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# LATTICE–THEORETICALLY CHARACTERIZED CLASSES OF FINITE BANDS

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Abstract. There are investigated classes of finite bands such that their subsemigroup lattices satisfy certain lattice-theoretical properties which are related with the cardinalities of the Green's classes of the considered bands, too. Mainly, there are given disjunctions of equations which define the classes of finite bands.

## 1. Introduction and Summary

For a semigroup $S$ let $L(S)$ be the subsemigroup lattice of $S$ with the usual lattice operations $\vee$ and $\wedge$.

In the following let $1 \leqq n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of the natural numbers) and $SD_\vee(n)$ be the class of all finite bands (i.e., finite idempotent semigroups) $S$ such that for $T, A, B_0, \dots, B_n \in L(S)$ the following implication holds: If

$$T = A \vee B_0 = \ldots = A \vee B_n\,,$$

then

$$T = A \vee \bigvee\{B_i \wedge B_j\colon\ 0 \leqq i < j \leqq n\}\,.$$

Obviously, the class $SD_\vee(1)$ is equal to the class of all finite bands such that their subsemigroup lattices are $\vee$-semidistributive or satisfy the so-called Jónsson condition (cf. [3, 5, 6]), respectively.

Moreover, let $AE(n)$ be the class of all finite bands $S$ such that for $A \in L(S)$ and $i_0, \dots, i_n \in S \setminus A$ the following implication holds: If

$$A \vee i_0 = A \vee i_1 = \ldots = A \vee i_n\,,$$

then

$$|\{i_0, \dots, i_n\}| \ \leqq n$$

where $A \vee i$ denotes $A \vee \{i\}$ for $i \in S$.

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The class $AE(1)$ is equal to the class of all finite bands such that their subsemigroup closure operators have the anti-exchange property (cf. [1]). Therefore, the class $AE(1)$ concides with the class of all finite so-called filtered bands, i.e., the class of all finite bands $S$ such that each $T \in L(S)$ has the least generating set with respect to inclusion (cf. [5, 6]). A set $U$ is called to be a *generating set* of $T$ if and only if $U \subseteqq T$ and $T = \langle U \rangle$ where $\langle U \rangle$ is the subsemigroup of $S$ generated by $U$.

For example, the finite left zero semigroups, right zero semigroups and semilattices are finite filtered bands.

In the following it is proved that the classes $SD_\vee(n)$ and $AE(n)$ coincide.

Moreover, there are given disjunctions of equations which define the classes $SD_\vee(n)$ and $AE(n)$, respectively.

For this let $\mathbb{X} := \{z\} \cup \{x_i : i \in \mathbb{N}\} \cup \{y_i : i \in \mathbb{N}\}$ be a (countable) set of variables and $\mathbb{X}^+$ be the free semigroup on $\mathbb{X}$. Let $S$ be a band and $\mathcal{A} \subseteqq \mathbb{X}^+ \times \mathbb{X}^+$, i.e., $\mathcal{A}$ is a set of equations. Then it is said that $\mathcal{A}$ *holds in* $S$ *disjunctively*, in symbols: $\mathcal{A} \varphi S$, if and only if for each homomorphism $f$ from $\mathbb{X}^+$ into $S$ there exists an equation $(p,q) \in \mathcal{A}$ such that the equality $f(p) = f(q)$ is fulfilled (cf. [2]).

Let $\mathfrak{V}$ be a class of bands. Then $\mathfrak{V}$ is called to be *disjunctively defined* if and only if there exists a system $\mathfrak{A}$ of sets $\mathcal{A} \subseteqq \mathbb{X}^+ \times \mathbb{X}^+$ such that $\mathfrak{V}$ is equal with the class of all bands $S$ where $\mathcal{A} \varphi S$ for each $\mathcal{A} \in \mathfrak{A}$, in symbols: $\mathfrak{V} = MOD(\mathfrak{A})$.

For $\mathbb{Y} \subseteqq \mathbb{X}$ and $e \in \mathbb{N}$ let $\mathbb{Y}^e \subseteqq \mathbb{X}^+$ defined as follows:

$$\mathbb{Y}^e := \{y_1 \dots y_i : y_1, \dots , y_i \in \mathbb{Y}, 1 \leqq i \leqq e\} \text{ for } e \geqq 1 \text{ and } \mathbb{Y}^0 := \emptyset\,.$$

Let

$$\begin{aligned}
F_1(n) &:= \{(x_m \dots x_0)(x_n \dots x_0)z\colon\ 0 \leqq m \leqq n\}\,,\\
F_2(n) &:= \{z(y_0 \dots y_n)(y_0 \dots y_m)\colon\ 0 \leqq m \leqq n\}\,,\\
F_3(n) &:= \{(x_m \dots x_0)(x_n \dots x_0)z(y_0 \dots y_n)(y_0 \dots y_m)\colon\ 0 \leqq m \leqq n\}\,.
\end{aligned}$$

Then for natural numbers $e$ and $i = 1, 2, 3$ let

$$\begin{aligned}
\mathcal{A}_i(n) &:= \{(p,q)\colon\ p,q \in F_i(n),\ p \neq q\}\,,\\
\mathcal{B}_1(n,e) &:= \{(x_n \dots x_0)z\} \times \{x_0, \dots , x_n\}^e\,,\\
\mathcal{B}_2(n,e) &:= \{z(y_0 \dots y_n)\} \times \{y_0, \dots , y_n\}^e\,,\\
\mathcal{B}_3(n,e) &:= \{(x_n \dots x_0)z(y_0 \dots y_n)\} \times \{x_0, \dots , x_n,\ y_0, \dots , y_n\}^e\,,\\
\mathcal{C}_i(n,e) &:= \mathcal{A}_i(n) \cup \mathcal{B}_i(n,e)\,,\\
\mathfrak{C}(n,e) &:= \{\mathcal{C}_1(n,e),\ \mathcal{C}_2(n,e),\ \mathcal{C}_3(n,e)\}\,.
\end{aligned}$$

Let

$$\mathfrak{W}_{n,e} := MOD(\mathfrak{C}(n,e))\,.$$

Then for each finite band $S$ it holds $S \in SD_\vee(n)$ if and only if there is a natural number $e$ such that $S \in \mathfrak{W}_{n,e}$.

Let $\mathcal{D}$ be that Green's relation on a band S defined as follows: For $a, b \in S$ it is $a\mathcal{D}b$ if and only if $a = aba$ and $b = bab$ (cf. [4]). Moreover, let $S/\mathcal{D}$ be the system of the $\mathcal{D}$-classes or Green's classes (with respect to $\mathcal{D}$) of $S$, respectively.

Then for each finite band $S$ the following holds: It is $|D| \leqq n$ for each $D \in S/\mathcal{D}$ if and only if $S \in \mathfrak{W}_{n,0}$ or $S \times F \in SD_{\vee}(n)$ for each finite semilattice $F$, respectively.

Consequently, for a finite band $S$ it is $S \in \mathfrak{W}_{n,0}$ if and only if (with respect to the notations in Petrich's structural theorem) $S$ is a finite semilattice $Y$ of rectangular bands $S_\gamma$ (which are the Green's classes of $S$) such that $|S_\gamma| \leqq n$ for $\gamma \in Y$ (cf. Theorem 1, [4]).

## 2. Results

At first it is proved that for each natural number $n \geqq 1$ the classes $SD_{\vee}(n)$ and $AE(n)$ coincide.

For this let $GEN(T)$ be the system of all generating sets of some $T \in L(S)$.

**Proposition 1.** *Let $S$ be a finite band and $1 \leqq n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $S \in SD_{\vee}(n)$.

(ii) $S \in AE(n)$.

**Proof.** (i) $\Longrightarrow$ (ii): Let $S \in SD_{\vee}(n)$, $A \in L(S)$ and $i_0, \ldots, i_n \in S \setminus A$. Then for

$$T := A \vee i_0 = A \vee i_1 = \ldots = A \vee i_n$$

it is $T \neq A$. By the assumption it follows

$$T = A \vee \bigvee \{\{i_k\} \wedge \{i_l\} \colon\ 0 \leqq k < l \leqq n\}\,.$$

Therefore, $|\{i_0, \ldots, i_n\}| \leqq n$ and $S \in AE(n)$.

(ii) $\Longrightarrow$ (i): Let $S \in AE(n)$, $T \in L(S)$. Obviously, for $X \subseteqq T$ it holds $X \in GEN(T)$ if and only if for each maximal subsemigroup $T' \subseteqq T$ with $T \setminus T' \neq \emptyset$ it is $(T \setminus T') \cap X \neq \emptyset$.

Moreover, $|T \setminus T'| \leqq n$. Otherwise, there exist $i_0, \ldots, i_n \in T \setminus T'$ such that $|\{i_0, \ldots, i_n\}| = n + 1$ and

$$T' \vee i_0 = T' \vee i_1 = \ldots T' \vee i_n,$$

contradicting $S \in AE(n)$.

Let $T, A, B_0, \ldots, B_n \in L(S)$ with

$$T = A \vee B_0 = \ldots = A \vee B_n\,,$$

i.e., $A \cup B_0, \ldots, A \cup B_n \in GEN(T)$ and

$$(T \setminus T') \cap (A \cup B_0) \neq \emptyset, \ldots, (T \setminus T') \cap (A \cup B_n) \neq \emptyset.$$

Then $(T \setminus T') \cap (A \cup \bigcup\{B_i \cap B_j\colon\ 0 \leqq i < j \leqq n\}) \neq \emptyset$: If $(T \setminus T') \cap A \neq \emptyset$, the assertion follows, directly. If $(T \setminus T') \cap A = \emptyset$, then $(T \setminus T') \cap B_i \neq \emptyset$ for $i = 0, \dots, n$ by the assumption. Because $|T \setminus T'| \leqq n$ it holds $(T \setminus T') \cap B_i \cap B_j \neq \emptyset$ for some $i, j$ with $0 \leqq i < j \leqq n$ and the assertion follows.

Consequently,

$$A \cup \bigcup\{B_i \cap B_j\colon\ 0 \leqq i < j \leqq n\} \in GEN(T)$$

and the statement holds. $\square$

**Example.** For $1 \leqq n \in \mathbb{N}$ let $L_n$ be the semigroup $(\{0, \dots, n\}, \circ)$ with $a \circ b = a$ for $a, b \in \{0, \dots, n\}$. Moreover, let $F$ be the semigroup $(\{0, 1\}, \cdot)$ with the usual multiplication. Then the direct product $S_n := L_n \times F$ is a finite band with $S_n \in AE(n+1)$ and $S_n \notin AE(n)$.

(a) It holds $S_n \in AE(n+1)$: Otherwise, there exist some $A \in L(S_n)$ and $i_0, \dots, i_{n+1} \in S_n \setminus A$ with $|\{i_0, \dots, i_{n+1}\}| = n+2$ and $i \in A \vee j$ for $i, j \in \{i_0, \dots, i_{n+1}\}$.

It is $\{i_0, \dots, i_{n+1}\} \subseteqq L_n \times \{0\}$ or $\{i_0, \dots, i_{n+1}\} \subseteqq L_n \times \{1\}$. Otherwise, there exist some $i \in L_n \times \{1\}$, $j \in L_n \times \{0\}$ with $i, j \notin A$ and $i \in A \vee j$, a contradiction.

Therefore, $|\{i_0, \dots, i_{n+1}\}| \leqq n+1$, contradicting the assumption.

(b) It holds $S_n \notin AE(n)$: For this let $A = L_n \times \{1\}$ and $i_0 = (0, 0), \dots, i_n = (n, 0) \in S_n \setminus A$. Then $|\{i_0, \dots, i_n\}| = n+1$ and $i \in A \vee j$ for $i, j \in \{i_0, \dots, i_n\}$. From this it follows the assertion. $\square$

From the Example and Proposition 1 it follows

**Proposition 2.** *For each natural number $n \geqq 1$ it is*

$$SD_\vee(n) \subsetneqq SD_\vee(n+1)\,. \qquad \square$$

Now it is shown that the class $SD_\vee(n)$ is equal to the class of all finite bands $S$ such that $S \in \mathfrak{W}_{n,e}$ for some natural number $e$.

**Proposition 3.** *Let $S$ be a finite band and $1 \leqq n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $S \in SD_\vee(n)$.

(ii) $S \in \mathfrak{W}_{n,e}$ *for some* $e \in \mathbb{N}$.

**Proof.** (i) $\Longrightarrow$ (ii): Let $S \in SD_\vee(n)$. Clearly, $S \in AE(n)$ by Proposition 1. Moreover, $\mathcal{C}_1(n,e) \varphi S$ and $\mathcal{C}_2(n,e) \varphi S$ and $\mathcal{C}_3(n,e) \varphi S$ for $e = |S|$.

Otherwise, for $k = 1, 2$ or $3$ it holds: There exists a homomorphism $f_k$ from $\mathbb{X}^+$ into $S$ such that for each $(p, q) \in \mathcal{C}_k(n,e)$ it is $f_k(p) \neq f_k(q)$.
Let $f_k(x_i) = s_i \in S$, $f_k(y_i) = t_i \in S$ for $i \in \mathbb{N}$ and $f_k(z) = c \in S$.

For $k = 1$ let

$$i_0 = (s_n \dots s_0)c\,,$$

$$i_m = (s_{m-1} \dots s_0)(s_n \dots s_0)c$$

where $1 \leqq m \leqq n$ and $A = \langle s_0, \dots, s_n \rangle$.

For $k = 2$ let

$$i_0 = c(t_0 \dots t_n)\,,$$
$$i_m = c(t_0 \dots t_n)(t_0 \dots t_{m-1})$$

where $1 \leqq m \leqq n$ and $A = \langle t_0, \dots , t_n\rangle$.

For $k = 3$ let

$$i_0 = (s_n \dots s_0)c(t_0 \dots t_n)\,,$$
$$i_m = (s_{m-1} \dots s_0)(s_n \dots s_0)c(t_0 \dots t_n)(t_0 \dots t_{m-1})$$

where $1 \leqq m \leqq n$ and $A = \langle s_0, \dots , s_n,\ t_0, \dots , t_n\rangle$.

Because of $\langle U\rangle = \{u_1 \dots u_i\colon u_1, \dots , u_i \in U, 1 \leqq i \leqq e\}$ for $U \subseteqq S$ and $e = |S|$ it follows $|\{i_0, \dots , i_n\}| = n + 1$ with $i_0, \dots , i_n \in S \setminus A$ and

$$A \vee i_0 = A \vee i_1 = \dots = A \vee i_n\,,$$

contradicting $S \in AE(n)$. Consequently,

$$S \in MOD(\mathfrak{C}(n, e)) = \mathfrak{W}_{n,e}$$

for $e = |S|$.

(ii)$\Longrightarrow$(i): Let $S \in \mathfrak{W}_{n,e}$ with $e \in \mathbb{N}$, $A \in L(S)$ and $i_0, \dots , i_n \in S \setminus A$ such that

$$A \vee i_0 = A \vee i_1 = \dots = A \vee i_n\,.$$

In the following it is proved that $|\{i_0, \dots , i_n\}| \leqq n$.

There hold the following implications (I), (II) and (III).

(I) If

$$i_0 = s_n i_n h_n \text{ with } h_n \in A \vee i_n \text{ and } s_n \in A,$$
$$i_m = s_{m-1} i_{m-1} h_{m-1} \quad \text{with} \quad h_{m-1} \in A \vee i_{m-1} \quad \text{and} \quad s_{m-1} \in A$$

where $1 \leqq m \leqq n$, then $(s_n \dots s_0)i_0 = i_0$ and from

$$(s_k \dots s_0)(s_n \dots s_0)i_0 = (s_l \dots s_0)(s_n \dots s_0)i_0$$

for some integer numbers $k, l$ with $-1 \leqq k < l \leqq n - 1$ it follows that $\{i_{k+1}, i_{l+1}\}$ is a right zero semigroup : By successively substitutions of the $i_0, \dots , i_n$ in the right hand sides of the above equations one gets

$$i_0 = (s_n \dots s_0)i_0(h_0 \dots h_n)\,,$$
$$i_m = (s_{m-1} \dots s_0)(s_n \dots s_0)i_0(h_0 \dots h_n)(h_0 \dots h_{m-1})$$

where $1 \leqq m \leqq n$.

Clearly, $(s_n \dots s_0)i_0 = i_0$ and

$$\begin{aligned} i_{k+1} &= (s_k \dots s_0)(s_n \dots s_0)i_0(h_0 \dots h_n)(h_0 \dots h_k)\,, \\ i_{k+1} &= (s_k \dots s_0)(s_n \dots s_0)i_0(h_0 \dots h_n)^2(h_0 \dots h_k)\,, \\ i_{l+1} &= (s_l \dots s_0)(s_n \dots s_0)i_0(h_0 \dots h_n)(h_0 \dots h_l) \\ &= (s_k \dots s_0)(s_n \dots s_0)i_0(h_0 \dots h_n)(h_0 \dots h_l)\,. \end{aligned}$$

Therefore, $i_{k+1}i_{l+1} = i_{l+1}$ and $i_{l+1}i_{k+1} = i_{k+1}$, i.e., $\{i_{k+1}, i_{l+1}\}$ is a right zero semigroup.

(II) If

$$\begin{aligned} i_0 &= g_n i_n t_n \text{ with } g_n \in A \vee i_n \text{ and } t_n \in A\,, \\ i_m &= g_{m-1} i_{m-1} t_{m-1} \quad \text{with} \quad g_{m-1} \in A \vee i_{m-1} \quad \text{and} \quad t_{m-1} \in A \end{aligned}$$

where $1 \leqq m \leqq n$, then $i_0(t_0 \dots t_n) = i_0$ and from

$$i_0(t_0 \dots t_n)(t_0 \dots t_k) = i_0(t_0 \dots t_n)(t_0 \dots t_l)$$

for some integer numbers $k, l$ with $-1 \leqq k < l \leqq n-1$ it follows that $\{i_{k+1}, i_{l+1}\}$ is a left zero semigroup: By successively substitutions of the $i_0, \dots, i_n$ in the right hand sides of the above equations one gets

$$\begin{aligned} i_0 &= (g_n \dots g_0)i_0(t_0 \dots t_n)\,, \\ i_m &= (g_{m-1} \dots g_0)(g_n \dots g_0)i_0(t_0 \dots t_n)(t_0 \dots t_{m-1}) \end{aligned}$$

where $1 \leqq m \leqq n$.

Clearly, $i_0(t_0 \dots t_n) = i_0$ and

$$\begin{aligned} i_{k+1} &= (g_k \dots g_0)(g_n \dots g_0)i_0(t_0 \dots t_n)(t_0 \dots t_k)\,, \\ i_{k+1} &= (g_k \dots g_0)(g_n \dots g_0)^2 i_0(t_0 \dots t_n)(t_0 \dots t_k)\,, \\ i_{l+1} &= (g_l \dots g_0)(g_n \dots g_0)i_0(t_0 \dots t_n)(t_0 \dots t_l) \\ &= (g_l \dots g_0)(g_n \dots g_0)i_0(t_0 \dots t_n)(t_0 \dots t_k)\,. \end{aligned}$$

Therefore, $i_{l+1}i_{k+1} = i_{l+1}$ and $i_{k+1}i_{l+1} = i_{k+1}$, i.e., $\{i_{k+1}, i_{l+1}\}$ is a left zero semigroup.

(III) If

$$\begin{aligned} i_0 &= s_n i_n q_n i_n t_n \quad \text{with} \quad q_n \in A \vee i_n \quad \text{and} \quad s_n, t_n \in A\,, \\ i_m &= s_{m-1} i_{m-1} q_{m-1} i_{m-1} t_{m-1} \quad \text{with} \quad q_{m-1} \in A \vee i_{m-1} \quad \text{and} \quad s_{m-1}, t_{m-1} \in A \end{aligned}$$

where $1 \leqq m \leqq n$, then $(s_n \dots s_0)i_0(t_0 \dots t_n) = i_0$ and from

$$(s_k \dots s_0)(s_n \dots s_0)i_0(t_0 \dots t_n)(t_0 \dots t_k) = (s_l \dots s_0)(s_n \dots s_0)i_0(t_0 \dots t_n)(t_0 \dots t_l)$$

for some integer numbers $k, l$ with $-1 \leqq k < l \leqq n-1$ it follows that $\{i_{k+1}, i_{l+1}\}$ is a left zero semigroup as well as a right zero semigroup, i.e., $i_{k+1} = i_{l+1}$:

By successively substitutions of the $i_0, \dots, i_n$ in the right hand sides of the above equations one gets

$$i_0 = (s_n \dots s_0) i_0 (h_0 \dots h_n)\,,$$
$$i_m = (s_{m-1} \dots s_0)(s_n \dots s_0) i_0 (h_0 \dots h_n)(h_0 \dots h_{m-1})$$

where $h_j = q_j i_j t_j$, $0 \leqq j \leqq n$, $1 \leqq m \leqq n$ and

$$i_0 = (g_n \dots g_0) i_0 (t_0 \dots t_n)\,,$$
$$i_m = (g_{m-1} \dots g_0)(g_n \dots g_0) i_0 (t_0 \dots t_n)(t_0 \dots t_{m-1})$$

where $g_j = s_j i_j q_j$, $0 \leqq j \leqq n$, $1 \leqq m \leqq n$.

Obviously,

$i_0 = (s_n \dots s_0) i_0$, $i_0 = i_0(t_0 \dots t_n)$ and $i_0 = (s_n \dots s_0) i_0 (t_0 \dots t_n)$.

Therefore,

$$\begin{aligned}
i_{k+1} &= (s_k \dots s_0)(s_n \dots s_0) i_0 (h_0 \dots h_n)(h_0 \dots h_k)\,,\\
i_{k+1} &= (s_k \dots s_0)(s_n \dots s_0) i_0 (h_0 \dots h_n)^2 (h_0 \dots h_k)\,,\\
i_{l+1} &= (s_l \dots s_0)(s_n \dots s_0) i_0 (h_0 \dots h_n)(h_0 \dots h_l)\\
&= (s_l \dots s_0)(s_n \dots s_0) i_0 (t_0 \dots t_n)(t_0 \dots t_l)(t_0 \dots t_n)(h_0 \dots h_n)(h_0 \dots h_l)\\
&= (s_k \dots s_0)(s_n \dots s_0) i_0 (t_0 \dots t_n)(t_0 \dots t_k)(t_0 \dots t_n)(h_0 \dots h_n)(h_0 \dots h_l)\\
&= (s_k \dots s_0)(s_n \dots s_0) i_0 (h_0 \dots h_n)(h_0 \dots h_l)\,.
\end{aligned}$$

Consequently, $i_{k+1} i_{l+1} = i_{l+1}$ and $i_{l+1} i_{k+1} = i_{k+1}$, i.e., $\{i_{k+1}, i_{l+1}\}$ is a right zero semigroup.

Analogously, it follows

$$\begin{aligned}
i_{k+1} &= (g_k \dots g_0)(g_n \dots g_0) i_0 (t_0 \dots t_n)(t_0 \dots t_k)\,,\\
i_{k+1} &= (g_k \dots g_0)(g_n \dots g_0)^2 i_0 (t_0 \dots t_n)(t_0 \dots t_k)\,,\\
i_{l+1} &= (g_l \dots g_0)(g_n \dots g_0) i_0 (t_0 \dots t_n)(t_0 \dots t_k)\,.
\end{aligned}$$

Hence, $i_{l+1} i_{k+1} = i_{l+1}$ and $i_{k+1} i_{l+1} = i_{k+1}$, i.e., $\{i_{k+1}, i_{l+1}\}$ is a left zero semigroup. Finally, $i_{k+1} = i_{l+1}$.

Furtherly, it hold the following statements:

(a) If there exist $i, j \in \{i_0, \dots, i_n\}$ such that $i = jpj$, $j = s(iqi)$ with $p \in A \vee j$, $q \in A \vee i$ and $s \in A$, then $i = s(iqi)pj = s(jpjqjpj)$.

(b) If there exist $i, j \in \{i_0, \dots, i_n\}$ such that $i = jpj$, $j = (iqi)t$ with $p \in A \vee j$, $q \in A \vee i$ and $t \in A$, then $i = jp(iqi)t = (jpjqjpj)t$.

(c) If there exist $i, j \in \{i_0, \dots, i_n\}$ such that $i = s(jpj)$, $j = (iqi)t$ with $p \in A \vee j$, $q \in A \vee i$ and $s, t \in A$, then
$i = sjp(iqi)t = sjp(sjpj)q(sjpj)t = s(jpjqsjpj)t$,
$j = s(jpj)qit = s(iqit)p(iqit)qit = s(iqitpiqi)t$.

Therefore, the following four cases are possible, exactly.

*Case 1.* There hold the following equations:

$$i_0 = i_n q_n i_n \text{ with } q_n \in A \vee i_n,$$
$$i_m = i_{m-1} q_{m-1} i_{m-1} \text{ with } q_{m-1} \in A \vee i_{m-1}$$

where $1 \leqq m \leqq n$.

Obviously, it is $\{i_0, \dots, i_n\}$ a right zero semigroup and a left zero semigroup, too. Therefore $i_0 i_1 = i_1$ and $i_0 i_1 = i_0$, i.e., $i_0 = i_1$ and $|\{i_0, \dots, i_n\}| \leqq n$.

*Case 2.* There hold the following equations:

$$i_0 = s_n i_n q_n i_n \text{ with } q_n \in A \vee i_n \text{ and } s_n \in A,$$
$$i_m = s_{m-1} i_{m-1} q_{m-1} i_{m-1} \text{ with } q_{m-1} \in A \vee i_{m-1} \text{ and } s_{m-1} \in A$$

where $1 \leqq m \leqq n$.

By (I) it is $(s_n \dots s_0) i_0 = i_0$. Because $S \in MOD(\{\mathcal{A}_1(n) \cup \mathcal{B}_1(n,e)\})$ and $i_0 \notin A$ it follows

$$(s_k \dots s_0)(s_n \dots s_0) i_0 = (s_l \dots s_0)(s_n \dots s_0) i_0$$

for some integer numbers $k, l$ with $-1 \leqq k < l \leqq n-1$. Then $\{i_{k+1}, i_{l+1}\}$ is a right zero semigroup by (I). Clearly, it is $\{i_0, \dots, i_n\}$ a left zero semigroup. Consequently, $i_{k+1} i_{l+1} = i_{l+1}$ and $i_{k+1} i_{l+1} = i_{k+1}$ , i.e., $i_{k+1} = i_{l+1}$ and $|\{i_0, \dots, i_n\}| \leqq n$.

*Case 3.* There hold the following equations:

$$i_0 = i_n q_n i_n t_n \text{ with } q_n \in A \vee i_n \text{ and } t_n \in A,$$
$$i_m = i_{m-1} q_{m-1} i_{m-1} t_{m-1} \text{ with } q_{m-1} \in A \vee i_{m-1} \text{ and } t_{m-1} \in A$$

where $1 \leqq m \leqq n$.

Because $S \in MOD(\{\mathcal{A}_2(n) \cup \mathcal{B}_2(n,e)\})$ and $i_0 \notin A$ it follows $|\{i_0, \dots, i_n\}| \leqq n$ by (II), analogously to Case 2.

*Case 4.* There hold the following equations:

$$i_0 = s_n i_n q_n i_n t_n \quad \text{with} \quad q_n \in A \vee i_n \quad \text{and} \quad s_n, t_n \in A,$$
$$i_m = s_{m-1} i_{m-1} q_{m-1} i_{m-1} t_{m-1} \quad \text{with} \quad q_{m-1} \in A \vee i_{m-1} \quad \text{and} \quad s_{m-1}, t_{m-1} \in A$$

where $1 \leqq m \leqq n$. By (III) it is $(s_n \dots s_0) i_0 (t_0 \dots t_n) = i_0$.

Because $S \in MOD(\{\mathcal{A}_3(n) \cup \mathcal{B}_3(n,e)\})$ and $i_0 \notin A$ it follows

$$(s_k \dots s_0)(s_n \dots s_0) i_0 (t_0 \dots t_n)(t_0 \dots t_k) = (s_l \dots s_0)(s_n \dots s_0) i_0 (t_0 \dots t_n)(t_0 \dots t_l)$$

for some integer numbers $k, l$ with $-1 \leqq k < l \leqq n-1$. Consequently, $i_{k+1} = i_{l+1}$ by (III). Therefore, $|\{i_0, \dots, i_n\}| \leqq n$.

From the Cases 1, 2, 3 and 4 it follows that $S \in AE(n)$ and $S \in SD_\vee(n)$ by Proposition 1.

$\square$

Finally, it is shown that a finite band $S$ belongs to $\mathfrak{W}_{n,0}$ if and only if $|D| \leqq n$ for each Green's class $D$ with respect to the relation $\mathcal{D}$ on $S$.

For this let $J$ be the *ideal closure operator on* $S$, i.e., for each $U \subseteq S$ it is

$$J(U) := U \cup S \cdot U \cup U \cdot S \cup S \cdot U \cdot S.$$

Let $a, b \in S$. Then it follows easily that $a\mathcal{D}b$ if and only if $J(\{a\}) = J(\{b\})$.

**Proposition 4.** *Let $S$ be a finite band and $1 \leqq n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $S \in \mathfrak{W}_{n,0}$.

(ii) $S \times F \in SD_\vee(n)$ *for each finite semilattice* $F$.

(iii) $|D| \leqq n$ *for each* $D \in S/\mathcal{D}$.

**Proof.** It is easy to check that (i) if and only if (ii) by Proposition 3.

(ii) $\Longrightarrow$ (iii): Let $\{0,1\}$ be that semilattice with respect to multiplication. Then $S \times \{0,1\} \in SD_\vee(n)$ by (ii) and $S \times \{0,1\} \in AE(n)$ by Proposition 1. Now let $D \in S/\mathcal{D}$ and $i_0, \ldots, i_n \in D$. Then $J(\{i_0\}) = \ldots = J(\{i_n\})$. Therefore

$$S \times \{1\} \vee (i_0, 0) = S \times \{1\} \vee (i_1, 0) = \ldots = S \times \{1\} \vee (i_n, 0)$$

and

$$(i_0, 0), \ldots, (i_n, 0) \in S \times \{0,1\} \setminus S \times \{1\}\,.$$

Because of $S \times \{0,1\} \in AE(n)$ it follows $|D| \leqq n$.

(iii) $\Longrightarrow$ (ii): Clearly, if $|D| \leqq n$ for each $D \in S/\mathcal{D}$, then $|D'| \leqq n$ for each $D' \in (S \times F)/\mathcal{D}$ and each finite semilattice $F$, too. Hence, if $J(\{i'_0\}) = \ldots = J(\{i'_n\})$ for $i'_0, \ldots, i'_n \in S \times F$, then $|\{i'_0, \ldots, i'_n\}| \leqq n$.

Consequently, $S \times F \in AE(n)$ and $S \times F \in SD_\vee(n)$ by Proposition 1. $\square$

From Proposition 4 it follows that each finite band $S \in \mathfrak{W}_{n,0}$ is characterized by Petrich's structural theorem restricted to a finite semilattice $Y$ of rectangular bands $S_\gamma$ under the condition $|S_\gamma| \leqq n$ for $\gamma \in Y$.

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