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# A FUNCTIONAL MODEL FOR A FAMILY OF OPERATORS INDUCED BY LAGUERRE OPERATOR

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Abstract. The paper generalizes the instruction, suggested by B. Sz.-Nagy and C. Foias, for operatorfunction induced by the Cauchy problem

$$T_t : \begin{cases} th''(t) + (1-t)h'(t) + Ah(t) = 0 \\ h(0) = h_0 (th')(0) = h_1 \end{cases}$$

A unitary dilatation for $T_t$ is constructed in the present paper. then a translational model for the family $T_t$ is presented using a model construction scheme, suggested by Zolotarev, V., [3]. Finally, we derive a discrete functional model of family $T_t$ and operator $A$ applying the Laguerre transform

$$f(x) \to \int_0^\infty f(x)\, P_n(x)\, e^{-x} dx$$

where $P_n(x)$ are Laguerre polynomials [6, 7]. We show that the Laguerre transform is a straightening transform which transfers the family $T_t$ (which is not semigroup) into discrete semigroup $e^{-itn}$.

## Introduction

Functional models for contraction semigroups $Z_t = \exp(itA)$ and $T^n$, $(t \geq 0,\, n \in \mathbb{Z}^+)$ have been constructed by B. Sz.-Nagy and C. Foias [2] at the beginning of 70-s. The bases of this method is a significant concept of dilatation of contraction semigroup. A spectral realization of the dilatation and subsequent narrowing upon the original space leads to a functional model of the contraction semigroup. As a result an operator $A(T)$ in this case is realized by operators which carry out multiplication by independent variable in a specific functional space. The basis of the concept is the Fourier transform of space $L^2$.

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## 1. Preliminary information on the functional model in a Fourier representation

**1.1.** We recall [1] that operator collegation $\Delta$,

$$\Delta = (A, H, \phi, E, \sigma) \tag{1}$$

is a collection of Hilbert spaces $H$ and $E$ and of linear operators $A : H \to H$, $\phi : H \to E$, $\sigma : E \to E$ $(\sigma^* = \sigma)$ where the collegation condition holds:

$$A - A^* = i\phi^* \sigma\phi\,. \tag{2}$$

It is customary to associate with the collegation (1) an open system [1] which is defined by relations

$$\begin{cases} i\frac{d}{dt}\,h(t) + Ah(t) = \phi^*\sigma u(t)\,; \\ h(0) = h_0,\ (t \geq 0)\,; \end{cases}$$
$$v(t) = u(t) - i\phi h(t) \tag{3}$$

where $h(t)$, $u(t)$, $v(t)$ are vector functions from Hilbert spaces $H$ and $E$ respectively. An important role in the further construction of the model representation plays the conservation Law [1].

**Theorem 1.1.** *For the open system* (3) *associated with the collegation* $\Delta$ (1) *the conservation Law holds*

$$\|h_0\|^2 + \int_0^T \langle\sigma u(\zeta)\,, u(\zeta)\rangle\, d\zeta = \|h(T)\|^2 + \int_0^T \langle\sigma v(\zeta), v(\zeta)\rangle\, d\zeta \tag{4}$$

*for any* $T$, $0 \leq T \leq \infty$.

If operator $A$ is selfadjoint then $\phi = 0$, $\sigma = 0$, and Cauchy problem (3) in induced by the semigroup

$$Z_t = \exp(itA)\,, \quad \text{i.e.} \quad h(t) = Z_t h_0$$

and the conservation Law (4) yields $Z_t$.

**1.2.** Let us consider a contractive semigroup $Z_t = \exp(itA)$ $(t \geq 0)$, which has a property $\|Z_t h\| \leq \|h\|$ for all $h \in H$.

A unitary dilatation of contractive semigroup $Z_t$ in $H$ is said to be a unitary semigroup $U_t$ in $\mathcal{H}$ [2] such that the following relation holds:

$$\mathcal{H} \supseteq H\,;\ P_H U_t|_H = Z_t \qquad (t \geq 0) \tag{5}$$

where $P_H$ is an orthoprojector on $H$. The dilatation $U_t$ in $H$ is said to be minimal if

$$\mathcal{H} = \operatorname{span}\{U_t h;\ t \in \mathbb{R},\ h \in H\} \tag{6}$$

where span in (6) denotes a closed linear span of the vectors $U_t h$ for any $t \in \mathbb{R}$ and any $h \in H$.

A significant role in the theory of dilatation of contractive semigroup $Z_t$ plays the following Theorem 1.2.

**Theorem 1.2.** *Any contracting semigroup $Z_t$ in $H$ has a unitary dilatation $U_t$ in $H$. Moreover the minimal dilatation $U_t$ is defined up to isomorphism.*

We present a construction of the dilatation $U_t$ according to the paper [3]. A contractibility of the semigroup $Z_t$ means [2, 3] that $A$ is dissipative, i.e. $-i(A-A^*) \geq 0$. Consequently including $A$ into the collegation $\Delta$ (1) we can assume that $\sigma = I$. Therefore the conservation law (4) has the form

$$\|h_0^2\| + \int_0^T \|u(\zeta)\|^2 d\zeta = \|h(T)\|^2 + \int_0^T \|v(\zeta)\|\, d\zeta \tag{7}$$

We defined [3] a dilatation space $\mathcal{H}$, which forms vector-functions $f(\zeta) = \big(u_+(\zeta), h, u_-(\zeta)\big)$ so that $u_\pm(\zeta) \in E$ and Supp $u_\pm(\zeta) \in \mathbb{R}_\mp$ for a finite norm

$$\|f\|^2 = \int_{-\infty}^0 \|u_+(\zeta)\|^2\, d\zeta + \|h\|^2 + \int_0^\infty \|u_-(\zeta)\|^2\, d\zeta < \infty\,. \tag{8}$$

We define a dilatation $U_t$ in $\mathcal{H}$ by the formula

$$(U_t f)(\zeta) = \big(u_+(t,\zeta), h_t, u_-(t,\zeta)\big) \tag{9}$$

where $u_-(t,\zeta) = P_{\mathbb{R}_+} u_-(\zeta+t)$; $h_t = y_t(0)$, and $y_t(\zeta)$ is a solution of the Cauchy problem

$$\begin{cases} i\frac{d}{d\zeta}\, y_t(\zeta) + A y_t(\zeta) = \phi^* u_-(\zeta+t)\,; \\ y_t(-t) = 0\,, \quad \zeta \in (-t,0)\,; \end{cases}$$

and at last $u_+(t,\zeta) = u_+(t+\zeta) + P_{(-t,0)}\{u_-(\zeta+t) - i\phi y_t(\zeta)\}$ where $P_{\mathbb{R}_+}$ and $P_{(-t,0)}$ are operators of narrowing (projection operators at set $\mathbb{R}_+$ and $(-t,0)$ respectively), $t \geq 0$.

It is not difficult to show that unitary of $U_t$ (9) in $\mathcal{H}$ is a consequence of the conservation law (1). By the dilatation construction $U_t$ one can see that the space $\mathcal{H}$ has the form

$$\mathcal{H} = D_+ \oplus H \oplus D_- \tag{10}$$

where the subspace $D_+$ is found by vector-function of the form $\big(u_+(\zeta), 0, 0\big) \in \mathcal{H}$ and the subspace $D_-$ is formed by vector-function $\big(0, 0, u_-(\zeta)\big)$ from $\mathcal{H}$, respectively.

The subspaces $D_\pm$ have the following properties:

$$\begin{aligned} U_t D_+ &\subseteq D_+ \qquad (t \geq 0)\,, \\ U_t D_- &\subseteq D_- \qquad (t \leq 0)\,. \end{aligned} \tag{11}$$

Thus $D_+$ is outgoing subspace and $D_-$ is incomming subspace in the sense of P. D. Lax and R. S. Phillips [4]. In accordance with the paper [3], we define a free unitary group $V_t$ in the space $L^2_{\mathbb{R}}(E)$, which will act as

$$(V_t g)(\zeta) = g(\zeta + t) \tag{12}$$

and vector-function $g(\zeta) \in E$, $\zeta \in \mathbb{R}$ is such that

$$\int_{-\infty}^{\infty} \|g(\zeta)\|^2 \, d\zeta < \infty \, .$$

It is evidently that $D_\pm$ after identification belongs to $L^2_{\mathbb{R}}(E)$ also.

Wave operators $W_\pm$ play a significant role in the scattering theory. They are defined [3, 4] as

$$W_\pm = s - \lim_{t \to \mp\infty} U_+ P_{D_\pm} V_{-t} \tag{13}$$

where $P_{D_\pm}$ are orthoprojectors on subspaces $D_\pm$. The following theorem holds [3].

**Theorem 1.3.** *The wave operators $W_\pm$ exist as strong limits* (13) *are isometries from $L^2_{\mathbb{R}}(E)$ to $\mathcal{H}$, and the relations*

$$W_\pm V_t = U_t W_\pm \, , \quad (\forall t) \, , \quad W_\pm P_{D_\pm} = P_{D_\pm} \tag{14}$$

*are valid.*

The scattering operator $S$ is defined by the wave operator $W_\pm$ in a conventional way [3, 4]:

$$S = W_+^* W_- \, . \tag{15}$$

From Theorem 1.3 there follows a proposition.

**Theorem 1.4.** *The operator $S$* (15) *is a contraction, i.e. $\|S\| \le 1$ and has the properties:*

$$\begin{aligned} &SV_t = V_t S \, ; \quad SL^2_{\mathbb{R}_+} \subseteq L^2_{\mathbb{R}_+}(E) \, ; \\ &\overline{SL^2_{\mathbb{R}}(E)} = L^2_{\mathbb{R}}(E) \end{aligned} \tag{16}$$

**1.3.** We recall that the collegation $\Delta$ (1) is simple [1–3] if $H = \operatorname{span}\{A^n \phi^* g;\ n \in \mathbb{Z}_+,\ g \in E\}$. Let us define the following subspaces in $\mathcal{H}$,

$$\Re_\pm = \overline{W_\pm L^2_{\mathbb{R}}(E)} \, .$$

The following theorem gives a sufficient condition for the completeness of the wave operators $W_\pm$, [3].

**Theorem 1.5.** *If the collegation $\Delta$ is simple then the relation $\mathcal{H} = \operatorname{span}\{f_+ + f_-;\ f_\pm \in \Re_\pm\}$ holds.*

Now we construct a translational model [3]. Let $f_k(\zeta) \in L^2_{\mathbb{R}}(E)$, $(k = 1, 2)$. We define a mapping

$$\begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix} \to \Psi_p(\zeta) = W_- f_1(\zeta) + W_+ f_2(\zeta) \in \mathcal{H} \, .$$

Then using isometry of $W_\pm$ and the form of operator $S$ (15) it is not difficult to show that

$$\|\Psi_p(\zeta)\|^2 = \int_{-\infty}^{\infty} < \begin{bmatrix} I & S^* \\ S & I \end{bmatrix} \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix}, \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix} > d\zeta\,, \tag{17}$$

Using Theorem 1.5 we may assert, that space $H$ is isomorphic to the space $L^2\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix}$ which is formed by vector-functions $f(\zeta) = \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix}$ for which the norm (17) is finite. By virtue of conditions (14) the dilatation $U_t$ on $\Psi_p$ will act as a shift. Therefore if $f(\zeta) \in L^2\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix}$ then the dilatation $U_t$ is transformed into

$$\widehat{U}_t f(\zeta) = f(\zeta + t)\,. \tag{18}$$

Applying again (14), one can easily deduce that the spaces $D_\pm$ are realized now in the form

$$\widehat{D}_- = \begin{pmatrix} L^2_{\mathbb{R}_+}(E) \\ 0 \end{pmatrix}, \quad \widehat{D}_+ = \begin{pmatrix} 0 \\ L^2_{\mathbb{R}_-}(E) \end{pmatrix}. \tag{19}$$

Thus the initial space $H$ acquires such model form

$$\begin{aligned} \widehat{H}_p &= L^2\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} \ominus \begin{pmatrix} L^2_{\mathbb{R}_+}(E) \\ L^2_{\mathbb{R}_-}(E) \end{pmatrix} \\ &= f = \left( \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix};\ \begin{matrix} f_1 + S^* f_2 \in L^2_{\mathbb{R}_-}(E) \\ S f_1 + f_2 \in L^2_{\mathbb{R}_+}(E) \end{matrix} \right) \end{aligned} \tag{20}$$

and in the virtue of the dilatation the action of semigroup $Z_t$ is transformed to the shift semigroup

$$\widehat{Z} f(\zeta) = P_{\widehat{H}_P} f(\zeta + t) \tag{21}$$

where $f(\zeta) \in \widehat{H}_p$ (20). Thus the following theorem is proved.

**Theorem 1.6.** *A minimal unitary dilatation $U_t$ in $\mathcal{H}$ of the contraction semigroup $Z_t = \exp(itA)$ in $H$, where $A$ is dissipative operator of a simple collegation $\Delta$ is unitary equivalent to a translation group $\widehat{U}_t$ (18) in the space $L^2\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix}$, and the contraction semigroup $Z_t$ is unitary equivalent to the shift semigroup $\widehat{Z}_t$ (21) in the space $\widehat{H}_p$ respectively.*

The Fourier transform by formula

$$\widetilde{f}(\lambda) = \int_{-\infty}^{\infty} f(\zeta)\, e^{-i\lambda\zeta}\, d\zeta \tag{22}$$

in the virtue of Plancherel theorem [2, 3] is a unitary operator in $L^2_{\mathbb{R}}(E)$. By the virtue of Wiener-Paley theorem

$$\widetilde{L}^2_{\mathbb{R}_+}(E) = H^2_-(E)\,; \quad \widetilde{L}^2_{\mathbb{R}_-}(E) = H^2_+(E)$$

where $H^2_\pm(E)$ are Hardy spaces of $E$-value function from $L^2_{\mathbb{R}}(E)$ which are holomorphically continued into lower (upper) half-plane. Let us apply the Fourier transform (22) to translational model (18) – (21) and take advantage of the following Theorem 1.7

**Theorem 1.7.** *The Fourier transform of the scattering operator $S$ (15) transfers the operator $S$ into operator performing multiplication by characteristic function*

$$\begin{aligned} &S_\Delta(\lambda) = I - \phi(A-\lambda I)^{-1}\phi^*\,, \quad \textit{i.e.} \\ &(\widetilde{Sf})(\lambda) = S_\Delta(\lambda)\widetilde{f}(\lambda)\,. \end{aligned} \tag{23}$$

*As it is known $\widetilde{f}(\lambda+t) = e^{i\lambda t}\widetilde{f}(\lambda)$, therefore we derive such functional model.*

**Theorem 1.8.** *A minimal unitary dilatation $U_t$ in $H$ of the contraction semigroup $Z_t = \exp(itA)$ in $H$, where $A$ is dissipative operator of a simple collegation $\Delta$ is unitary equivalent to the group*

$$\widetilde{U}_t f(\lambda) = e^{i\lambda t} f(\lambda) \tag{24}$$

*where $f(\lambda) \in L^2\begin{pmatrix} I & S^*_\Delta(\lambda) \\ S_\Delta(\lambda) & I \end{pmatrix}$ and contraction semigroup $Z_t$ is unitary equivalent to semigroup $\widetilde{Z}_t f(\lambda) = P_{\widetilde{H}_p} e^{i\lambda t} f(\lambda)$, where $f(\lambda)$ belongs to the space*

$$\widetilde{H}_p = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(\lambda) \in \begin{pmatrix} I & S^*_\Delta(\lambda) \\ S_\Delta(\lambda) & I \end{pmatrix} \begin{array}{l} ;\ f_1 + S^*_\Delta(\lambda) f_2 \in H^2_+(E) \\ ;\ S_\Delta(\lambda) f_1 + f_2 \in H^2_-(E) \end{array} \right\}$$

*Here the main operator $\widetilde{A}$ in $\widetilde{H}_p$ act as multiplication operator by independent variable*

$$\widetilde{A} f(la) = P_{\widetilde{H}_p} \lambda f(\lambda)\,,\ f(\lambda) \in \widetilde{H}_p\,. \tag{26}$$

In the next section we will generalize this construction on the case of the Laquerre transform.

## 2. A functional model for the Laguerre representation

**2.1.** Let us consider a differential operator

$$\ell = t\frac{d^2}{dt^2} + (1-t)\frac{d}{dt} \tag{27}$$

in what follows called the Laguerre operator; it acts on functions form $C^2 = (\mathbb{R}_+)$. We denote by $L^2_{\mathbb{R}_+}(e^{-t}\,dt)$ the following space:

$$L^2_{\mathbb{R}_+}(e^{it}dt) = \left\{ f(t),\ t \in \mathbb{R}_+;\, \int_0^\infty |f(t)|^2 e^{-t}\,dt < \infty \right\} \tag{28}$$

**Proposition 2.1.** *An operator $\ell$ is symmetric in the space $L^2_{\mathbb{R}_+}(e^{-t}dt)$ under the self-adjoint boundary conditions, i.e. $\langle \ell x, y\rangle = \langle y, \ell y\rangle$ for all $x, y \in \mathbb{C}^2(\mathbb{R}_+)$ such that $tx(t)|_{t=0} = 0$, $ty(t)|_{t=0} = 0$ and $ty'(t)|_{t=0} < \infty$, $tx'(t)|_{t=0} < \infty$.*

**Proof.** We calculate

$$\begin{aligned}\langle \ell x, y\rangle - \langle x, \ell y\rangle &= \int_0^\infty \big\{(tx'' + (1-t)x')\overline{y} - x(t\overline{y}'' + (1-t)\overline{y}')\big\}e^{-t}\,dt \\ &= \int_0^\infty \big\{te^{-t}(x'\overline{y} - \overline{y}'x)\big\}'\,dt = \big\{te^{-t}(x'\overline{y} - \overline{y}'x)\big\}|_0^\infty = 0\end{aligned}$$

by virtue of the boundary conditions. □

Let us consider now an open system of special form, generated by the Laguerre operator (27) and corresponding to the collegation $\Delta$ (1):

$$\begin{cases} \ell h(t) + Ah(t) = \phi^*\sigma u(t)\,; \\ h(0) = h_0 (th')(0) = h_1\,; \\ v(t) = u(t) - i\phi h(t)\,. \end{cases} \tag{29}$$

The following assertion is valid, similar to Theorem (1.1).

**Theorem 2.1.** *For the open system* (29) *associated with collegation $\Delta$ the law of conservation of energy is valid, i.e.*

$$\begin{aligned}&\int_0^T \langle \sigma u(\zeta), u(\zeta)\rangle e^{-\zeta}\,d\zeta + \langle I\widehat{h}_0, \widehat{h}_0\rangle \\ &= \int_0^T \langle \sigma v(\zeta), v(\zeta)\rangle e^{-\zeta}\,d\zeta + \langle I\widehat{h}_T, \widehat{h}_T\rangle\end{aligned} \tag{30}$$

*where* $I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ *and* $h_0 = \begin{pmatrix} h_0 \\ h_t \end{pmatrix}$, $h_T = \begin{pmatrix} h(T) \\ e^{-T}Th'(T) \end{pmatrix}$ *for any finite $T > 0$.*

**Proof.** We calculate

$$\begin{aligned}\langle \ell h, h\rangle - \langle h, \ell h\rangle &= \langle \phi^*\sigma u - Ah, h\rangle - \langle h, \psi^*\sigma u - Ah\rangle \\ &= \langle \sigma u, \frac{u-v}{i}\rangle - \langle \frac{u-v}{i}, \sigma u\rangle - \langle (A - A^*)h, h\rangle \\ &= i\langle \sigma u, u - v\rangle + i\langle u - v, \sigma u\rangle - i\langle \phi^*\sigma\phi h, h\rangle \\ &= i\langle \sigma u, u - v\rangle + i\langle u - v, \sigma u\rangle - i\langle \sigma(u-v), u - v\rangle \\ &= i\langle \sigma u, u\rangle - i\langle \sigma v, v\rangle\,.\end{aligned}$$

Now we integrate the derived equality:

$$\begin{aligned}&\int_0^T \langle \sigma v, v\rangle e^{-t}\,dt - \int_0^T \langle \sigma u, u\rangle e^{-t}\,dt \\ &\quad = i\int_0^T \big[\langle \ell h, h\rangle - \langle h, \ell h\rangle\big]e^{-t}\,dt \\ &\quad = i\big\{e^{-t}t\big[\langle h'', h\rangle - \langle h, h'\rangle\big]\big\}|_0^T \\ &\quad = \langle I\widehat{h}_0, \widehat{h}_0\rangle - \langle I\widehat{h}_T, \widehat{h}_T\rangle\end{aligned}$$

which proves our assertion. □

**2.2.** Let us make use of the energy conservation law (30) to construct a dilatation for operator $T_t$ generated by the Cauchy problem

$$\begin{cases} \ell h(t) + Ah(t) = 0\,; \\ h(0) = h_0\,;\ (th')(0) = h_1\,; \end{cases} \tag{31}$$

where $T_t(h_0, h_1) = \big(h(t), th'(t)\big)$. We will call an unitary operator-function $U_t$ in $\mathcal{H}$ a dilatation of family $T_t$ in $H$, if $\mathcal{H} \supseteq H$, $T_t = P_H U_t|_H$.

Here we do not suppose that $T_t$ and $U_t$ is semigroup. Moreover, the unitary property of $U_t$ may hold not necessarily in Hilbert metric but in indefinite one. The following analog of Theorem 1.2 is valid.

**Theorem 2.2.** *The operator-function $T_t$ generated by the Cauchy problem* (31) *with dissipative operator $A$ of collegation $\Delta$* (1) *(i.e. $\sigma = I$) possesses the unitary (in indefinite metric) dilatation $U_t$, where the minimal dilatation is determined up to isomorphism.*

**Proof.** To prove the theorem we bring a construction of dilatation $U_t$ by analog with (8), (9).

Let us consider a Hilbert space

$$\begin{aligned} \mathcal{H} =& \big\{ f = \big(u(\zeta), \widehat{h}, v(\zeta)\big);\ u(\zeta), v(\zeta) \in E\,,\ \operatorname{supp} v \in \mathbb{R}_-\,,\ \operatorname{supp} u \in \mathbb{R}_+\,, \\ \widehat{h} =& \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}, h_k \in H\,;\ \|f\|^2 = \int_{-\infty}^{0} \|v(\zeta)\|^2 e^{-\zeta}\, d\zeta + \|\widehat{h}\|^2 \\ &+ \int_0^\infty \|u(\zeta)\|^2 e^{-\zeta}\, d\zeta < \infty \big\}\,. \end{aligned} \tag{32}$$

We set indefinite metric $\mathcal{H}$

$$\langle f \rangle_I^2 = \int_{-\infty}^{0} \|v(\zeta)\|^2 e^{-\zeta}\, d\zeta + \langle I\widehat{h}, \widehat{h}\rangle + \int_0^\infty \|u(\zeta)\|^2 e^{-\zeta}\, d\zeta \tag{33}$$

where $I$ has the form indicated in Theorem 2.1.

We construct the dilatation $U_t$ in $\mathcal{H}$,

$$U_t f = f_t\big(u(t,\zeta), \widehat{h}_t, v(t,\zeta)\big)\,. \tag{34}$$

Let us consider further the Cauchy problem

$$\begin{cases} \big(i\frac{\partial}{\partial t} + \ell_\zeta\big)\, \widehat{u}(t,\zeta) = 0\,; \\ \widehat{u}(0,\zeta) = u(\zeta)\,;\ \zeta \in \mathbb{R}_+\,; \end{cases} \tag{35}$$

where $\ell_\zeta$ is operator $\ell$ (27) with respect to $\zeta$.

Solution of the problem is easily obtained. In fact, let

$$\widehat{u}(t,\zeta) = \sum_{n\in z_+} e^{-itn} C_n g_n(\zeta)$$

where $g_n(\zeta)$ are the Laguerre polynomials [5] which are the solutions of equation $\ell_\zeta g_n(\zeta) + n g_n(\zeta) = 0$ and have the form

$$g_n(\zeta) = \frac{1}{n!}\, e^{\zeta} \frac{d^n}{d\zeta^n}\,(\zeta e^{-\zeta})$$

and make a complete system of orthogonal polynomials in $L^2_{\mathbb{R}_+}(e^{-\zeta}\,d\zeta)$. The coefficients $C_n$ are obtained from the initial condition $\sum C_n g_n(\zeta) = u(\zeta)$.

Therefore $\widehat{u}(t,\zeta)$ possesses the property $\operatorname{supp}\widehat{u}(t,\zeta) = \operatorname{supp}\widehat{u}(\zeta) \subseteq \mathbb{R}_+$. Now we determine $u(t,\zeta)$ in (34) by the formula

$$u(t,\zeta) = P_{\mathbb{R}_+}\widehat{u}(t,\zeta+t)e^{-\frac{t}{2}}\,. \tag{36}$$

To set $\widehat{h}_t$ (34), we consider the following Cauchy problem

$$\begin{cases} \ell_\zeta y(\zeta) + Ay(\zeta) = \phi^*\widehat{u}(t,\zeta+t)e^{-\frac{t}{2}}\,;\ \zeta\in(-t,0)\,; \\ y(-t) = h_0\,; \\ (-t)e^t\,y(-t) = h_1\,; \end{cases} \tag{37}$$

and put $\widehat{h}_t = \begin{pmatrix} y(0) \\ (ty')(0) \end{pmatrix}$.

Finally, to set $v(t,\zeta)$ (34) we consider the similar equation

$$\begin{cases} (i\frac{\partial}{\partial t} + \ell_\zeta)\,\widehat{v}(t,\zeta) = 0\,; \\ \widehat{v}(0,\zeta) = v(\zeta)\,;\ \zeta\in\mathbb{R}_-\,; \end{cases} \tag{38}$$

and put $v(t,\zeta) = e^{-\frac{t}{2}}\widehat{v}(t,\zeta+t) + P_{\mathbb{R}_-}\{\widehat{u}(t,\zeta+t)e^{-\frac{t}{2}} - i\phi y(\zeta)\}$. We show that $U_t$ (34) has property of isometry in the metric (33). To this end we calculate,

$$\begin{aligned}
\langle f_t\rangle_I^2 &= \int_{-\infty}^0 \|v(t,\zeta)\|^2 e^{-\zeta}\,d\zeta + \langle I\widehat{h}_t, \widehat{h}_t\rangle + \int_0^\infty \|u(t,\zeta)\|^2 e^{-\zeta}\,d\zeta \\
&= \int_{-\infty}^{-t} \|\widehat{v}(t,\zeta+t)\|^2 e^{-\zeta-t}\,d\zeta + \int_{-t}^0 \|\widehat{u}(t,\zeta+t)e^{-\frac{t}{2}} - i\phi y(\zeta)\|^2 e^{\zeta}\,d\zeta \\
&\quad + \langle I\widehat{h}_t, \widehat{h}_t\rangle + \int_0^\infty \|u(t,\zeta+t)\|^2 e^{-\zeta-t} d\zeta \\
&= \int_{-\infty}^{-t} \|\widehat{v}(t,\zeta+t)\|^2 e^{-\zeta-t}\,d\zeta + \langle I\widehat{h}_0, \widehat{h}_0\rangle + \int_{-t}^\infty \|\widehat{u}(t,\zeta+t)\|^2 e^{-\zeta-t}\,d\zeta \\
&= \int_{-\infty}^0 \|\widehat{v}(t,\zeta)\|^2 e^{-\zeta}\,d\zeta + \langle I\widehat{h}_0, \widehat{h}_0\rangle + \int_0^\infty \|\widehat{u}(t,\zeta)\|^2 e^{-\zeta}\,d\zeta \\
&= \langle f\rangle_I^2
\end{aligned}$$

In this calculation we have made use of the conservation law (30) and of the fact that norms of solutions of Cauchy problems $\widehat{u}(t,\zeta)$, $\widehat{v}(t,\zeta)$ (35) and (38) coincide with norms of initial data $u(\zeta)$ and $v(\zeta)$ in the spaces $L^2_{\mathbb{R}_+}(e^{-t}\,dt)$ and $L^2_{\mathbb{R}_-}(e^{-t}\,dt)$ by virtue of selfadjointness of operators $\ell_\zeta$ in the spaces.

In order to prove that $U_t$ has the property of being unitary, it is necessary to ascertain that from $U_t^*f=0$ implies $f=0$. It is easy to show that $U_t^*$ will act by the formula

$$U_t^*f=\left(u(t,\zeta),\widehat{h}_t,v(t,\zeta)\right). \tag{39}$$

Here $v(t,\zeta)=P_{\mathbb{R}_-}\widehat{v}(t,\zeta-t)e^{\frac{t}{2}}$ where $\widehat{v}(t,\zeta)$ is a solution of problem (38).

In order to obtain $\widehat{h}_t$, it is necessary to consider dual to (37) problem

$$\begin{cases}\ell_\zeta y(\zeta)+A^*y(\zeta)=\phi^*\widehat{v}(\zeta,\zeta-t)e^{\frac{t}{2}}\,;\\ y(t)=h_0\,;\\ e^{-t}ty'(t)=h_1\,;\end{cases} \tag{40}$$

and put $\widehat{h}_t=\begin{pmatrix}y(0)\\(ty')(0)\end{pmatrix}$. Finally,

$$u(t,\zeta)=\widehat{u}(t,\zeta-t)e^{\frac{t}{2}}+P_{\mathbb{R}_+}\{\widehat{v}(t,\zeta-t)e^{\frac{t}{2}}+i\phi y(\zeta)\}\,,$$

where $\widehat{u}(t,\zeta)$ is the solution of Cauchy problem (35).

Thus let $U_t^*f=0$, then $\widehat{u}(t,\zeta)=0$ and so $\widehat{u}(t,\zeta)=0$ and $\widehat{v}(t,\zeta-t)e^{\frac{t}{2}}+i\phi y(\zeta)=0$ therefore $u(\zeta)\equiv 0$. Now, by substituting $\widehat{v}(t,\zeta-t)=-i\phi y(\zeta)e^{-\frac{t}{2}}$ in (40) we obtain a homogeneous equation

$$\ell_\zeta y+A^*y+i\phi^*\phi y=0$$

with zero condition in the origin $\widehat{h}_t=0$. By virtue of uniqueness of Cauchy problem solution, this yields that $y(\zeta)\equiv 0$, therefore $\widehat{v}(t,\zeta-t)=0$ on interval $(0,t)$. Accounting that $\widehat{v}(t,\zeta-t)=0$ with $(-\infty,0)$, finally we conclude that $v(\zeta)=0$. Thus $f=0$. This proves the property of being unitary for $U_t$ (34) and completes the proof of the theorem. □

**2.3.** Let us pass to constructing wave operators. To this end we define a "free" group by analogy with (38)

$$V_tg(\zeta)=g(t,\zeta)\,, \tag{41}$$

where $g(t,\zeta)$ is a solution of Cauchy problem

$$\begin{cases}\left(i\frac{\partial}{\partial t}+\ell_\zeta\right)\,g(t,\zeta)=0\,;\\ g(0,\zeta)=g(\zeta)\in L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)\,.\end{cases} \tag{42}$$

It is evident that $V_t$ (41) is unitary. Now we define the operators

$$\begin{aligned}W_-&=s-\lim_{t\to+\infty}U_tP_{\mathbb{R}_+}V_{-t}\,,\\ W_+&=s-\lim_{t\to-\infty}U_t^*P_{\mathbb{R}_-}V_{-t}^*\,.\end{aligned} \tag{43}$$

By analogy with Theorem 1.3 we have

**Theorem 2.3.** *The wave operators* $W_\pm$ *exist as strong limits* (43), *are isometries from* $L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)$ *to* $\mathcal{H}$, *and the following relations are valid:*

$$\begin{aligned} &U_t W_- = W_- V_t, U_t^* W_+ = W_+ V_t^*\,, \quad (t \geq 0) \\ &W_\pm P_{\mathbb{R}_\mp} = P_{\mathbb{R}_\mp} \end{aligned} \tag{44}$$

**Proof.** We prove the assertion of the theorem for $W_-$ (for $W_+$ the proof is similar). The main matter of the theorem consists of existence proof of $W_-$ since the relation (44) is proved by analogy with arguments given in Section 1; sec [2, 3]. Let

$$f_t = U_t P_{\mathbb{R}_+} V_{-t} g = \big(v(t,\zeta), h_t, u(t,\zeta)\big)$$

then $u(t,\zeta) = P_{\mathbb{R}_+} g(\zeta)$. We consider the Cauchy problem

$$\begin{cases} \ell_\zeta y(\zeta) + Ay(\zeta) = \phi^* g(\zeta)\,; \\ y(-t) = 0\,;\ y'(-t) = 0\,,\ \zeta \in (-t,0)\,. \end{cases} \tag{45}$$

Then $\widehat{h}_t = \begin{pmatrix} y(0) \\ (ty')(0) \end{pmatrix}$.

We denote by $K(\zeta,\eta)$ a Cauchy function of the problem (45) (i.e. $K(\zeta,\zeta) = 0$, $K'(\zeta,\zeta) = I$), then a solution $y(\zeta)$ of (45) has the form

$$y_t(\zeta) = \int_{-t}^{\zeta} K(\zeta,\eta)\phi^*\, g(\eta)\, d\eta\,.$$

Therefore $V(t,\zeta)$ has the form

$$V(t,\zeta) = P_{(-t,0)}\big\{g(\zeta) - i\phi y(\zeta)\big\}\,.$$

Thus,

$$f_t = \left(P_{(-t,0)}\big\{g(\zeta) - i\phi\int_{-t}^{0} K(\zeta,\eta)\phi^* g(\eta) d\eta\big\}, \begin{pmatrix} \int_{-t}^{0} K(0,\eta)\phi^* g(\eta) d\eta \\ \int_{-t}^{0} K'(0,\eta)\phi^* g(\eta) d\eta \end{pmatrix}, P_{\mathbb{R}_+} g(\zeta)\right).$$

We show that $f_t$ is a Cauchy sequence, i.e $\|f_{t+\Delta} - f_t\|^2 \to 0$ as $t \to \infty$. Since

$$\|f_{t+\Delta} - f_t\|^2 = \int_{-\infty}^{0} \|v_t(t+\Delta,\zeta) - v(t,\zeta)\|^2 e^{-\zeta}\, d\zeta + \|\widehat{h}_{t+\Delta} - \widehat{h}_t\|^2\,. \tag{46}$$

It is sufficient to show that each summand approaches to zero as $t \to \infty$. We show that $\|\widehat{h}_{t+\Delta} - \widehat{h}\| \to 0$ when $t \to \infty$ and we will prove this property component by component. It is obvious that

$$\begin{aligned} \|\widehat{h}_{t+\Delta} - \widehat{h}\|^2 - &\left\| \int_{(-t-\Delta)}^{-t} K(0,\eta)\phi^* g(\eta)\, d\eta \right\|^2 \\ &\leq \int_{-t-\Delta}^{-t} \|K(0,\eta)\|^2 e^{\eta}\, d\eta \cdot \int_{-t-\Delta}^{-t} e^{-\eta} \|\phi^*\|^2\, \|g(\eta)\|^2\, d\eta \end{aligned}$$

and since the function $K(0,\eta)e^{\eta}$ is bounded (see [6, 7]), we obtain that

$$\|h_{t+\Delta} - h_t\|^2 \leq \Delta C \|\phi^*\|^2 \int_{-t-\Delta}^{-t} \|g(\eta)\|^2 e^{-\eta}\, d\eta \to 0 \quad \text{as} \quad t \to \infty$$

since $g(\eta) \in L^2_{\mathbb{R}}(e^{-\eta}\, d\eta)$.

The convergence of second components $\widehat{h}_{t+\Delta} - \widehat{h}_t$ to zero is proved in a similar way. We show that the first summand in (46) approaches to zero too.

In fact,

$$\begin{aligned}
A &= \int_{-\infty}^{0} \|P_{(-t-\Delta,-t)} g(\zeta) - iP_{(-t-\Delta,0)}\phi \int_{-t-\Delta}^{\zeta} K(\zeta,\eta)\phi^* g(\eta)\, d\eta \\
&\quad + i \int_{-t}^{\zeta} \phi K(\zeta,\eta)\phi^* g(\eta)\, d\eta\|^2 e^{-\zeta}\, d\zeta \\
&= \int_{-t-\Delta}^{-t} \|g(\zeta)\|^2 e^{-\zeta}\, d\zeta + \int_{-\infty}^{0} \|P_{(-t-\Delta,0)}\phi y_{t+\Delta}(\zeta) - P_{(-t,0)}\phi y_t(\zeta)\|^2 e^{-\zeta}\, d\zeta \\
&\quad + 2\mathrm{Im} \int_{-t-\Delta}^{-t} \langle g(\zeta), P_{(-t-\Delta,0)}\phi y(\zeta) - P_{(-t,0)}\phi y(\zeta)\rangle e^{-\zeta}\, d\zeta
\end{aligned}$$

It is obvious that the first and third summands in the given sum approaches to zero as $t \to \infty$ because $g(\zeta) \in L^2_{\mathbb{R}}(e^{-\zeta}\, d\zeta)$. We evaluate the second summand:

$$\begin{aligned}
B &= \int_{-\infty}^{0} \|P_{(-t-\Delta,0)}\phi y_{t+\Delta}(\zeta) - P_{(-t,0)})\phi y_t(\zeta)\|^2 e^{-\zeta}\, d\zeta \\
&= \int_{-\infty}^{0} \langle \phi\Delta y, \phi\Delta y\rangle e^{-\zeta}\, d\zeta\,,
\end{aligned}$$

where

$$\Delta y = P_{(-t-\Delta,0)} y_{t+\Delta}(\zeta) - P_{(-t,0)} y_{-t}(\zeta)\,.$$

Then

$$\begin{aligned}
A &= \int_{-\infty}^{0} \langle \phi^*\phi\Delta y, \Delta y\rangle e^{-\zeta}\, d\zeta = \int_{-\infty}^{0} \left\langle \frac{A - A^*}{i}\Delta y, \Delta y \right\rangle e^{-\zeta}\, d\zeta \\
&= 2\mathrm{Im} \int_{-\infty}^{0} \langle \phi^* g - \ell\Delta y, \Delta y\rangle e^{-\zeta}\, d\zeta \\
&= 2\mathrm{Im} \int_{-\infty}^{0} \langle \phi^* g, \Delta y\rangle e^{-\zeta}\, d\zeta + 2\mathrm{Im} \int_{-\infty}^{0} \langle \ell\Delta y, \Delta y\rangle e^{-\zeta}\, d\zeta
\end{aligned}$$

the first summand approaches to zero again on account of $g(\zeta) \in L^2_{\mathbb{R}}(e^{-\zeta}\, d\zeta)$, and the second one yields after integration by parts

$$\|\zeta e^{-\zeta}\Delta y\|\,|_{\zeta=0} \to 0 \qquad (t \to \infty)$$

since $\Delta\widehat{h}_t \to 0$. The theorem is proved. □

As before, we define the operator $S$ by the formula (15). Then the following theorem holds.

**Theorem 2.4.** *The operator $S$ (15) is a contraction from $L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)$ to $L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)$ and possesses the following properties:*

$$SV_t = V_tS\,;\; SL^2_{\mathbb{R}_+}(e^{-\zeta}\,d\zeta) \subset L^2_{\mathbb{R}_+}(e^{-\zeta}\,d\zeta)\,;$$
$$\overline{SL^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)} = L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)\,.$$

**2.4.** Further we suppose that the collegation $\Delta$ (1) is simple and as in subsection 1.3 we set a mapping

$$\Psi_p(\zeta = W_-f_1(\zeta)) + W_+f_2(\zeta)$$

from $L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta) + L^2_{\mathbb{R}}(e^{-\zeta}\,d\zeta)$ to $\mathcal{H}$. It is obvious that

$$\Psi_p(\zeta) \in L^2\left(\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}, e^{-\zeta}\,d\zeta\right)$$

Action of dilatation in this space again reduces to a translation

$$\widehat{U}_t f(\zeta) = f(\zeta + t)\,, \tag{47}$$

since

$$\begin{aligned} U_t\Psi_p(\zeta) &= W_-f_1(\zeta+t) + U_tW_+f_2(\zeta) \\ &= W_-f_1(\zeta+t) + U_tW_+V_t^*V_tf_2(\zeta) \\ &= W_-f_1(\zeta+t) + U_tU_t^*W_+V_tf_2(\zeta) = \Psi_P(\zeta+t)\,. \end{aligned}$$

As earlier, it is obvious that

$$D_- = \begin{pmatrix} L^2_{\mathbb{R}_+}(e^{-\zeta}\,d\zeta) \\ 0 \end{pmatrix}, \quad D_+ = \begin{pmatrix} 0 \\ L^2_{\mathbb{R}_-}(e^{-\zeta}\,d\zeta) \end{pmatrix}$$

and the model space $H_p$ has the form

$$H_p = L^2\left(\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} e^{-\zeta}\,d\zeta\right) \ominus \begin{pmatrix} L^2_{\mathbb{R}_+}(e^{-\zeta}\,d\zeta) \\ L^2_{\mathbb{R}_-}(e^{-\zeta}\,d\zeta) \end{pmatrix} \tag{48}$$

and in addition $T_t$ passes to shift semigroup

$$\widehat{T}_t f(\zeta) = f(\zeta + t)\,. \tag{49}$$

Now we consider a Laguerre transform

$$L_n = \int_0^\infty e^{-x} P_n(x)\, f(x)\, dx \tag{50}$$

where $P_n(x) = \frac{1}{n!}e^{-x}\frac{d^n}{dx^n}(xe^{-x})$ are a Laguerre polynomials, and $f(x) \in L^2_{\mathbb{R}_+}(e^{-x}dx)$. The transform (50) ascertains isomorphism between $L^2_{\mathbb{R}_+}(e^{-x}dx)$ and $\ell^2$.

We extend the Laguerre transform (50) on $\mathbb{R}_-$ in a symmetric way. Then an image of this map yields a space $\ell^2_-$. Let $\ell^2_{\mathbb{Z}} = \ell^2_- + \ell^2_+$ is a space of square summable two-sided sequences. Just as for the case of Fourier transform (see Theorem 1.7 in Section 1) a theorem the proof of which repeats the reasonings brought out in [3] holds.

**Theorem 2.5.** *The Laguerre transform of scattering operator $S$ transfers the operator $S$ into an operator of multiplication by a characteristic function $S_\Delta(n) = I - i\phi(A - nI)^{-1}\phi^*$, $n \in \mathbb{Z}$, i.e.*

$$L_n(Sg) = S_\Delta(n) g_n \tag{51}$$

*where $g_n = L_n(g)$.*

After realizing the Laguerre transform, the space $L^2\left(\begin{pmatrix} I & S^* \\ S & I \end{pmatrix} e^{-\zeta}\, d\zeta\right)$ passes into the space $\ell^2_{\mathbb{Z}}\begin{pmatrix} I & S^*_\Delta(n) \\ S_\Delta(n) & I \end{pmatrix}$ and dilatation $\widehat{U_t}$ (47) is converted into

$$\widehat{U}_t(n) f_n = e^{-itn} f_n\,. \tag{52}$$

Supspaces $D_\pm$ will have the form

$$D_- = \begin{pmatrix} \ell^2_- \\ 0 \end{pmatrix}, \quad D_+ = \begin{pmatrix} 0 \\ \ell^2_+ \end{pmatrix}.$$

Therefore $H_p$ is converted to the form

$$\widetilde{H}_p = \left\{ f_n = \begin{pmatrix} f^1_n \\ f^2_n \end{pmatrix} \in \ell^2_{\mathbb{Z}}\begin{pmatrix} I & S^*_\Delta(n) \\ S_\Delta(n) & I \end{pmatrix}; \begin{matrix} f^1_n + S^*_\Delta(n) f^2_n \in \ell^2_+ \\ S_\Delta(n) f^1_n + f^2_n \in \ell^2_- \end{matrix} \right\} \tag{53}$$

and a "semigroup" $T_t$ will have the form

$$\widetilde{T}_t(n) f_n = P_{\widetilde{H}_p} e^{-itn} f_n\,. \tag{54}$$

Thus the following theorem is proved.

**Theorem 2.6.** *The minimal unitary dilatation $U_t$ (34) in $\mathcal{H}$ (32) of the family of operators $T_t$ (31) with a scattering operator $A$ of collegation $\Delta$ (1) is unitary equivalent to $\widetilde{U}_t(n)$ (52) in the space $\ell^2_{\mathbb{Z}}\begin{pmatrix} I & S^*_\Delta(n) \\ S_\Delta(n) & I \end{pmatrix}$, and the family $T_t$ (31) is unitary equivalent to $\widetilde{T}_t(n)$ (54) in the space $\widetilde{H}_p$.*

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