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NATURAL OPERATORS IN THE VIEW OF CARTAN GEOMETRIES

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Abstract. We prove, that r-th order gauge natural operators on the bundle of Cartan connections with a target in the gauge natural bundles of the order $(1,0)$ ("tensor bundles") factorize through the curvature and its invariant derivatives up to order $r-1$. On the course to this result we also prove that the invariant derivations (a generalization of the covariant derivation for Cartan geometries) of the curvature function of a Cartan connection have the tensor character. A modification of the theorem is given for the reductive and torsion free geometries.

In [P] we have shown that Cartan connections on principal fibered bundles with a given structure group, say H, with values in $\mathfrak{g}$ ($H \subset G$ Lie groups, $\mathfrak{h}$, $\mathfrak{g}$ their Lie algebras) are (all) sections of a gauge natural bundle which we call *the bundle of Cartan connections* and we will write C for it. In fact it is a bundle of elements of Cartan connections. It is a functor on the category $\mathcal{PB}_m(H)$ of principal bundles with a structure group H and principal bundle morphisms with local diffeomorphisms as base maps. For each principal bundle P the bundle CP can be viewed as a subbundle of the bundle of principal connections on the associated bundle $P \times_H G$. We use the terms gauge natural bundle and gauge natural operator in the sense of [KMS].

We will study r-th order gauge natural operators on the bundle of Cartan connections with gauge natural bundles of the order $(1,0)$ as target spaces. The bundles of the order $(1,0)$ are bundles on which the induced action of morphisms from $\mathcal{PB}_m(H)$ depends only on 1-jets of underlying maps and only on values of morphisms in fibers. The notion of a natural sheaf will be used to describe the results. The natural sheaf is in a way the simplest structure we can introduce on the kernel or on the image of a natural operator. If we say that natural operators factorize through the curvature we have to specify what we mean by that: in

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general we mean that natural operators on Cartan connections factorize through the natural sheaf of their curvatures. Thus we reduce the problem of finding natural operators on Cartan connections to the problem of finding natural operators on the natural sheaf of curvatures of Cartan connections. This is in general still a complicated task. However in some specific cases, like torsion free geometries, we are able to say more about the structure of the natural sheaf of the curvatures. It is a subsheaf in the sheaf which is formed by all sections of an affine bundle.

One of the key results en route to the final theorem is that the natural sheaf of r-th order invariant jets of curvature functions of Cartan connections is of order $(1,0)$, i.e. it has a tensor character. The invariant jet is an object built with the help of the invariant derivation, which could be understood as a generalization of the covariant derivation for Cartan geometries. The tensor character of the invariant jets (among other nice properties of these) shows that they really worth to notice.

Thus our results generalize the theorems from [KMS], sections 28 and 52. The first one says that r-th order natural operators on the bundle of symmetric linear connections factorize through the curvature operator and its covariant derivations up to order $r-1$, while the second one shows that first order gauge natural operators on the bundle of principal connections factorize through the curvature operator. The same result (as in the second theorem) on the first order gauge natural operators but on the bundle of Cartan connections is obtained in [P].

If not explicitly defined the terminology used is taken from [KMS].

1. Preliminaries

We present some basic definitions from [CSS] which we will need. We assume we have a subbalgebra $\mathfrak{g}_- \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h}$.

A Cartan connection on a principal H-bundle P over a m-dimensional manifold is a $\mathfrak{g}$-valued one-form ω ($\mathfrak{g}$ is the Lie algebra of G, H is a closed subgroup of G) with the following properties:

(1) $\omega(X^*) = X$, for $X \in \mathfrak{h}$, where X^* is the fundamental vector field corresponding to X
(2) $(r^a)^*\omega = Ad(a^{-1}) \circ \omega$, where r^a is the right principal action of $a \in H$,
(3) $\omega : T_uP \to \mathfrak{g}$ is an isomorphism for each $u \in P$.

We point out, that $\dim G/H = m$. We talk about the Cartan connection of type $(\mathfrak{g}, H)$.

Curvature. We define the curvature of a Cartan connection by the structure equation:

$$K = \frac{1}{2}[\omega,\omega] + d\omega\,.$$

The curvature is a horizontal 2-form and satisfies the same invariance property as ω does: $(r^a)^*K = \mathrm{Ad}(a^{-1})K$.

Curvature function. Let ω be a Cartan connection, K its curvature. Then the formula

$$\kappa(X,Y) = K(\omega^{-1}(X), \omega^{-1}(Y))\,,$$

well defines a H-equivariant function $\kappa : P \to \mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}$, which is called the curvature function. The curvature function is H-equivariant with respect to the action $\lambda : H \to GL(\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g})$:

$$\lambda(a)\kappa(p) = \mathrm{Ad}(a)\kappa(p)(\mathrm{Ad}_-(a^{-1})(\_), \mathrm{Ad}_-(a^{-1})(\_))\,,$$

where $Ad_- : \mathfrak{h} \to gl(\mathfrak{g}_-)$ is an $\mathfrak{g}_-$-part of the Ad representation: $\mathrm{Ad}_-(a)(X) = [\mathrm{Ad}(a)X]_{g_-}$. And the curvature function is a section in the associated bundle $P \times_\lambda \mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}$.

The invariant differential corresponding to a Cartan connection ω is a mapping $\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathfrak{g}_-^* \otimes V)$ defined by

$$\nabla_X^w s = \mathcal{L}_{\omega^{-1}(X)} s\,.$$

See [CSS] for more details.

The "jet" prolongation of a representation (in fact we should rather use the term "jet-like", but we stick to the terminology introduced in [CSS]). Let $\lambda : H \to GL(V)$ be a representation of H on the vector space V. Then we define the representation λ of $\mathfrak{h}$ on $\mathcal{J}^1 V := V \oplus (\mathfrak{g}_-^* \otimes V)$:

$$\mathcal{J}^1\lambda(Z)(v, \varphi) = (\lambda(Z)(v), \lambda(Z) \circ \varphi - \varphi \circ \mathrm{ad}_-(Z) + \lambda(\mathrm{ad}_{\mathfrak{h}}(Z)(\_))(v))$$

where $\mathrm{ad}_-(Z) : \mathfrak{g}_- \to \mathfrak{g}_-$, $X \to [Z, X]_-$ and $\lambda(\mathrm{ad}_{\mathfrak{h}}(Z)(\_)(v) : \mathfrak{g}_- \to V$, $\lambda(\mathrm{ad}_{\mathfrak{h}}(Z)(\_)(v)(X) = \lambda([Z, X]_{\mathfrak{h}})(v)$.

One can quite easily verify that λ is really an action. The $\mathcal{J}^1$ can be extended to a functor on the category of $\mathfrak{h}-$*representations*: for an $\mathfrak{h}$-module homomorphism f we define $\mathcal{J}^1(f)(v, \varphi) := (f(v), f \circ \varphi)$.

The motivation for this definition comes from the invariant differentiation of a section: if we have a section s of a principal H-bundle P we would like to see the invariant derivation $\nabla_X^\omega s$ again as a section of a bundle. Unfortunately this is not possible, but we can view the object (s, ∇_X^ω) as a section of the bundle $P \times_{\mathcal{J}^1(\lambda)} (V \oplus (\mathfrak{g}_-^* \otimes V))$, where λ is the action above (the form of λ comes from the requirement, that the invariant derivation along vector fields from $\mathfrak{h}$ should give the action λ up to the minus sign (see [CSS] for details).

We can view the above construction also from a different angle. The action λ coincides with a canonical action of $\mathfrak{h}$ on the standard fiber of the first jet prolongation of the associated bundle $P \times_\lambda V$. Thus we get an identification $J^1(P \times_\lambda V) \simeq P \times_{\mathcal{J}^1(\lambda)} \mathcal{J}^1(V)$.

Higher jet prolongations. This procedure allows us to define higher jet (semi-holonomic) prolongations $\mathcal{J}^r(V) = \oplus_{i=0}^r (\otimes^i \mathfrak{g}_- \otimes V)$ with the representation λ of $\mathfrak{h}$ induced from the original λ on V: firstly we define $\mathcal{J}^2(V)$. $\mathcal{J}^2$ is the subspace of $\mathcal{J}^1(\mathcal{J}^1(V)) = (V \oplus (\mathfrak{g}_-^* \otimes V)) \oplus \mathfrak{g}_-^* \otimes (V \oplus (\mathfrak{g}_-^* \otimes V)) = V \oplus (\mathfrak{g}_-^* \otimes V) \oplus (\mathfrak{g}_-^* \otimes V) \oplus (\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes V)$, where the two middle components are equal. One can easily check that then λ really induces an action on $\mathcal{J}^2$.

Inductively we can construct the action on $\mathcal{J}^r(V)$.

Invariant jets. Let $s : P \to V$ be a H-invariant function, ω a Cartan connection on P. We define the mapping

$$j^r_\omega s := (s, \nabla^\omega s, \dots , (\nabla^w)^r s) : P \to \mathcal{J}^k(V)$$

and call it the *invariant r-jet prolongation* of s with respect to ω. However we are going to "misuse" the shorter term *invariant r-jet* for the same thing meaning the invariant r-jet prolongation rather than its value in one point only.

As in the first order case we get by induction

1.1. Lemma. *The invariant r-jet of a section $s \in C^\infty(P \to V)^\lambda$ is an equivariant mapping $j^r_w s : P \to \mathcal{J}^r V$ with respect to the action $\mathcal{J}^r(\lambda)$ constructed above.*

Thus we have an identification of the semiholonomic r-th jet prolongation $\overline{J}^r(P \times_\lambda V)$ of the associated vector bundle $P \times_\lambda V$ with the associated vector bundle $P \times_{\mathcal{J}^r(\lambda)} \mathcal{J}^r(V)$.

The orbit reduction theorem. In the study of natural operators often an orbit reduction theorem plays a key role. We present the following version from [KMS]:

1.2. Theorem. *Let $p : G \to H$ be a Lie group homomorphism with kernel K, S be a G-space, T be an H-space, and let $\pi : S \to T$ be a p-equivariant surjective submersion, i.e. $\pi(gx) = p(g)\pi(x)$ for all $x \in S$, $g \in G$. Given p, we can consider every H-space N to be G-space: $gy = p(g)y$, $g \in G$, $y \in N$. If each $\pi^{-1}(t)$, $t \in T$, is a K-orbit in S, then there is a bijection between the G-equivariant maps $f : S \to N$ and the H-equivariant maps $\varphi : T \to N$ given by $f = \varphi \circ \pi$.*

2. Curvature spaces

The above construction of the jet prolongations of $\mathfrak{h}$-modules can be applied to the vector space $\mathfrak{g}^*_- \wedge \mathfrak{g}^*_- \otimes \mathfrak{g}$ with the action $\lambda : H \to \mathfrak{g}^*_- \wedge \mathfrak{g}^*_- \otimes \mathfrak{g}$, $\lambda(a)\varphi = \mathrm{Ad}(a)\varphi((\mathrm{Ad}_-(a^{-1})(\_), \mathrm{Ad}_-(a^{-1})(\_))$, respectively to the derivation of this action $\lambda : \mathfrak{h} \to \mathfrak{g}^*_- \wedge \mathfrak{g}^*_- \otimes \mathfrak{g}$ (we denote it by the same letter λ). Thus the r-th invariant jet $(\nabla)^r\kappa$ of the curvature function is a section of the bundle $P \times_{\mathcal{J}^r(\lambda)} \mathcal{J}^r(\Lambda^2(\mathfrak{g}^*_-) \otimes \mathfrak{g})$.

But not any section of this bundle is the curvature function of some Cartan connection. Nevertheless the invariant jets of the curvature functions always form so called natural sheaf.

Natural sheafs

Sometimes it is useful not to work with natural bundles but with natural sheafs. This is a notion introduced by D.J. Eck in [Eck]. Some geometrical objects on a manifold are more likely to form a natural sheaf than a natural bundle. For example if we take the natural bundle of real functions on a manifold it is a natural sheaf but if we restrict ourselves to constant functions they cannot be considered as a natural bundle (they are sections in the bundle of all real functions but not all the sections) but they are still a natural sheaf. Similarly as in case of natural bundles we will define natural sheafs as functors and we give the definition just for the natural sheafs over the category $\mathcal{PB}_m(H)$ though they can be defined over different categories as well.

2.1. Definition. A natural sheaf $\mathcal{F}$ over the category $\mathcal{PB}_m(H)$ is a functor on $\mathcal{PB}_m(H)$ such that for each principal bundle $P \to M$, $\mathcal{F}P$ is a sheaf of modules over the sheaf of Lie algebras $\mathcal{X}^r(P)$ of right invariant vector fields on P, that is for any open subset U of M, there is a vector space $\mathcal{F}(U)$ on which the Lie algebra acts by continuous linear operators $\mathcal{L}_X$. Moreover for any open subsets $V \subset U$ the operators are equivariant with respect to the sheaf restriction $\mathcal{F}(U) \to \mathcal{F}(V)$.

Example. Let E be a natural vector bundle on a manifold M. Then the spaces $C^\infty(EU)$ of local sections of EU form a natural sheaf over M. Vector fields $\mathcal{X}(U)$ on U act on EU via Lie derivatives of section. The topology on sections is induced by vector bundle atlas of E. This is a natural sheaf over the category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms. There we have the action of the sheaf of Lie algebras of all vector fields on M.

Remark. We restrict ourselves to natural sheafs over $\mathcal{PB}_m(H)$, where the sheaf of right invariant vector fields act by Lie derivatives, i.e. to subsheafs of gauge natural bundles.

Remark. The definition of the natural sheaf enables us to speak about natural operators between natural sheafs even if they are not natural bundles: if the operators are even linear we substitute the naturality condition $\Phi_*(D(s)) = D(\Phi_* s)$ with its infinitesimal version $D(\mathcal{L}_X s) = \mathcal{L}_X(D(s))$ (D is a natural operator, s element of a sheaf, Φ a $\mathcal{PB}_m(H)$-morphism and X a right invariant vector field). However for nonlinear operators we are not able to compare $D(\mathcal{L}_X s)$ and $\mathcal{L}_X(D(s))$. We have to deal with the *vertical prolongation* of a natural operator: let $D : C^\infty(Y \to M) \to C^\infty(Y' \to M)$ be a natural operator, and $q \in C^\infty(VY \to M)$ a section. Then q is of the form $\frac{\partial}{\partial t}|_0 s_t$ for a family $s_t \in C^\infty(Y)$ and we define the vertical prolongation $VD : C^\infty(VY \to M) \to C^\infty(VY' \to M)$ as

$$VD(q) = VD(\frac{\partial}{\partial t}|_0 s_t) = \frac{\partial}{\partial t}|_0 (Ds_t) \in C^\infty(VY' \to M)\,.$$

Then the infinitesimal version of naturality reads as $VD(\mathcal{L}_X s) = \mathcal{L}_X(D(s))$. See [CS] for details on infinitesimal naturality.

2.2. Lemma. *Every gauge natural operator on $\mathcal{PB}_m(H)$ consists of infinitesimally gauge natural operators D_M (i.e. "every gauge natural operator is infinitesimally gauge natural").*

Proof. See [CS].

Remark. If H is connected, then infinitezimally gauge natural operator D_P can be uniquely extended to a gauge natural operator over $\mathcal{PB}_m(H)^+$ with oriented base manifolds and orientation preserving morphisms.

2.3. Corollary. *The kernel and the image of a finite order gauge natural operator is a (gauge) natural sheaf.*

Remark. If H is connected, then infinitezimally gauge natural operator D_P can be uniquely extended to a gauge natural operator over $\mathcal{PB}_m(H)^+$ with oriented base manifolds and orientation preserving morphisms.

2.4. Lemma. *Let ω be a Cartan connection of type $(\mathfrak{g}, H)$ on a principal bundle P, $s \in (P, V)^\lambda$ a section of an associated bundle $P \times_\lambda V$. Then the invariant jet $(s, \nabla^\omega s) : CP \oplus C^\infty(P, V)^\lambda \to C^\infty(P, V \oplus V \otimes \mathfrak{g}_-^*)^{\mathcal{J}^1(\lambda)}$ is a gauge natural operator.*

Proof. Let $\Phi \in \mathrm{Mor}(\mathcal{PB}_m(H))$ be a principal bundle morphism. The gauge naturality of $\nabla^\omega s$ means $\nabla^{\Phi_*\omega}\Phi_* s = \Phi_*(\nabla^\omega s)$, that is $((\Phi_*\omega)^{-1}(X)) \cdot \Phi_* s = \Phi_*(\omega^{-1}(X) \cdot s)$. Further we have

$$\begin{aligned}\Phi_*\omega(\Phi(u))(\Phi_*(\omega^{-1}(X)(\Phi(u))) &= \Phi_*\omega(\Phi(u))(T\Phi \circ \omega^{-1}(X)(u)) \\ &= (\omega \circ T(\Phi^{-1}) \circ T\Phi \circ \omega^{-1}(X))(u) = X\,.\end{aligned}$$

Thus $(\Phi_*\omega)^{-1}(X) = \Phi_*(\omega^{-1}(X))$ and we get

$$\begin{aligned}((\Phi_*\omega)^{-1}(X)) \cdot \Phi_* s &= \Phi_*(\omega^{-1}(X))\Phi_* s \\ &= Ts \circ T\Phi^{-1} \circ T\Phi \circ \omega^{-1}(X) \circ \Phi^{-1} = (\omega^{-1}(X) \cdot s) \circ \Phi^{-1} \\ &= \Phi_*(\omega^{-1}(X) \cdot s)\end{aligned}$$

□

2.5. Corollary. *The curvatures of the Cartan connections in the bundle of Cartan connections C form a natural sheaf. The r-th order invariant jets of the curvature function of Cartan connections on P form a natural sheaf. It is a subsheaf in the bundle $P \times_{\mathcal{J}^r(\lambda)} \oplus_{i=0}^r (\otimes^i(\mathfrak{g}_-^*) \otimes (\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}))$.*

Notation. We will write $\mathcal{K}_r$ for the natural sheaf of r-th order invariant jets of the curvature function.

Ricci and Bianchi identities. The curvature function and its invariant derivatives are related by Ricci and Bianchi identities:

2.6. Lemma.

$$(\nabla)^k s(u)(X, Y, \dots, Z) = (\mathcal{L}_{\omega^{-1}(Z)} \circ \cdots \circ \mathcal{L}_{\omega^{-1}(Y)} \circ \mathcal{L}_{\omega^{-1}(X)}) s(u)\,.$$

In particular we obtain

$$(\nabla^\omega_X \nabla^\omega_Y - \nabla^\omega_Y \nabla^\omega_X) s = (\nabla^\omega_{[X,Y]} - \nabla^\omega_{\kappa_-(X,Y)} + \lambda(\kappa_h(X, Y)) s\,,$$

the Ricci identity.

Proof. See [CSS]. The prove there is just for special groups and algebras but it works in the general case as well. □

2.7. Lemma. *The Bianchi identity.*

$$\sum_{\text{cyclic}} ([\kappa(X, Y), Z] - \kappa(\kappa_-(X, Y), Z) + \nabla^\omega_Z \kappa(X, Y) + \kappa([X, Y], Z)) = 0\,,$$

for $X, Y, Z \in \mathfrak{g}_-$.

Proof. See [CSS] (again just for a special case).

3. Action of morphisms on the standard fibre of the bundle of Cartan connections and the orbit reduction

We compute the action of $\mathcal{PB}_m(H)$-morphisms on the standard fibre of the bundle of Cartan connections in the coordinate form. Then we use the orbit reduction to obtain the main theorem of this article. To be able to do this we need the coordinate form of the curvature function and of the invariant derivation firstly.

Notation. We indicate the splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_-$ by indices. We use ^{p}, ^{q}, $\dots$ indices for values in the whole $\mathfrak{g}$ (or G), ^{v}, ^{w}, $\dots$ indices for values in $\mathfrak{h}$ (or H), and ^{i}, ^{j}, $\dots$ for values in $\mathfrak{g}_-$. Since T_xM and $\mathfrak{g}_-$ are isomorphic we use the ^{i}, ^{j}, $\dots$ indices also for coordinates on M (or $\mathbb{R}^m$). Thus, for example, we write $\omega^p_i dx^i = \omega^v_i dx^i + \omega^j_i dx^i$ for the $\mathfrak{h}$ and $\mathfrak{g}_-$ parts of the "horizontal" part of a Cartan connection.

3.1. Lemma. *Let ω be a Cartan connection on P, x^i, y^p be coordinates on P given by a local trivialization $P = \mathbb{R}^m \times H$. Then $\omega = \omega^p_v dy^v - \omega^p_i dx^i = \omega_H - \omega^p_i dx^i$, where w_H is the Maurer-Cartan form on H.*

Coordinate form of curvature and its invariant derivatives. We will write $\Gamma^p_i = \omega^p_i(0,e)$ for the Christoffel symbols of Cartan connections. The coordinates on the r-th jet prolongation of the standard fiber of the bundle of Cartan connections will be Γ^p_i, $\Gamma^p_{i_1\dots i_r} = \partial\Gamma^p_i/\partial x^{i_1}\dots\partial x^{i_r}$.

The curvature form K of a Cartan connection is given by the structure equation $K = d\omega + \frac{1}{2}[\omega,\omega]$. Since the curvature is a horizontal 2-form its values on the whole tangent space of the principle fiber bundle are given by values on any subbundle complementary to the vertical subbundle. In particular any local trivialization $P = \mathbb{R}^m \times H$ defines the horizontal bundle $T\mathbb{R}^m \times 0 \subset T\mathbb{R}^m \times TH = TP$ and the curvature is fully determined by values on this subbundle only. Moreover as we have already mentioned, the curvature is H-equivariant along the fiber and thus its value on $T\mathbb{R}^m \times 0$ is given by its value on $\mathbb{R}^m \times 0_e$. In the local trivialization $P = \mathbb{R}^m \times H$ with x^i coordinates on $\mathbb{R}^m$ and y^v on H we can write according to Lemma 3.1 $\omega = \omega^p_v dy^v - \omega^p_i dx^i$. If we substitute this local expression to the structure equation we get the coordinate expression of the horizontal part of the curvature

$$K^p_{ij} dx^i \wedge dx^j = \Gamma^p_{[ij]} dx^i \wedge dx^j + c^p_{qr}\Gamma^q_i\Gamma^r_j dx^i \wedge dx^j \,, \tag{1}$$

where the c^p_{qr} are the structure constants of G: $[e_q, e_r] = c^p_{qr}e_p$, e_p is a base in $\mathfrak{g}$.

For each $u \in P$, $\pi(u) = x$ we have an isomorphism $(T\pi)\circ(w(u))^{-1} : \mathfrak{g}_- \to T_xM$ ($\pi : P \to M$ projection to the base manifold).

Now if we write $\rho = \omega^p_i dx_i = \omega^v_i dx^i + \omega^j_i dx^i$ according to splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_-$, the matrix ω^i_j represents the inverse isomorphism $T_xM \to \mathfrak{g}_-$ in given bases. We can choose the base in $\mathfrak{g}_-$ in such a way that ω^i_j is the identity matrix at some point in the fiber, let's say at $(0,e)$, that is $\Gamma^i_j = \delta^i_j$. Now we can interpret the K^p_{ij} as the coordinate expression of the curvature function $\kappa \in C^\infty(P, \mathfrak{g}^*_- \otimes \mathfrak{g}^*_- \otimes \mathfrak{g})$, $\kappa(u)(X,Y) = K(\omega^{-1}(u)(X), \omega^{-1}(u)(Y))$. Since in the local trivialization $P = \mathbb{R}^m \times H$ vectors $((T\pi)\omega^{-1}(u)(X), 0)$ and $\omega^{-1}(u)(X)$ differ only by a vertical vector, the curvature form has the same value on both of them. If we take the base in $\mathfrak{g}_-$ as above, we can write $\kappa(0,e)(X,Y) = K(\omega^{-1}(0,e)(X),\omega^{-1}(0,e)(Y)) = K((X,0)_{(0,e)},(Y,0)_{(0,e)}) = K^p_{ij}X^iY^j$. And K^p_{ij} are the coordinates of $\kappa(0,e)(X,Y)$.

The invariant differentiation of the curvature function in local coordinates. We express the invariant derivative of the r-th invariant jet of the curvature function in terms of local coordinates on J^r_0C, namely the Christoffel symbols and their partial derivatives. As before we can assume that $\Gamma^i_j = \delta^i_j$, let further $(K^p_{ij},\, K^p_{iji_1},\dots,\, K^p_{iji_1\dots i_r})$ be the coordinate expression of the invariant jet of the curvature function in $(0,e)$. That is $K^p_{iji_1\dots i_r} \simeq (\nabla)^r\kappa(0,e)(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})$, and $(K^p_{ij},\, K^p_{iji_1},\dots,\, K^p_{iji_1\dots i_r}) \simeq j^r_\omega\kappa(0,e)$. We know that the invariant r-jet of the curvature function is a section of the associated bundle $P\times_{(\mathcal{J}^r\lambda)}\oplus^r_{i=1}((\otimes^i\mathfrak{g}^*_-)\otimes(\mathfrak{g}^*_-\wedge\mathfrak{g}^*_-)\otimes\mathfrak{g})$. Further let $\kappa\simeq k^p_{iji_1\dots i_{r+1}}$ be the coordinate expression of the r-th invariant derivative of the curvature function $(\nabla)^r\kappa\in C^\infty(P,\otimes^r\mathfrak{g}^*_-\otimes(\mathfrak{g}^*_-\wedge\mathfrak{g}^*_-\otimes\mathfrak{g}))$ (then $K^p_{iji_1\dots i_r} = k^p_{iji_1\dots i_r}(0,e)$). We also need a coordinate expression of the $(\omega(0,e))^{-1}|_{\mathfrak{g}_-}:\mathfrak{g}_-\to T_{(0,e)}(\mathbb{R}^m\times H)$. Let $X\simeq(X^j)\in\mathfrak{g}_-$. Then $\omega^{-1}(0,e)(X)\simeq\widetilde{\Gamma}^i_jX^j\frac{\partial}{\partial x^i}+\Gamma^v_k(\widetilde{\Gamma}^k_jX^j)\frac{\partial}{\partial y^v}$, where $\widetilde{\Gamma}^i_j$ is the inverse matrix to Γ^i_j.

$$\begin{aligned}\nabla_{(\_)}((\nabla)^r\kappa)(0,e) &\simeq K^p_{iji_1\dots i_{r+1}} = (\omega^{-1}(\_)\cdot\kappa^p_{iji_1\dots i_r})(0,e)\\ &= \frac{\partial K^p_{iji_1\dots i_r}}{\partial x^k}\widetilde{\Gamma}^k_{i_{r+1}} - \lambda(\Gamma^v_j\widetilde{\Gamma}^j_{i_{r+1}})\circ(K^p_{ij},\,K^p_{iji_1},\dots,\,K^p_{iji_1\dots i_r})\\ &= \frac{\partial K^p_{iji_1\dots i_r}}{\partial x^{i_{r+1}}} - \lambda(\Gamma^v_{i_{r+1}})\circ(K^p_{ij},\,K^p_{iji_1},\dots,\,K^p_{iji_1\dots i_r})\,,\end{aligned}$$

where λ is the action of $\mathfrak{h}$ on the invariant jet of the curvature function discussed in the previous section, and the term $\lambda(\Gamma^v_{i_{r+1}})$ stands for the action of the $\mathfrak{h}$-part of the vector $\omega(0,e)(\frac{\partial}{\partial x^{r+1}})\in\mathfrak{g}$ on the invariant r-th jet.

Notation. We will write K_r for the bundle $P\times_{\mathcal{J}^r(\lambda)}(\oplus^r_{i=1}\otimes^i\mathfrak{g}^*_-\otimes(\Lambda^2\mathfrak{g}^*_-\otimes\mathfrak{g}))$.

3.2. Lemma. *The coordinate form of the Bianchi identity. The Bianchi identity from the Lemma 2.6 can be expressed in the terms of local coordinates as*

$$\sum_{\text{cyclic}} c^p_{qk}K^q_{ij} - K^p_{lk}K^l_{ij} + K^p_{lk}c^l_{ij} + K^p_{ijk} = 0\,, \tag{2}$$

where c^p_{qr} are structure constants of G.

Proof. The equations (2) are just transcription of Lemma 2.6 in local coordinates. □

A well-known theorem says that r-th order gauge natural operators between gauge natural bundles correspond to W^r_mG equivariant maps between appropriate jet prolongations of their standard fibers, see [KMS], Section 51.15. We are going to exploit this theorem thus we need to express the action of W^r_mG on the standard fiber of (the jet prolongations of) the bundle of Cartan connection. We present a technical lemma firstly.

3.3. Lemma. *For the curvature function κ^ω of the Cartan connection ω on the principal bundle P and $\mathcal{PB}_m(H)$-morphism Φ holds*

$$\Phi_*\kappa^\omega = \kappa^{(\Phi_*\omega)} = \kappa^\omega \circ \Phi^{-1}\,.$$

That is the action of $\mathcal{PB}_m(H)$-morphisms on the curvature function coincides with the action of $\mathcal{PB}_m(H)$-morphisms on frame forms of sections of an associated bundle $P \times_\lambda V$.

Proof. It is easy to verify that the curvature function can be expressed as $\kappa^\omega(X,Y) = [X,Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)])$. Then

$$\begin{aligned}\kappa^{\Phi_*(\omega)}(u)\;(X,Y) \\ &= [X,Y] - \omega \circ T\Phi^{-1}([T\Phi\circ\omega^{-1}(\Phi^{-1}(u))(X), T\Phi\circ\omega^{-1}(\Phi^{-1}(u))(Y)]) \\ &= [X,Y] - \omega([\omega^{-1}(\Phi^{-1}(u))(X), \omega^{-1}(\Phi^{-1}(u))(Y)]) = \kappa^\omega\circ\Phi^{-1}\,. \qquad \square\end{aligned}$$

The action of $\mathcal{PB}_m(G)$-morphisms on the standard fiber of the bundle of principal connections. Explicitly we compute only the action of W^2_mG on $J^1_0(\mathcal{Q})$ ($\mathcal{Q}$ is the bundle of principal connections). We more or less copy the calculations made in [KMS, Section 52]. Knowing this action we will be able to compute the action of $\mathcal{PB}_m(H)$-morphisms on the standard fiber of Cartan connections quite easily.

For the standard fiber S of the bundle of principal connections we use the identification $S = J^1_0(\mathbb{R}^m \times G)/G \simeq J^1_0(\mathbb{R}^m, G)_e \simeq \mathbb{R}^{m*}\otimes\mathfrak{g}$. In this identification we take $\Gamma(x,e) \in \mathbb{R}^{m*}\otimes\mathfrak{g}$ to represent a connection Γ in the fiber over x. In coordinates we get Γ^p_i, the Christoffel symbols (for a fixed base e_p in $\mathfrak{g}$); then the connection form ω of the connection Γ is related with the coordinates Γ^p_i in the standard fiber via the left Maurer-Cartan form ω_G of G by the formula $\Gamma^p_i dx^i = \omega_G(e) - \omega(x,e)$.

Coordinates on $W^1_m(G)$: for $\mathcal{PB}_m(G)$-isomorphism $\Phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$; $\Phi(x,g) = (f(x), \varphi(x)\cdot g)$, $f(0) = 0$, $\varphi : \mathbb{R}^m \to G$, $\varphi(x) = \Phi(x,e)$, the corresponding element in $W^1_m(G)$ has the coordinates $a = \varphi(0) \in G$, $a^p_i \sim j^1_0(a^{-1}\cdot\varphi(x)) \in \mathbb{R}^{m*}\otimes\mathfrak{g}$, $a^i_j \sim j^1_0 f \in G^1_m$.

The action of Φ on $\Gamma(x) \simeq j^1_x s \in \mathcal{Q}(\mathbb{R}^m\times G)$, is given by the formula $\mathcal{Q}\Phi(j^1_x s) = j^1_{f(x)}\big(y \mapsto (r^{(\varphi(x))^{-1}} \circ \Phi \circ s \circ f^{-1}(y))\big) = j^1_{f(x)}\big(y \mapsto (\mathrm{conj}(\varphi(x)) \circ \mu \circ (l^{(\varphi(x))^{-1}} \circ \varphi, s) \circ f^{-1}(y))\big)$, where r^a is the right multiplication by an element a in G, l is the left multiplication by a in G, and μ is the multiplication in G. This yields then in coordinates (we overline the coordinates changed by the action of Φ):

$$\overline{\Gamma}^p_i(f(x)) = A^p_q(\varphi(x))\big(\Gamma^q_j(x) + a^q_j(x)\big)\widetilde{a}^j_i\,,$$

where $A^p_q(a)$ is a coordinate expression of the adjoint representation of G, $\widetilde{a}^j_i$ is the inverse matrix to a^j_i, that's $\widetilde{a}^i_j \sim j^1_0(f^{-1})$. Especially for $\Gamma(0) \sim j^1_0(s) \in S$ we get

$$\overline{\Gamma}^p_i = A^p_q(a)(\Gamma^q_j + a^q_j)\widetilde{a}^j_i\,.$$

On $W^1_m H$ we have coordinates (a^i_j, a, a^v_i) and these coordinates describe the embedding of $W^1_m H$ into $W^1_m G$ (in $W^1_m G$ has the element $(a^i_j, a, a^v_i) \in W^1_m H$ coordinates $(a^i_j, a, a^v_i, a^m_i = 0)$, where we write $a^p_i \simeq (a^v_i, a^m_i)$ for the $\mathfrak{h}$ and $\mathfrak{g}_-$-part of the coordinates in $W^1_m G$). The action of $W^1_m H$ on Cartan connections is then given by the same formula (one can easily see that it really preserves Cartan connections, that is the regularity of the matrix Γ^m_i).

The action of $W^2_m H$ on $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$. On $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ we have coordinates Γ^p_j and $\Gamma^p_{jk} = \partial\Gamma^p_j/\partial x^k$.

On $W^2_m(G)$ we have the coordinates $(a^i_j,\ a^i_{jk} \simeq \frac{\partial}{\partial x^k} a^i_j,\ a,\ a^p_i,\ a^p_{ij} \simeq \frac{\partial}{\partial x^j} a^p_i)$ (recall the coordinates on $W^1_m(G)$).

The action of $W^2_m(G)$ on $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ is then given by the formula $(j^2_0\Phi)(\Gamma) = j^1_0(\mathcal{Q}\Phi \circ \Gamma \circ f^{-1}) \sim j^1_0(\mathcal{Q}\Phi \circ j^1 s \circ f^{-1})$.

In coordinates we get

$$\begin{aligned} \overline{\Gamma}^p_{ij} = \frac{\partial \overline{\Gamma}^p_i}{\partial x^j} &= \frac{\partial}{\partial x^j}(A^p_q(\varphi \circ f^{-1}(x))(\Gamma^q_k(f^{-1}(x)) + a^q_k(f^{-1}(x)))\widetilde{a}^k_i) \\ &= A^p_q(a)\Gamma^q_{kl}\widetilde{a}^k_i\widetilde{a}^l_j + A^p_q(a)a^q_{kl}\widetilde{a}^k_i\widetilde{a}^l_j + A^p_{qr}(a)\Gamma^q_k a^r_l\widetilde{a}^k_i\widetilde{a}^l_j \\ &\quad + A^p_{qr}(a)a^q_k a^r_l\widetilde{a}^k_i\widetilde{a}^l_j + A^p_q(a)(\Gamma^q_k + a^q_k)\widetilde{a}^k_{ij}\,, \end{aligned} \tag{1}$$

where $A^p_q(a)$ is a coordinate expression of the adjoint representation of G, $\widetilde{a}^j_i$ is the inverse matrix to a^j_i, that's $\widetilde{a}^i_j \sim j^1_0(f^{-1})$, $A^p_{qr}(a) = \frac{\partial}{\partial x^r}A^p_q(a)$ are some functions on G and since A^p_q is a coordinate expression of the adjoint representation we have $A^p_{qr}(e) = c^p_{qr}$.

The coordinates of the image of the canonical inclusion of $W^2_m H$ into $W^2_m G$ are $(a^i_j, a^i_{jk}, a, a^v_i, a^m_i = 0, a^v_{ij}, a^m_{ij} = 0)$. And the action of $W^2_m H$ on $J^1_0(C)$ is again given by the same formulas.

The action of the kernel of the projection $W^2_m H \to G^1_m \times H$ on the curvature, Christoffels and their symmetrizations:

$$\overline{\Gamma}^v_i = \Gamma^v_i + a^v_i\,, \tag{2}$$

$$\overline{\Gamma}^m_i = \Gamma^m_i\,, \tag{3}$$

$$\overline{\Gamma}^v_{(ij)} = \Gamma^v_{(ij)} + a^v_{ij} + A^v_{qr}(e)\Gamma^q_{(i}a^r_{j)} + A^v_{qr}(e)a^q_{(i}a^r_{j)} + (\Gamma^v_k + a^v_k)\widetilde{a}^k_{ij}\,, \tag{4}$$

$$\overline{\Gamma}^m_{(ij)} = \Gamma^m_{(ij)} + A^m_{qr}(e)\Gamma^q_{(i}a^r_{j)} + A^m_{qr}(e)a^q_{(i}a^r_{j)} + \Gamma^m_k\widetilde{a}^k_{ij}\,, \tag{5}$$

$$\overline{K}^p_{ij} = K^p_{ij}\,. \tag{6}$$

The last equality can be justified by the tensoriality of the curvature or it can be verified by direct computation (we substitute (1) into the coordinate expression of the curvature: $K^p_{ij} = \Gamma^p_{[ij]} + c^p_{qr}\Gamma^q_i\Gamma^r_j$ the coordinate expression).

The action of $W^r_m H$ on the r-th invariant jet of the curvature function. Let us recall we use the value of the jet in $(0,e)$ (in the local trivialization) to represent the values of the curvature function in the fiber. For the $\Phi : \mathbb{R}^m \times H \to \mathbb{R}^m \times H$ we use the notation $\Phi \simeq (f, \varphi)$, f the base map, $\varphi : \mathbb{R}^m \to H$ (see above). Then we have

$$\begin{aligned}(\Phi_* j^r_\omega \kappa^\omega)(0,e) &= j^r_{(\Phi_*\omega)} \kappa^{(\Phi_*\omega)}(0,e) \\ &= j^r_{(\Phi_*\omega)}(\Phi_*\kappa^\omega)(0,e) = j^r_\omega \kappa^\omega \circ \Phi^{-1}(0,e) \\ &= j^r_\omega \kappa^\omega(0, \varphi^{-1}(0)) = \mathcal{J}^r(\lambda)(\varphi(0))(j^r_\omega \kappa^\omega(0,e)) \,,\end{aligned}$$

where $\lambda : H \to gl(\mathfrak{g}^*_- \otimes \mathfrak{g}^*_- \otimes \mathfrak{g})$ is just tensor product of the Ad representation (see earlier) and $\mathcal{J}^r(\lambda)$ is the r-th jet prolongation of the representation λ. Especially we can see that the group $W^r_m H = G^r_m \rtimes T^r_m H$ acts on the invariant r-jet of curvature function only by the values in H ($H \subset W^r_m H$, see later for the explicit form of the injection in reductive, locally effective geometries). On the other hand we will show that the kernel of the projection $W^r_m H = G^r_m \rtimes T^r_m H \to G^1_m \times H$ acts transitively on the symmetrizations of Christoffel symbols of Cartan connections.

Applying the invariant derivation formula we can see that in the coordinate expression of the first invariant derivation of the curvature function, the highest order canonical coordinates on $J^2_0(C)$ appears as $\Gamma^p_{[ij]k}$, the antisymmetrization in first two subscripts. Inductively we deduce that in the coordinate expression of $K^p_{iji_1\dots i_r}$, the r-th invariant derivation of the curvature function κ, the $\Gamma^p_{iji_1\dots i_r}$ are present just as antisymetrizations in the first two subscripts. That is the r-th invariant jet (K^p_{ij}, $K^p_{iji_1}$, … , $K^p_{iji_1\dots i_r}$), the symmetrizations $\Gamma^p_{(ij)}$, $\Gamma^p_{(iji_1)}$, … $\Gamma^p_{(iji_1\dots i_r)}$ of the derivations of the Christoffels symbols and the Christoffels symbols Γ^p_i determine uniquely an r-th jet in the $J^r_0 C$, the r-th jet prolongation of the standard fiber of the bundle of the Cartan connections.

From the above computations it is also clear that the kernel of the projection $W^r_m H \to G^1_m \rtimes H$ acts transitively on the symmetrizations and on the Γ^v_i, the $\mathfrak{h}$ part of the Cartan connection. Thus on the set level we are already ready to exploit the orbit reduction theorem and we get

3.4. Proposition. *All $W^{r+1}_m(H)$-equivariant maps from $J^r_0 C$ to any $G^1_m \times H$ space (it is also a $W^{r+1}_m(H)$ space – we define the action of the kernel of the projection $W^{r+1}_m H \to G^1_m \times H$ to be the trivial one) can be factorized through the formal $(r-1)$-st invariant jet of the curvature function mapping $J^r_0 C \to \oplus^{r-1}_{i=0}(\otimes^i \mathfrak{g}^*_- \otimes \mathfrak{g})$ and w_- part of the connection $\omega_- : J^r_0 C \to \mathfrak{g}^*_- \otimes \mathfrak{g}_-$. All maps here are considered as maps between sets without any additional structure.*

We can formulate the previous result in the language of natural sheafs. There is a smooth structure on the natural sheaf $\mathcal{K}_r$ of r-th invariant jet prolongation of curvature function: we know $\mathcal{K}_r \subset C^\infty(K_r)$ and there is a canonical smooth structure on sections of K_r (smooth maps are smoothly parametrized maps, see [KM]). Smooth local maps $\psi : \mathbb{R}^\alpha \to \mathcal{K}_r$ are then those maps which are also smooth as maps to $C^\infty(K_r)$.

There is also the second smooth structure on the stalk of $\mathcal{K}_r$ as it is a factor space of $J^r_0 C$. It remaines unaswered to what extent these two structures coincide. Nevertheless smooth maps with respect to the second structure are also smooth with respect to the first one.

We use the term "the gauge natural sheaf of the order (r, s)" in the sense of the order of gauge natural bundles described in [KMS].

3.5. Theorem. *Let $\mathcal{Z}$ be a tensor gauge natural bundle (bundle of the order $(1, 0)$, see* [KMS]*). Then any r-th order gauge natural operator $D : \mathcal{C} \to \mathcal{Z}$ can be factorized through the $(r-1)$-st invariant jet of the curvature function and g_--part of the connections, that is $D = E \circ (j^{r-1}_\omega \kappa, \omega_-)$, $j^{r-1}_\omega \kappa : J^r(C) \to \mathcal{K}_{r-1}$.*

Remark. Let us summarize what we have proved: there is a natural sheaf in the sheaf of sections of K_r where the values of r-th invariant jets of curvature functions of Cartan connections on P take place. We know this sheaf is limited by Ricci and Bianchi identities and their invariant derivatives. We further know, that this sheaf is $W^r_m(H)$ equivariant and that all gauge natural operators of the order r from CP to some bundle of the order $(1, 0)$ have to factor through it.

The role of ω_- can be well interpreted in the reductive geometries (see later).

Thus we have reduced the problem of finding gauge natural operators of the order r on the bundle of Cartan connections to the problem of searching $W^{r+1}_m H$-equivariant maps from a $W^{r+1}_m(H)$-equivariant subspace of $P \times_{\mathcal{J}^r(\lambda)} \oplus^r_{i=0}(\otimes^i \mathfrak{g}^*_- \otimes V)$ (has to be specified from case to case) to a standard fiber of the bundle to which the natural operators aim. Further the group $W^{r+1}_m H$ acts on the curvatures nontrivially only by values in H. Unfortunately then such $W^r_m(H)$-equivariant maps can give rise to zero operators.

Further there is a question whether all smooth operators on $\mathcal{K}_r$ comes from the smooth maps from the stalk of $\mathcal{K}_r$ (with the smooth structure as factor space of $J^1_0 C$) to the standard fibre of the bundle in question.

There is a hope, that the subbundle in $C^\infty(K_r)$ given by Bianchi and Ricci identities is $\mathcal{K}_r$. In that case there would be a bijective correspondence between $G^1_m \times H$-equivariant maps from the algebraic submanifold of the standard fibre of K_r given by the Ricci and Bianchi identies and r-th order natural operators on the bundle of Cartan connections.

This is true for the bundle of torsion free affine connections: the Ricci and Bianchi identities have exactly the same form (after some identifications) as those for symmetric linear connections. Thus the proof of the theorem 51.16 from [KMS] can be modified for the torsion free affine connections.

4. Reductive geometries

If the algebra $\mathfrak{g}$ allows an $\mathfrak{h}$-module splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_-$ ($\mathfrak{g}_-$ is an ideal in $\mathfrak{g}$), we talk about *reductive* geometries.

If $\mathfrak{g}_-$ is an abelian ideal we can do even more. For $h \in \mathfrak{h}$ we have $\mathrm{ad}(h) : \mathfrak{g}_- \to \mathfrak{g}_-$ and thus we have a homomorphism $\mathfrak{h} \to gl(\mathfrak{g}_-)$, and let us suppose that it is injective, i.e. $\mathfrak{h} \subset gl(\mathfrak{g}_-)$ through the ad representation. In other words we will deal with locally effective models only (see [S]). Further $\omega = \omega_\mathfrak{h} \oplus \omega_-$ and it is known that the form ω_- on a principal H-bundle P over M enables us to identify P with a cover of a subbundle in P^1M the bundle of linear frames on M (see [K], Chapter 2, also for next lemmas; the notation in [K] is however a little different). For a comfort of a reader let us quickly recall this identification: let us fix an isomorphism $\mathfrak{g}_- \simeq \mathbb{R}^m$ (as vector spaces, m is the dimension of M) and let $X \in T_x(M)$, $u \in P$. Then the coordinates of X in the frame u are given by $\omega_-(\gamma(u)(X))$, where $\gamma(u)(X)$ is a horizontal lift of X to u (because ω_- is a horizontal form, we can use any horizontal lift). Let us denote by i the inclusion of P into P^1M.

Since the ω_- has the same equivariant properties as the solder form on P^1M does, this identification is really a morphism of principal bundles.

Remark. Let us spend a few words on the principal bundle structure of the linear frame bundle P^1M of a manifold M. Let $v \in P^1M$ be a linear frame. It can be identified with the mapping $v : T_xM \to \mathbb{R}^m$ which assigns to a vector $X \in T_xM$ his coordinates in the base v. The group $\mathrm{Ad}(H) \subset GL(g_-) = GL(\mathbb{R}^m)$ then acts on P^1M in the following way: for $\mathfrak{h} \in H$ we have $(vh)(X) = Ad(h^{-1})(v(X))$. (A linear frame v can be as well considered as a mapping $\mathbb{R}^m \to T_xM$ and then $A \in GL(\mathbb{R}^m)$ acts as $vh = v \circ h$; we will use either identification according to a situation)

4.1. Lemma. *The inclusion i of P into P^1M is a morphism of principal fiber bundles.*

Proof. Let $u \in P$, then

$$i(uh)(X) = \omega_-(\gamma(uh)(X)) = Ad(h^{-1})\omega_-(u)(X) = ((i(u))h)(X)\,,$$

where we used the "first" principal bundle structure of the linear frame bundle from the previous remark. □

4.2. Lemma. *For the embedding $i : P \to P^1M$ we have*

$$i_*(\omega_-) = \theta\,,$$

where θ is the solder form on P^1M, $\theta(X_u) = u(T\pi(X))$ ($u : T_xM \to \mathbb{R}^m$ a linear frame, $\pi : P^1M \to M$ the projection on the base).

Proof.

$$(i_*\omega_-)(X_u) = \omega_-(T(i^{-1})X_u) = \omega_-(\pi(T(i^{-1})X_u)) = \omega_-(T\pi(X_u)) = u(T\pi(X))\,,$$

where in the second equality we have used the identification $w_- : TP \to \mathbb{R}^m \simeq TM \to (P \times_{\mathrm{Ad}} \mathbb{R}^m)$ and further we know that $i : P \to P^1M$ is fibered over identity. □

Thus ω_h is a connection on an G-structure corresponding to the group H. Further we have the following characterization of the morphisms preserving the form ω_-:

4.3. Lemma. *Let ω be a Cartan connection on the principal H-bundle P over the base manifold M. Then the (local) automorphisms of P preserving the form ω_- are exactly coverings of morphisms of the form $i^*(P^1 f)$, where f is a morphism of the H-structure $i(P) \subset P^1(M)$.*

Proof. Let $\Phi : P \to P$, $\Phi^*(\omega_-) = \omega_-$. For technical reasons let us indicate the inclusion $i : P \to P^1 M$ described above as i_{ω_-}. Then $i_{\omega_-} = i_{\Phi^*(\omega_-)} = i$ and $i_*(\Phi) : i(P) \to i(P)$ is according to previous lemma a $P^1 M$ morphism, even a morphism of the H-structure $i(P) \subset P^1 M$ and therefore it is of the form $P^1 f$, where $f : M \to M$. □

Remark 1. A morphism of linear frame bundles is uniquely determined by its base map, thus it looses sense to speak about the orders of bundles as about two numbers and we describe the order of the natural bundle by a single non-negative integer only. Further we will speak about natural bundles on a given G-structure (an H principal subbundle P in $P^1 M$). These will be the bundles invariant uder the actions of morphisms f of the base manifold preserving the given G-structure ($P^1 f(P) \subset P$).

Remark 2. We know that a Cartan connection on a principal bundle P in a reductive geometry splits to $\omega = \omega_{\mathfrak{h}} + \omega_-$, where $\omega_{\mathfrak{h}}$ is a principal connection on P. Then $i_*(\omega_{\mathfrak{h}})$, where the $i : P \to P^1 M$ is the embedding described above, can be uniquely extended to a linear connection belonging to the H-structure, and with the help of the previous lemma we are able to reformulate the theorem 3.5 for the H-structures.

4.4. Theorem. *Let $H \subset GL(m, \mathbb{R})$ be a linear subgroup. Then any r-th order natural operator D on the bundle of linear connections belonging to a given H-structure $P \subset P^1 M$ with values in a first order natural bundle Z on the H-structure factorizes through up to the r-th order invariant (covariant) derivatives of the curvature and torsion operators.*

We present two proofs. In the first proof we show it is the consequence of the theorem 3.5. The second proof is the straight calculation.

First proof. The theorem 3.5 talks about the gauge natural operators over the category $\mathcal{PB}_m(H)$. We will introduce new, in a way equivalent notion of naturality: let $\mathcal{PB}_m^\theta(H)$ be a category of principal bundles whose objects are H-principal bundles over m-dimensional manifolds together with a horizontal form $\rho : TP \to \mathfrak{g}_-$ with the invariance property $(r^h)^*\rho = Ad(h^{-1})\rho$ and such that $\rho(T_u P) = \mathfrak{g}_-$ for any $u \in P$. We will write P_ρ for the objects. Morphisms between P_ρ and P'_σ are (local) principal bundle morphisms $\Phi : P \to P'$ such that $\Phi^*(\sigma) = \rho$. Let us notice, that for a Cartan connection ω on an H-principal bundle P the object P_{ω_-} is from $\mathcal{PB}_m^\theta(H)$, and on $\mathcal{PB}_m^\theta(H)$ there is a counterpart of the gauge natural bundle of Cartan connections: a functor C_θ which appoints to each object P_ρ a bundle of principal connections on P.

Now let $D : C \to Z$ be a gauge natural operator between the bundle of Cartan connections and a gauge natural bundle Z, i.e. $D(\Phi^*(\omega)) = \Phi^*(D(\omega))$ for a Cartan connection ω on P and a $\mathcal{PB}_m(H)$-morphism Φ. Then we define $D_\theta : C_\theta \to Z$, $D_{P_{\omega_-}}(\omega_h) = D_{\omega_-}(\omega_h) = D(\omega_h \oplus \omega_-)$, where ω_h is an arbitrary principal connection on P and ω_- a horizontal form with the needed properties. And for Cartan connections ω on P and ω' on P' such that $\Phi^*(\omega') = \omega$, $\Phi : P \to P'$ we have:

$$\begin{aligned} D_{\omega_-}(\Phi^*(\omega'_{\mathfrak{h}})) &= D_{\omega_-}(\omega_{\mathfrak{h}}) = D(\omega_{\mathfrak{h}} \oplus \omega_-) \\ &= D(\Phi^*(\omega')) = \Phi^*(D(\omega')) = \Phi^*(D_{\omega'_-}(\omega'_{\mathfrak{h}}))\,, \end{aligned}$$

and the operator D_θ is natural in $\mathcal{PB}^\theta_m(H)$.

Conversely let $D_\theta : C_\theta \to Z$ be an operator commuting with the action of $\mathcal{PB}^\theta_m(H)$-morphisms. Then we have

$$\begin{aligned} D(\Phi^*(\omega')) = D(\omega) &= D_{\omega_-}(\omega_{\mathfrak{h}}) = D_{\omega_-}(\Phi^*(\omega'_{\mathfrak{h}})) \\ &= \Phi^*(D_{\omega_-}(\omega_{\mathfrak{h}})) = \Phi^*(D(\omega))\,, \end{aligned}$$

and D is a gauge natural operator.

Moreover the naturality in $\mathcal{PB}^\theta_m(H)$ is equivalent to the naturality in the sense of G-structures, that is we call an operator on a bundle natural if it commutes with the actions (pullbacks and pushouts) of morphisms of the form $P^1 f$, $f : M \to M$ a local diffeomorphism.

Let D be a natural operator over $\mathcal{PB}^\theta_m(H)$ on C_θ. Then it is trivially a natural operator on linear connections on $P^1(M)$ (we just restrict the set of morphisms).

Conversely let D be a natural operator on linear connections over $P^1(M)$. Then we extend the operator over the whole $\mathcal{PB}^\theta_m(H)$ as follows: $D_{\omega_-}(\omega_{\mathfrak{h}}) = \Phi^* D_\theta(\Phi_*(w_{\mathfrak{h}})) = \Phi^* D(\Phi_*(w_{\mathfrak{h}}))$, where θ is the canonical solder form on $P^1(M)$ and $\Phi : P \to P^1(M)$ is the principal bundle morphism over identy on M such that $\Phi_*(\omega_-) = \theta$ (under these conditions it is unique). Let us verify that this extention is natural: let ω' be a Cartan connection on P', $\Psi : P' \to P$, $\Psi_*(\omega') = \omega$, and let $\Phi' : P' \to P^1(M)$ be the unique morphism such that $\Phi'_*(\omega'_-) = \theta$, then

$$\begin{aligned} D_{\omega_-}(\Psi_*(\omega'_{\mathfrak{h}})) &= \Phi^* D(\Phi_* \Psi_*(\omega'_{\mathfrak{h}})) \\ &= \Phi^* D(\Phi_* \Psi_* (\Phi')^* \Phi'_*(\omega'_{\mathfrak{h}})) \\ &= \Phi^* \Phi_* \Psi_* (\Phi')^* D(\Phi'_*(\omega'_{\mathfrak{h}})) = \Psi_* D_{\omega'_-}(\omega'_{\mathfrak{h}})\,. \qquad \square \end{aligned}$$

Second proof. Theorem 3.5 gives us that the r-th order gauge natural operators from the bundle of the Cartan connections of the type $(\mathfrak{h}, \mathfrak{h} \oplus \mathbb{R}^m)$ to a gauge natural bundle over the category $\mathcal{PB}_m(H)$ factorize through the r-th invariant jet of the curvature function and the ω_--part of the Cartan connections.

In the statement of this theorem we restrict both the set of morphisms to which we require the invariance and the domain of natural operator. Thus it is theoretically possible that the set of natural operators invariant under these new conditions could be different.

We have to make some considerations in local coordinates again. Let us first describe the action of jets of morphisms of the given G-structure on the connections of the G-structure or rather the action of jets of the morphisms on the corresponding jets of germs of connections (derivations of Christoffel symbols).

Let us describe first the set of jets of the morphisms in question as a subgroup of $W^r_m H$. Let $g : \mathbb{R}^m \to \mathbb{R}^m$, $g(0) = 0$ be a local diffeomorphism. Then we have $P^1 g : P^1\mathbb{R}^m \to P^1\mathbb{R}^m$, further $(P^1\mathbb{R}^m)_0 = \mathrm{inv} J^1_0(\mathbb{R}^m, \mathbb{R}^m)_0 = G^1_m$ and $P^1\mathbb{R}^m$ has the canonical structure of the trivial bundle $\mathbb{R}^m \times G^1_m$. Thus $j^1_{(0,e)} P^1 g$ lies in $W^1_m(G^1_m)$. Recall that for $\Phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$ we write $\Phi = (f, \varphi)$, $\Phi(x, g) = (f(x), \varphi(x)g)$ and on $W^1_m(G)$ we have the coordinates $a^i_j = (\frac{\partial}{\partial x^j} f^i)(0), a = \varphi(0)$, $a^p_i = (\frac{\partial}{\partial x^i}(\varphi(0))^{-1}\varphi^p)(0)$, and on $W^2_m(G)$ then we add the coordinates $a^i_{jk} = (\frac{\partial}{\partial x^j \partial x^k} f^i)(0)$,
$a^p_{ij} = (\frac{\partial}{\partial x^i \partial x^j}(\varphi(0))^{-1}\varphi^p)(0)$. Then we have

$$(P^1 g)(x, j^1_0 h) = (g(x), j^1_0(h \circ g(x)))\,,$$

$h : \mathbb{R}^m \to \mathbb{R}^m$, $h(0) = 0$, and for 1-jet in $(0, e)$ we get in coordinates

$$\begin{aligned} j^1_{(0,e)} g &\simeq (b^i_j = \frac{\partial}{\partial x^j} g^i(0), b^i_j = \mathrm{pr}_2((P^1 g)(0, e))\,, \\ \widetilde{b}^i_l b^l_{jk} &= (\frac{\partial}{\partial x^j \partial x^k}(\mathrm{pr}_2((P^1 g)(0, e)))^{-1} g^i)(0))\,. \end{aligned} \tag{7}$$

Since both in G^1_m and $W^1_m G^1_m$ the group operation is defined by jet composition and P^1 is a functor we get the inclusion $i : G^2_m \to W^1_m G^1_m$, $i(b^i_j, b^i_{jk}) = (b^i_j, b^i_j, \widetilde{b}^i_l b^l_{jk})$. And if we consider the equation (7) also for higher jets we obtain the inclusion $i : G^{r+1}_m \to W^r_m G^1_m$,

$$i(a^i_j, a^i_{jk}, \dots, a^i_{j_1 j_2 \dots j_r + 1}) = (a^i_j, a^i_j, \widetilde{a}^i_l a^l_{jk}, \dots, \widetilde{a}^i_l a^l_{j_1 j_2 \dots j_r + 1})\,.$$

Now for an H-principal bundle P, Cartan connections with values in $\mathfrak{g}$ with a fixed ω_- form a subbundle in CP. In coordinates it is given by Γ^m_i constant and Γ^v_k arbitrary. Now we write $\Gamma^i_{j;k}$ instead of Γ^v_k since $\Gamma^v_k \in \mathfrak{h} \otimes \mathbb{R}^m$ and $\mathfrak{h} \subset gl(\mathbb{R}^m)$. The equations (2) can thus be split in two equations:

$$\begin{aligned} \overline{\Gamma}^i_{(j;k)} &= \Gamma^i_{(j;k)} + a^i_{kj}\,, \\ \overline{\Gamma}^i_{[j;k]} &= \Gamma^i_{[j;k]}\,, \end{aligned}$$

that is the kernel of the composition of the inclusion $i : G^1_m \to W^1_m G^1_m$ with the projection of $W^1_m G^1_m \to G^1_m \times G^1_m$ acts transitively on the symmetric part $\Gamma^i_{(j;k)}$ of the Christoffel symbols and leaves the antisymmetrizations (that is the formal torsion) intact. It leaves Γ^m_i, that is the $\mathfrak{g}_-$-part of the connection intact as well. The situation is similar on the first jet prolongation of the standard fiber of the bundle of Cartan connections: the equations (4) and (5) gives us that the kernel of the projection of $i(G^3_m)$ on $G^1_m \times G^1_m$, that is in coordinates $(\delta^i_j, \delta^i_j, a^i_{jk}, a^i_{jkl})$, acts transitively on the symmetrizations of the two jets of the $\mathfrak{h}$-part of the elements of

Cartan connections (that is in coordinates on $\Gamma^i_{(j;kl)}$) and acts trivially on the curvature (in (5) we act transitively by the term $\Gamma^m_k \widetilde{a}^k_{ij}$, in (4) then by the term a^v_{ij}, which corresponds to a^k_{lij}) in the "reductive" notation). Further derivations of (4) and (5) give us that $\ker(\pi^r \circ i(G^{r+1}_m))$, $\pi^r : W^r_m G^1_m \to G^1_m \times G^1_m$ acts transitively on $\Gamma^i_{(j;i_1\dots i_r)}$ and the tensor character of r-th invariant jets of the curvature function implies it leaves the invariant (covariant) derivatives of the curvature intact.

Thus the orbit reduction theorem says that any $i(G^{r+1}_m)$-map from the $J^r_0 C$ to $G^1_m \times G^1_m$-space factors through the antisymetrization of the $\mathfrak{h}$-part of the Christofel symbols, up to r-th order covariant derivatives of the curvature and through up to r-th derivatives of the $\mathfrak{g}_-$-part of the Christoffel symbols and this is in the language of natural operators the statement of the theorem. □

Remark. The action $\mathcal{J}^1(\lambda)$ of $\mathfrak{h}$ on the invariant jet of the curvature function κ can be in reductive geometries simplified. Namely we have

$$\mathcal{J}^1\lambda(Z)(v,\varphi) = (\lambda(Z)(v), \lambda(Z)\circ\varphi - \varphi\circ \mathrm{ad}_-(Z))\,,$$

and this is the "tensor" action corresponding to the following action of the group H:

$$\mathcal{J}^1\lambda(h)(v,\varphi) = (\lambda(h)(v), \lambda(h)(\varphi)\circ \mathrm{Ad}(h^{-1}))\,.$$

It shows also the invariance of the invariant derivative itself. This also gives the higher prolongations of the action: for $\varphi \in \otimes^r \mathfrak{g}^*_- \otimes V$ and $h \in H$ we have

$$\mathcal{J}^r\lambda(h)(\varphi) = \lambda(h)(\varphi)\circ \otimes^r(\mathrm{Ad}(h^{-1})))\,.$$

Torsion free geometries

A Cartan connection (geometry) is *torsion free* if its curvature function takes values in the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ only.

4.5. Lemma. *The functor which appoints to a principal B-bundle the bundle of germs of Cartan connections on P with values in $\mathfrak{g}$ without torsion is a gauge natural bundle. It is a subbundle of the bundle of Cartan connections.*

Proof. If we fix a principal B-bundle P, then Cartan connections without torsion on P correspond to the condition $\kappa_- = \kappa_{\mathfrak{g}_-} = 0$. This condition determines a subbundle in the first jet prolongation of the bundle of Cartan connections on P (in local coordinates it is in each $x \in M$ a system of linear or quadratic equations with constant coefficients). Moreover it is preserved by the action of morphisms of principal fiber bundles (for a Cartan connection ω with curvature form κ and for $\varphi : P \to R$ we have for the curvature form $\kappa_{\varphi_*(\omega)}$ of the image connection $\varphi(\omega)$:

$$\begin{aligned}(\kappa_{\varphi_*(\omega)}) &= (\kappa_{\varphi_*(\omega)})_{\mathfrak{h}} = (\varphi_*\kappa)_{\mathfrak{h}}\\ &= (\varphi_*\kappa_{\mathfrak{h}})_{\mathfrak{h}} = \varphi_*\kappa_{\mathfrak{h}}\,.\end{aligned}$$

The last equality comes from the fact that $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$. See later for the explicit description of the action of $\mathcal{PB}_m(H)$-morphisms on κ. Consequently $(\kappa_{\varphi_*(\omega)})_- = 0$. □

4.6. Lemma. *The coordinate form of Ricci and Bianchi identities. We get then two identities, the first and the second Bianchi identity*

$$K^i_{jkl} + K^i_{klj} + K^i_{ljk} = 0\,, \tag{3}$$

$$K^i_{jklm} + K^i_{jlmk} + K^i_{jmkl} = 0\,. \tag{4}$$

For the Ricci identity we have

$$K^i_{jkl[m_1m_2]} = \sum_n -\lambda(K^n_{m_1m_2})K^i_{jkl}\,. \tag{5}$$

Note that unlike in (2) *only the fourth subscript stands for the invariant derivation.*

Proof. Let us first mention what the homogeneity degree of a multilinear mapping between graded Lie algebras is: let $\mathfrak{g} = \sum_{i=-l}^{k} \mathfrak{g}_i$ be a grading of a Lie algebra $\mathfrak{g}$. Then for a multilinear mapping $f : \Pi_{r=1}^{s}\mathfrak{g}_{i_r} \to \mathfrak{g}_t$ we define the homogeneity degree of f as the number $t - \sum_{r=1}^{s} i_r$.

Now we just consider the equations (2) in torsion free geometry: The term $K^p_{lk}K^l_{ij}$ vanishes, $\mathfrak{g}_-$ is an abelian ideal and $K^p_{lk}c^l_{ij}$ vanishes as well, and the term $c^p_{qk}K^q_{ij}$ has the homogeneity degree 2, the term K^p_{ijk} has the homogeneity degree 3, and thus both of them have to vanish. For $\mathfrak{h} \subset gl(\mathfrak{g}_-)$ then $c^p_{qk}K^q_{ij}$ yields (3) and K^p_{ijk} gives (4). The (5) can be obtained easily from the non-coordinate form of the Ricci identity. □

Remark. Thus Ricci and Bianchi identities define an affine subbundle in $P \times_\lambda (\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g})$ and their invariant derivatives (up to appropriate order) define affine subbundle in $P \times_{\mathcal{J}^r(\lambda)} \otimes^r \mathfrak{g}_-^* \otimes (\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g})$ (which we denote by K^r). We have $\sum_{i=0}^{r} K^r = K_r$.

4.7. Corollary. *Let $H \subset GL(m,\mathbb{R})$ be a linear subgroup. Then any r-th order natural operator D on the bundle of linear connections belonging to a given H-structure $P \subset P^1M$ with values in a first order natural bundle Z on the H-structure factorizes through up to the $(r-1)$-st order invariant (covariant) derivatives of the curvature, i.e. there exists a natural transformation $\tau : K_{r-1} \to Z$ such that $D = \tau \circ j^{\omega}_{r-1}\gamma$, γ is the curvature operator.*

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