s_0! s_1! \ldots s_{q-1}!} K(\overline{p}; \overline{s}) \overline{K(\overline{t}; \overline{s})}$$
$$= \frac{q^n}{p_0! p_1! \ldots p_{q-1}!} \prod_{i=0}^{q-1} \delta_{p_i, t_i} .$$

Hence, from Lemma 4.4 of [1],

$$\sum_{\overline{s} \in V(n,q)} \frac{1}{s_0! s_1! \ldots s_{q-1}!} K(\overline{p}; \overline{s}) \overline{\left[\frac{s_0! s_1! \ldots s_{q-1}!}{t_0! t_1! \ldots t_{q-1}!} K(\overline{s}, \overline{t}) \right]} = \frac{q^n}{p_0! p_1! \ldots p_{q-1}!} \prod_{i=0}^{q-1} \delta_{p_i, t_i}$$

$$\therefore \sum_{\overline{s} \in V(n,q)} \frac{1}{t_0! t_1! \ldots t_{q-1}!} K(\overline{p}; \overline{s}) \overline{K(\overline{s}; \overline{t})} = \frac{q^n}{p_0! p_1! \ldots p_{q-1}!} \prod_{i=0}^{q-1} \delta_{p_i, t_i}$$

and the result follows. □

4. Krawtchouk expansion of a polynomial

The coefficients for the expansion of a polynomial in terms of Krawtchouk polynomials have been obtained (c.f. MacWilliams and Sloane (1978)). We obtain a similar result for generalized Krawtchouk polynomials.

Theorem 4.1. *Let $\overline{x} = (x_0, x_1, \ldots, x_{q-1})$ be a q-tuple of real numbers. If the Krawtchouk expansion of a polynomial $\alpha(\overline{x})$ is*

$$\text{(I)} \qquad \alpha(\overline{x}) = \sum_{\overline{s} \in V(n,q)} \alpha_{\overline{s}} K(\overline{x}; \overline{s})$$

then

$$\alpha_{\overline{l}} = q^{-n} \sum_{\overline{i} \in V(n,q)} \alpha(\overline{i}) \overline{K(\overline{l}; \overline{i})}, \ \forall \ \overline{l} \in V(n,q) .$$

Proof. Setting $\overline{x} = \overline{i}$ and multiplying both sides of Equation (I) by $\overline{K(\overline{l};\overline{i})}$, we obtain

$$\alpha(\overline{i})\ \overline{K(\overline{l};\overline{i})} = \sum_{\overline{s}\in V(n,q)} \alpha_{\overline{s}} \overline{K(\overline{l};\overline{i})}\ K(\overline{i};\overline{s})\,.$$

Summing both sides over all $\overline{i} \in V(n,q)$, we have

$$\begin{aligned}\sum_{\overline{i}\in V(n,q)} \alpha(\overline{i})\overline{K(\overline{l};\overline{i})} &= \sum_{\overline{s}\in V(n,q)} \alpha_{\overline{s}} \sum_{\overline{i}\in V(n,q)} \overline{K(\overline{l};\overline{i})}\ K(\overline{i};\overline{s}) \\ &= \sum_{\overline{s}\in V(n,q)} \alpha_{\overline{s}} q^n \prod_{j=0}^{q-1} \delta_{l_j,s_j} \overset{\text{(Theorem 3.1)}}{=} \alpha_{\overline{l}} q^n\,.\end{aligned}$$

□

5. Recurrence relations

Lemma 5.1. *Let $p_0, p_1, \ldots, p_{q-1}$ be nonnegative integers. The number of ways we can obtain $(p_0 - i_0, p_1 - i_1, \ldots, p_{q-1} - i_{q-1})$ from $(p_0, p_1, \ldots, p_{q-1})$ by subtracting 1 from a single coordinate of $(p_0, p_1, \ldots, p_{q-1})$ and each successive tuple in turn is*

$$\frac{j!}{i_0! i_1! \ldots i_{q-1}!}$$

where $i_0, i_1, \ldots, i_{q-1}$ and j are nonnegative integers satisfying $i_k \le p_k$; $k = 0, 1, \ldots, q-1$ and $\sum_{k=0}^{q-1} i_k = j$.

Proof. We note that the required number of ways is equal to the number of different words that we can form from the letters $L_0, L_1, \ldots, L_{q-1}$, where L_m occurs i_m times $(m = 0, 1, \ldots, q-1)$, and the result follows. □

Theorem 5.2. *Let $u_1 = 1$, $u_l = \chi(\alpha^{l-1})$; $l = 1, 2, \ldots, q-1$, and let j be a nonnegative integer satisfying $j \le s_h$, $h \in \{1, 2, \ldots, q-1\}$. Then*

$$\begin{aligned}&K(p_0, p_1, \ldots, p_{q-1}; s_0, s_1, \ldots, s_{q-1}) \\ &\quad = \sum_{i_0, i_1, \ldots, i_{q-1}} \Big\{ \frac{j!}{i_0! i_1! \ldots i_{q-1}!} u_h^{i_1} u_{h+1}^{i_2} \ldots u_{h-1}^{i_{q-1}} \\ &\qquad \times K(p_0 - i_0, p_1 - i_1, \ldots, p_{q-1} - i_{q-1}; s_0, s_1, \ldots, s_h - j, \ldots, s_{q-1}) \Big\}\end{aligned}$$

where the summation is taken over all nonnegative integers $i_0, i_1, \ldots, i_{q-1}$ satisfying $i_k \le p_k$; k=0,1, ..., q-1 and

$$i_0 + i_1 + \cdots + i_{q-1} = j\,.$$

Proof. From Theorem 4.3 of [3]

$$\begin{aligned}
&K(p_0,p_1,\dots,p_{q-1};s_0,s_1,\dots,s_{q-1})\\
&\quad=K(p_0-1,p_1,\dots,p_{q-1};s_0,s_1,\dots,s_h-1,\dots,s_{q-1})\\
&\qquad+u_hK(p_0,p_1-1,\dots,p_{q-1};s_0,s_1,\dots,s_h-1,\dots,s_{q-1})\\
&\qquad+u_{h+1}K(p_0,p_1,p_2-1,\dots,p_{q-1};s_0,s_1,\dots,s_h-1,\dots,s_{q-1})\\
&\qquad\vdots\\
&\qquad+u_{h-1}K(p_0,p_1,p_2,\dots,p_{q-1}-1;s_0,s_1,\dots,s_h-1,\dots,s_{q-1})\,.
\end{aligned}$$

We apply Theorem 4.3 of [3] repeatedly. In so doing, suppose that we reduce by one the term involving p_l $(l\in\{1,2,\dots,q-1\})$ in one of the Krawtchouk polynomials, e.g.

$$K(p_0-r_0,p_1-r_1,\dots,p_{q-1}-r_{q-1};s_0,s_1,\dots,s_h-r_0-r_1-\dots-r_{q-1},\dots,s_{q-1})$$

is changed to

$$\begin{aligned}
&K(p_0-r_0,p_1-r_1,\dots,p_l-r_l-1,\dots,p_{q-1}-r_{q-1};\\
&\quad s_0,s_1,\dots,s_h-r_0-r_1-\dots-r_{q-1}-1,\dots,s_{q-1})\,.
\end{aligned}$$

We must then multiply the latter by a factor u_{h+l-1}. The coefficient of

$$K(p_0-i_0,p_1-i_1,\dots,p_{q-1}-i_{q-1};s_0,s_1,\dots,s_h-j,\dots,s_{q-1})$$

must therefore be $u_h^{i_1}u_{h+1}^{i_2}\dots u_{h-1}^{i_{q-1}}$ multiplied by the number of ways that we can start with $(p_0,p_1,\dots,p_{q-1})$ and end up with $(p_0-i_0,p_1-i_1,\dots,p_{q-1}-i_{q-1})$, subtracting one each time from one of the entries of each successive tuple.

This coefficient is therefore

$$\frac{j!}{i_0!i_1!\dots i_{q-1}!}u_h^{i_1}u_{h+1}^{i_2}\dots u_{h-1}^{i_{q-1}}\,. \qquad \square$$

Corollary 5.3.

$$\begin{aligned}
&K(p_0,p_1,\dots,p_{q-1};s_0,s_1,\dots,s_{q-1})\\
&=\sum_{i_{ab}}\prod_{a=1}^{q-1}\frac{j_a!}{i_{a0}!i_{ai}!\dots i_{a(q-1)}!}\times\prod_{a=1}^{q-1}u_{h_a}^{i_{a1}}u_{h_a+1}^{i_{a2}}\dots u_{h_a-1}^{i_{a(q-1)}}\\
&\times K(p_0-\sum_{r=1}^{q-1}i_{r0},p_1-\sum_{r=1}^{q-1}i_{r1},\dots,p_{q-1}-\sum_{r=1}^{q-1}i_{r(q-1)};\\
&\quad s_0,s_1-j_1,s_2-j_2,\dots s_{q-1}-j_{q-1})
\end{aligned}$$

where $u_{h_a},u_{h_a+1},\dots,u_{h_a-1}\in\{u_1,u_2,\dots,u_{q-1}\}, u_{h_a}=u_a$ *and when* $u_{h_a}=u_1$, *we take* $u_{h_a}-1$ *as* u_{q-1}, *and the summation is taken over*

$$i_{ab}(a=1,2,\dots,q-1;b=0,1,\dots,q-1)\ni\sum_{b=0}^{q-1}i_{ab}=j_a;\quad a=1,2,\dots,q-1\,.$$

Proof. We apply the previous theorem to $K(p_0,p_1,\dots,p_{q-1};s_0,s_1,\dots,s_{q-1})$ repeatedly, subtracting j_1 from s_1, j_2 from $s_2,\dots,j_{q-1}$ from s_{q-1} in turn. $\square$

6. Summating over $V(n,q)$

We use the generating function to evaluate the summation of $K(\overline{p};\overline{s})$ as $\overline{p}$ varies over $V(n,q)$.

Theoren 6.1.

$$\sum_{\overline{p}\in V(n,q)} K(\overline{p};\overline{s}) = 0 \quad \forall\ \overline{s}\in V(n,q)\,.$$

Proof. From MacWilliams' theorem for complete weight enumerators (c.f. MacWilliams, Sloane and Goethals (1972)) and Theorem 4.1 of [3],

$$\left(\sum_{i=0}^{q-1} z_i\right)^{s_0} \prod_{j=1}^{q-1}\left(Z_0 + \sum_{i=1}^{q-1}\chi(\alpha^{i+j-2})z_i\right)^{s_j} = \sum_{\overline{p}\in V(n,q)} K(\overline{p};\overline{s}) \prod_{i=0}^{q-1} z_i^{p_i}\,.$$

Setting $z_0 = z_1 = \cdots = z_{q-1} = 1$, the result follows quickly, since $1+\chi(1)+\chi(\alpha)+\cdots+\chi(\alpha^{q-2}) = 0$. $\square$

Theorem 6.2. *Let $i\in\{1,2,\ldots,q-1\}$ and let r_i $(i=0,1,\ldots,q-1)$ be nonnegative integers whose sum is n. Then*

$$\begin{aligned}\sum_{\overline{s}\in V(n,q)} &\{K(r_0,\ldots,r_i+1,\ldots,r_{q-1};s_0,\ldots,s_i+1,\ldots,s_{q-1})\\ &\times\overline{K(r_0,r_1,\ldots,r_{q-1};s_0,s_1,\ldots,s_{q-1})}\,\}\\ &=\chi(\alpha^{i-1})\frac{q^n}{r_0!r_1!\ldots r_{q-1}!}\end{aligned}$$

where $\overline{s} = (s_0,s_1,\ldots,s_{q-1})$.

Proof. We use Theorem 4.3 of [3] to expand

$$K(r_0,r_1,\ldots,r_i+1,\ldots,r_{q-1};s_0,s_1,\ldots,s_i+1,\ldots,s_{q-1})$$

and then apply Theorem 4.2 of [3] to give the required result. $\square$

7. Special values

For the smallest values of s_i and p_j for which we get non-trivial generalized Krawtchouk polynomials, the values of these polynomials are given in the next result.

Theorem 7.1.

(i) *If $s_0 = p_0 = 1$, $s_i = 0$ $(i=1,2,\ldots,q-1)$ and $p_i = 0$ $(j=1,2,\ldots,q-1)$, then*

$$K(\overline{p};\overline{s}) = 1\,.$$

(ii) *If* $s_0 = 1$, $s_i = 0$ $(i = 1, 2, \ldots, q-1)$, $p_j = 0$ $(j = 0, 1, \ldots, b-1, b+1, \ldots, q-1)$ *and* $p_b = 1$, *then*

$$K(\overline{p}; \overline{s}) = 1\,.$$

(iii) *If* $s_i = 0$ $(i = 0, 1, \ldots, a-1, a+1, \ldots, q-1)$, $s_a = 1$, $p_0 = 1$ *and* $p_j = 0$ $(j = 1, 2, \ldots, q-1)$, *then*

$$K(\overline{p}; \overline{s}) = 1\,.$$

(iv) *If* $s_i = 0$ $(i = 0, 1, \ldots, a-1, a+1, \ldots, q-1)$, $s_a = 1$, $p_j = 0$ $(j = 0, 1, \ldots, b-1, b+1, \ldots, q-1)$ *and* $p_b = 1$, *then*

$$K(\overline{p}; \overline{s}) = \begin{cases} \chi(\alpha^{a+b-2}), & \text{if } a+b \le q \\ \chi(\alpha^{a+b-q-1}), & \text{if } a+b > q\,. \end{cases}$$

Proof. Let $s_i = r_{i0} + r_{i1} + \cdots + r_{i(q-1)}$, $p_j = r_{0j} + r_{ij} + \cdots + r_{(q-1)j}$, $i, j \in \{0, 1, \ldots, q-1\}$,

(i) **Case 1.** $r_{ij} = \begin{cases} 1, & \text{if } i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$

(ii) **Case 2.** $r_{ij} = \begin{cases} 1, & \text{if } i = 0, j = b; \\ 0, & \text{otherwise.} \end{cases}$

(iii) **Case 3.** $r_{ij} = \begin{cases} 1, & \text{if } i = a, j = 0; \\ 0, & \text{otherwise.} \end{cases}$

The result is clearly true in the above cases.

(iv) **Case 4.** $r_{ij} = \begin{cases} 1, & i = a, j = b; \\ 0, & \text{otherwise.} \end{cases}$

In the expression for $K(\overline{p}; \overline{s})$, the power of $\chi(\alpha^{i+j-2})$ is

$$\begin{aligned} \{r_{1(j+i-1)} &+ r_{2(i+j-2)} + r_{3(i+j-3)} + \cdots + r_{ij} + r_{(i+1)(j-1)} + \ldots \\ &+ r_{(i+j-1)1} + r_{(i+j)(q-1)} + r_{(i+j+1)(q-2)} + \ldots \\ &+ r_{(i+\frac{q-1}{2})(j+\frac{q-1}{2})} + r_{(i+\frac{q-1}{2}+1)(j+\frac{q-1}{2}-1)} + \ldots + r_{(q-1)(i+j)}\}\,. \end{aligned}$$

If $a + b \le q$, we set $a = i$, $b = j$ and obtain

$$K(\overline{p}; \overline{s}) = \chi(\alpha^{a+b-2})\,.$$

If $a + b > q$, we set

$$a = i + \frac{q-1}{2}, \quad b = j + \frac{q-1}{2}$$

$$\therefore \quad a + b = i + j + q - 1, \quad \therefore \quad i + j - 2 = a + b - q - 1$$

and so

$$K(\overline{p}; \overline{s}) = \chi(\alpha^{a+b-q-1})$$ □

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