ARCHIVUM MATHEMATICUM (BRNO)
Tomus 39 (2003), 123 – 127

ON THE CONVERGENCE OF THE ISHIKAWA ITERATES TO A COMMON FIXED POINT OF TWO MAPPINGS

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ABSTRACT. Let C be a convex subset of a complete convex metric space X, and S and T be two selfmappings on C. In this paper it is shown that if the sequence of Ishikawa iterations associated with S and T converges, then its limit point is the common fixed point of S and T. This result extends and generalizes the corresponding results of Naimpally and Singh [6], Rhoades [7] and Hicks and Kubicek [3].

In recent years several authors ([3], [7], [9], [11], [12]) have studied the convergence of the sequence of the Mann iterates [5] of a mapping T to a fixed point of T, under various contractive conditions.

The Ishikawa iteration scheme [4] was first used to establish the strong convergence for a pseudo contractive selfmapping of a convex compact subset of a Hilbert space. Very soon both iterative processes were used to establish the strong convergence of the respective iterates for some contractive type mappings in Hilbert spaces and then in more general normed linear spaces.

Recently, Naimpally and Singh [6] have studied the mappings which satisfy the contractive definition introduced in [2]. They proved the following:

Theorem 1 [6]. *Let X be a normed linear space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a selfmapping satisfying*

$$\text{(A)}\quad \|Tx - Ty\| \le h \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\}$$

for all x, y in C, where $0 \le h < 1$ and let $\{x_n\}$ be the sequence of the Ishikawa-scheme associated with T, that is, $x_0 \in C$,

$$y_n = (1-\beta_n)x_n + \beta_n T x_n\,, \qquad n \ge 0\,,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n\,, \qquad n \ge 0\,,$$

2000 *Mathematics Subject Classification*: 47H10, 54H25.
Key words and phrases: Ishikawa iterates, comon fixed point, convex metric space.
*The second author was supported by Korea Research Foundation Grant (KRF-2001-015-DP0025).
Received April 3, 2001.

where $0 \le \alpha_n,\ \beta_n \le 1$. *If* $\{\alpha_n\}$ *is bounded away from zero and if* $\{x_n\}$ *converges to* p*, then* p *is a fixed point of* T.

The purpose of this paper is to generalize the result of Naimpally and Singh to a pair of mappings S and T, defined on a convex metric space, which satisfy the following condition:

$$d(Sx,Ty) \le h\,[d(x,y) + d(x,Ty) + d(y,Sx)]\ , \tag{B}$$

where $0 < h < 1$.

It is clear that the condition (B) is very general, since by the triangle inequality, condition (B) is always satisfied with $h = 1$.

We need the following definition.

Definition 1 [13]**.** Let X be a metric space and $I = [0,1]$ the closed unit interval. A continuous mapping $W : X \times X \times I \to X$ is said to be a *convex structure on* X if for all x, y in X, λ in I, $d[u, W(x,y,\lambda)] \le \lambda d(u,x) + (1-\lambda)d(u,y)$ for all u in X. A space X together with a convex structure is called a convex metric space.

Clearly a Banach space, or any convex subset of it, is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x+(1-\lambda)y, 0) \le \lambda d(x,0)+(1-\lambda)d(y,0)$, then X is a convex metric space. There are many other examples but we consider these as paradigmatic.

Now we are in a position to state our main result.

Theorem 2. *Let* C *be a nonempty closed convex subset of a convex metric space* X *and let* $S,T : X \to X$ *be selfmappings satisfying* (B) *for all* x, y *in* C. *Suppose that* $\{x_n\}$ *is Ishikawa type iterative scheme associated with* S *and* T, *defined by*

$$x_0 \in C\,, \tag{1}$$

$$y_n = W(Sx_n, x_n, \beta_n)\,, \quad n \ge 0 \tag{2}$$

$$x_{n+1} = W(Ty_n, x_n, z_n)\,, \quad n \ge 0 \tag{3}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* $0 \le \alpha_n,\ \beta_n \le 1$ *and* $\{\alpha_n\}$ *is bounded away from zero. If* $\{x_n\}$ *converges to some point* $p \in C$, *then* p *is the common fixed point of* S *and* T.

Proof. It is clear that

$$d(x,y) \le d\,[x, W(x,y,\lambda)] + d\,[W(x,y,\lambda), y] \le (1-\lambda)d(x,y) + \lambda d(x,y) = d(x,y)$$

implies the following:

$$d\,[x, W(x,y,\lambda)] = (1-\lambda)d(x,y)\,; \quad d\,[y, W(x,y,\lambda)] = \lambda d(x,y)\,.$$

From (3) it follows that

$$d(x_n, x_{n+1}) = d\left[x_n, W(Ty_n, x_n, \alpha_n)\right] = \alpha_n d(x_n, Ty_n)\,.$$

Since $x_n \to p$, $d(x_n, x_{n+1}) \to 0$. Since $\{\alpha_n\}$ is bounded away from zero, it follows that

$$\lim_{n\to\infty} d(x_n, Ty_n) = 0\,. \tag{4}$$

Using (B) we get:

$$d(Sx_n, Ty_n) \le h\left[d(x_n, y_n) + d(x_n, Ty_n) + d(y_n, Sx_n)\right]\,.$$

From (2) and (3) we have

$$d(x_n, y_n) = d\left[x_n, W(Sx_n, x_n, \beta_n)\right] = \beta_n d(x_n, Sx_n)\,,$$
$$d(Sx_n, y_n) = d\left[Sx_n, W(Sx_n, x_n, \beta_n)\right] = (1-\beta_n) d(x_n, Sx_n)\,.$$

Thus we have

$$d(Sx_n, Ty_n) \le h\left[d(x_n, Sx_n) + d(x_n, Ty_n)\right]\,.$$

Since $d(x_n, Sx_n) \le d(Sx_n, Ty_n) + d(x_n, Ty_n)$, we get

$$d(Sx_n, Ty_n) \le h\left[d(Sx_n, Ty_n) + 2d(x_n, Ty_n)\right]\,.$$

Hence

$$d(Sx_n, Ty_n) \le \frac{2h}{1-h} d(x_n, Ty_n)\,.$$

Taking the limit as $n \to \infty$ we obtain, by (4),

$$\lim_{n\to\infty} d(Sx_n, Ty_n) = 0\,.$$

Since $Ty_n \to p$, it follows that $Sx_n \to p$. Since $d(x_n, y_n) = \beta_n d(x_n, Sx_n)$, it follows also that $y_n \to p$.

From (B) again, we have

$$d(Sx_n, Tp) \le h\left[d(x_n, p) + d(x_n, Tp) + d(p, Sx_n)\right]\,.$$

Taking the limit as $n \to \infty$, we obtain

$$d(p, Tp) \le hd(p, Tp)\,.$$

Since $h < 1$, $d(p, Tp) = 0$. Hence $Tp = p$. Similarly, from (B),

$$d(Sp, Tx_n) \le h\left[d(p, x_n) + d(p, Tx_n) + d(x_n, Sp)\right]\,.$$

Taking the limit as $n \to \infty$ we get

$$d(Sp, p) \le hd(p, Sp)\,.$$

Hence $Sp = p$. Therefore, $Sp = Tp = p$ and the proof is complete. $\square$

Corollary. *Let X be a normed linear space and C be a closed convex subset of X. Let $S,T : C \to C$ be two mappings satisfying (B) and $\{x_n\}$ be the sequence of Ishikawa-scheme associated with S and T; for $x_0 \in C$,*

$$y_n = (1-\beta_n)x_n + \beta_n T x_n\,, \quad n \geq 0\,,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n\,, \quad n \geq 0\,,$$

If $\{\alpha_n\}$ is bounded away from zero and $\{x_n\}$ converges to p, then p is a common fixed point of S and T.

Remark. Corollary with $S = T$ is a generalization of Theorem 1 of Naimpally and Singh [6]. Therefore, Theorem 9 of Rhoades [9] is also a special case of Corollary, where (B) is replaced with the following condition, introduced in [1],

(B') $$d(Tx,Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}\,,$$

where $0 < h < 1$.

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