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REGULAR HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. We introduce the concept of the regular (nonoscillatory) half-linear second order differential equation

$$(*) \qquad \big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0\,, \quad \Phi(x) := |x|^{p-2}x\,, \quad p > 1$$

and we show that if (*) is regular, a solution x of this equation such that $x'(t) \neq 0$ for large t is principal if and only if

$$\int^{\infty} \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty\,.$$

Conditions on the functions r, c are given which guarantee that (*) is regular.

1. Introduction and preliminaries

The aim of this paper is to investigate half-linear second order differential equations

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0\,, \quad \Phi(x) := |x|^{p-2}x\,, \quad p > 1\,, \tag{1}$$

where r, c are continuous functions, $r(t) > 0$, via the properties of solutions of the associated Riccati differential equation (related to (1) by the substitution $w = \frac{r(t)\Phi(x')}{\Phi(x)}$)

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0\,, \quad q := \frac{p}{p-1}\,. \tag{2}$$

In particular, we will focus our attention to the so-called regular half-linear equations and to the integral characterization of their principal solutions.

The notion of the principal solution of the nonoscillatory Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0 \tag{3}$$

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(which is the special case $p = 2$ of (1)) was introduced by Leighton and Morse [12] and this notion plays an important role in the oscillation theory of (3). Recall that a solution $\tilde{x}(t)$ of nonoscillatory equation (3) is said to be *principal* (at $t = \infty$) if

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{x(t)} = 0 \tag{4}$$

for any solution $x(t)$ which is linearly independent of $\tilde{x}(t)$. Differentiating the ratio $x(t)/\tilde{x}(t)$ and using the Wronskian identity $r(x'\tilde{x} - x\tilde{x}') = \text{const} \neq 0$, it follows that (4) is equivalent to

$$\int^\infty \frac{dt}{r(t)\tilde{x}^2(t)} = \infty\,. \tag{5}$$

Clearly, the principal solution $\tilde{x}(t)$ of (1) is unique up to a constant multiple. On the other hand, for any other solution $x(t)$, linearly independent of $\tilde{x}(t)$, the relation

$$\int^\infty \frac{dt}{r(t)x^2(t)} < \infty$$

holds.

Mirzov [15] extended the concept of the principal solution to half-linear equation (1) and defined this solution via the eventually minimal solution of the associated Riccati equation (2). This method is known to be the equivalent definition of the principal solution in the linear case, see [10, Chap. XI]. Elbert and Kusano [8] defined this concept as the "zero maximal" solution (which is in the linear case also equivalent characterization of the principal solution, see [13]), and showed that their definition is equivalent to Mirzov's one. The attempt to find an integral characterization of the principal solution of (1) which reduces to (5) in the linear case has been made in [3]. It was shown that the property being the principal solution is closely related to the divergence of the integral

$$I(x) := \int^\infty \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}}\,. \tag{6}$$

In this paper we continue in this investigation and we find conditions on the functions r, c which guarantee that a solution $\tilde{x}$ of (1) is principal if and only if $I(\tilde{x}) = \infty$.

2. Riccati equation, Picone's identity and principal solution

In this section we recall some basic facts concerning equations (1), (2), in particular, their principal and eventually minimal solutions.

It is well known that the classical Sturmian theory extends almost verbatim to (1), see e.g. [5, 14]. In particular, equation (1) can be classified as oscillatory or nonoscillatory according to the oscillatory nature of its solution near ∞. First we recall Mirzov's definition of the principal solution of (1). Suppose that this equation is nonoscillatory and let $\bar{x}$ be its solution for which $\bar{x}(t) \neq 0$ for $t > T$. Further, for $b > T$, let x_b be the solution of (1) given by the initial condition $x_b(b) = 0$, $x'(b) = -1$. Let $\bar{w} = \frac{r\Phi(\bar{x}')}{\Phi(\bar{x})}$, $w_b = \frac{r\Phi(x_b')}{\Phi(x_b)}$ be the corresponding solutions of (2). Then $w(b-) = -\infty$ and $w_b(t) < \bar{w}(t)$ on $[T, b)$. Moreover, if $T < b < \tilde{b}$ then $w_b(t) < w_{\tilde{b}}(t) < \bar{w}(t)$ on $[T, b)$. As $b \to \infty$, the functions w_b converge uniformly on every compact interval $[T, T_1] \subset [T, \infty)$ to a function $\tilde{w}$ which is also a solution (2). This solution has the property that any other solution w of (2) which is defined on the whole interval $[T, \infty)$ satisfies $w(t) > \tilde{w}(t)$ in this interval. Now, if

$$\tilde{x}(t) = \exp\left\{\int^t r^{1-q}(s)\Phi_q(\tilde{w}(s))\,ds\right\}, \quad \Phi_q(w) := |w|^{q-2}w\,,$$

then $\tilde{x}$ is a solution of (1) which is called the *principal solution* of this equation.

Elbert and Kusano [8] used a somewhat different construction based on the generalized Prüfer transformation and generalized trigonometric functions. They proved the following comparison theorem for (1) and for another differential equation of the same form

$$(R(t)\Phi(y'))' + C(t)\Phi(y) = 0\,. \tag{7}$$

Lemma 1. *Suppose that*

$$0 < R(t) \le r(t)\,, \quad C(t) \ge c(t)$$

hold for large t, i.e. (7) *is a Sturmian majorant of* (1). *Further suppose that equation* (7) *is nonoscillatory and let $\tilde{x}$, $\tilde{y}$ be principal solutions of* (1) *and* (7), *respectively. Denote by $\tilde{w} = r(t)\Phi(\tilde{x}'/\tilde{x})$, $\tilde{v} = R(t)\Phi(\tilde{y}'/\tilde{y})$ the corresponding eventually minimal solutions of* (2) *and of*

$$v' + C(t) + (p-1)R^{1-q}(t)|v|^q = 0\,. \tag{8}$$

Then $\tilde{v}(t) \ge \tilde{w}(t)$ for large t. Moreover, if t_0 is sufficiently large and w, v are solutions of (2) *and* (8), *respectively, which exist on the whole interval $[t_0, \infty)$ and satisfy $v(t_0) \le w(t_0)$, then $v(t) \le w(t)$ for $t \ge t_0$.*

Now we formulate (in a simplified form as needed in this paper) the recently found Picone type identity for (1), (see [1, 11]).

Lemma 2. *Suppose that w is a solution of* (2) *defined in the whole interval $I = [a, b]$. Then for any $y \in C^1(I)$ the following identity holds:*

$$[r(t)|y'|^p - c(t)|y|^p] = [w|y|^p]' + pr^{1-q}(t)P(r^{q-1}y', \Phi(y)w)\,,$$

where

$$P(u, v) = \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \ge 0 \tag{9}$$

for any $u, v \in \mathbb{R}$, with the equality if and only if $v = \Phi(u)$.

The next statement (proved in [3]) compares the function P with the quadratic part of its Taylor's expansion (with respect to v) at the center $v_0 = \Phi(u)$).

Lemma 3. *The function $P(u, v)$ defined in* (9) *satisfies the following inequalities*

$$P(u, v) \gtrless \frac{1}{2}|u|^{2-p}\,(v - \Phi(u))^2 \quad \text{for} \quad p \lessgtr 2\,, \quad v \ne \Phi(u)\,, \tag{10}$$

and

$$P(u,v) \lesseqgtr \frac{1}{2(p-1)}|u|^{2-p}\left(v-\Phi(u)\right)^2 \quad \text{for} \quad p \lesseqgtr 2\,,\ \Phi(|u|)>|v|\,,\ uv>0\,. \tag{11}$$

Of course, in case $p=2$ we have the equality in both relations (10) and (11). The previous two lemmas play the crucial role in the proofs of the main statements of [3] which are summarized in the next proposition.

Proposition 1. *Suppose that* (1) *is nonoscillatory and* $\tilde{x}$ *is its solution such that* $\tilde{x}'(t) \neq 0$ *for large* t.

(i) *Let* $p \in (1,2)$. *If*

$$I(\tilde{x}) := \int^{\infty} \frac{dt}{r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}} = \infty\,, \tag{12}$$

then $\tilde{x}$ *is the principal solution.*

(ii) *Let* $p>2$. *If* $\tilde{x}$ *is the principal solution then* (12) *holds.*

(iii) *Suppose that* $\int^{\infty} r^{1-q}(t)\,dt = \infty$, *the function*

$$\gamma(t) := \int_t^{\infty} c(s)\,ds$$

exists and $\gamma(t) \geq 0$ *but* $\gamma(t) \not\equiv 0$ *eventually. Then* $\tilde{x}(t)$ *is the principal solution if and only if* (12) *holds.*

Note that the proof of the part (i) is based on inequality (10) with $p \in (1,2)$, whereas (ii) leans on this inequality for $p>2$. The proof of (iii) uses the fact that under additional restrictions on r, c given there all solutions of (2) which are extensible up to ∞ are positive and one can then use also inequalities (11).

Generally, as pointed out in [3], the fact that (12) equivalently characterizes the principal solution $\tilde{x}$ of (1) can be proved whenever the inequalities of the form

$$P(u,v) > C_1|u|^{2-p}(v-\Phi(u))^2\,, \quad P(u,v) < C_2|u|^{2-p}(v-\Phi(u))^2\,,$$

C_1, C_2 being positive real constants, hold. This observation motivates the following statement.

Lemma 4. *Let* $T>0$ *be arbitrary. There exists a constant* $K>0$ *such that*

$$P(u,v) \lesseqgtr K|u|^{2-p}\left(v-\Phi(u)\right)^2 \quad \text{for} \quad p \lesseqgtr 2 \tag{13}$$

and every $u, v \in \mathbb{R}$ *satisfying* $\left|\frac{v}{\Phi(u)}\right| \leq T$, $v \neq \Phi(u)$.

Proof. Consider the case $p \in (1,2)$, i.e. $q>2$, the case $p>2$ can be treated in a similar way. For any $K>0$ and $u \neq 0$ we have

$$P(u,v) - K|u|^{2-p}(v-\Phi(u))^2 = |u|^p\left\{\frac{1}{q}\left|\frac{v}{\Phi(u)}\right|^q - \frac{v}{\Phi(u)} + \frac{1}{p} - K\left(\frac{v}{\Phi(u)}-1\right)^2\right\}.$$

Denote $t := \frac{v}{\Phi(u)}$, $f(t) := \frac{|t|^q}{q} - t + \frac{1}{p}$, $g(t) := K(t-1)^2$. If $K > \frac{q-1}{2}$, by a direct computation one can verify that $(f-g)(1) = 0 = (f-g)'(1)$, the function $f-g$ has the local maximum at $t=1$, two negative local minima attained at the numbers whose absolute value is greater than $\tilde{T} := \left(\frac{2K}{q-1}\right)^{\frac{1}{q-2}}$ (where the graph of $(f-g)$ has the inflection), and $(f-g)$ is concave for $|t| < \tilde{T}$ and convex for $|t| > \tilde{T}$. Consequently, if we chose K such that $(f-g)(-T) < 0$ then according to the previous considerations and the fact that $f(-t) = f(t) + 2t$ we have $(f-g)(t) < 0$ for $t \in [-T, T]$, i.e.

$$\frac{|t|^q}{q} - t + \frac{1}{p} < K(t-1)^2$$

which implies (13) for $q > 2$. The existence of a K with the required property follows from the second degree Taylor expansion formula for f at $t_0 = 1$ (with $\xi \in (1, T)$)

$$\begin{aligned} f(-T) - g(-T) &= f(T) + 2T - K(T+1)^2 \\ &= \frac{q-1}{2}(T-1)^2\left[1 + \frac{(q-2)(T-1)}{3}\xi^{q-3}\right] + 2T - K(T+1)^2, \end{aligned}$$

so, if we take K sufficiently large, then $(f-g)(-T) < 0$ really holds. □

3. Regular half-linear equations

We start with the basic definition of our paper.

Definition 1. Suppose that (1) is nonoscillatory. Equation (1) is said to be *regular* if there exists a positive constant $T \in \mathbb{R}$ such that

$$\limsup_{t\to\infty} \left|\frac{w_1(t)}{w_2(t)}\right| < T \tag{14}$$

for any pair of solution w_1, w_2 of (2) for which $w_2(t) > w_1(t)$ eventually.

Next theorem justifies the introduction of the concept of regular half-linear second order equations, for these equations we have the equivalent integral characterization of the principal solution of these equations.

Theorem 1. *Suppose that* (1) *is regular and x is its solution such that $x' \neq 0$ eventually. Then x is the principal solution if and only if*

$$I(x) := \int^{\infty} \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty\,. \tag{15}$$

Proof. We distinguish the cases $p \in (1,2)$ and $p > 2$. In the first case we have the implication "$I(x) = \infty \implies x$ is the principal solution" by Proposition 1, part (i). To prove the opposite implication we borrow some ideas used in [3]. Suppose, by contradiction, that x is the principal solution and $I(x) < \infty$. Let T be the constant from (14). By Lemma 4 there exists $K > 0$ such that (13) holds if $\left|\frac{v}{\Phi(u)}\right| < T$. Further, let $t_0 \in \mathbb{R}$ be such that

$$\int_{t_0}^{\infty} \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} < \frac{1}{Kp}$$

and consider the solution w of (2) given by the initial condition $w(t_0) = \tilde{w}(t_0) - (2x^p(t_0))^{-1}$, where $\tilde{w} = r\Phi\left(\frac{x'}{x}\right)$ is the eventually minimal solution of (2). Substituting the principal solution x into the Picone identity with the above defined solution w of (2), we have

$$r(t)|x'|^p - c(t)x^p = [x^p w]' + pr^{1-q}(t)x^p(t)P(\Phi_q(\tilde{w}), w)\,,$$

where $\Phi_q(s) := |s|^{q-2}s$ is the inverse function of Φ, and at the same time, replacing w by $\tilde{w}$ and using the fact that $P(\Phi_q(\tilde{w}), \tilde{w}) = 0$,

$$r(t)|x'|^p - c(t)|x|^p = [x^p\tilde{w}]' + pr^{1-q}(t)x^p(t)P(\Phi_q(\tilde{w}), \tilde{w}) = [x^p\tilde{w}]'.$$

Subtracting the last two equalities, we get

$$[x^p(\tilde{w} - w)]' = pr^{1-q}(t)x^p P(\Phi_q(\tilde{w}), w)\,. \tag{16}$$

By (13)

$$P(\Phi_q(\tilde{w}), w) < K|\Phi_q(\tilde{w})|^{2-p}\,(\tilde{w} - w)^2\,.$$

This implies, in view of (16),

$$\begin{aligned}[x^p(\tilde{w} - w)]' &< Kpr^{1-q}x^p \left|r^{1-q}\frac{x'}{x}\right|^{2-p} (\tilde{w} - w)^2 \\ &= \frac{Kp}{rx^2|x'|^{p-2}}\,[x^p(\tilde{w} - w)]^2\,.\end{aligned}$$

Denote $h := x^p(\tilde{w} - w)$. Then $h(t_0) = \frac{1}{2}$ and the last inequality reads

$$\frac{h'(t)}{h^2(t)} < \frac{Kp}{r(t)x^2(t)|x'(t)|^{p-2}}. \tag{17}$$

Integrating this inequality from t_0 to t we have

$$\frac{1}{h(t_0)} - \frac{1}{h(t)} < \int_{t_0}^{t} \frac{Kp}{r(s)x^2(s)|x'(s)|^{p-2}}\,ds\,,$$

which means, taking into account that $h(t_0) = \frac{1}{2}$,

$$h(t) < \frac{1}{2 - Kp\int_{t_0}^{\infty} \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}}\,dt}\,.$$

Consequently, $\frac{1}{2} < h(t) < 1$, for $t \in [t_0, \infty)$, i.e. $h(t)$ can be continued up to ∞. Hence $w(t)$ is a solution of (2) which is extensible up to ∞ and $w(t) < \tilde{w}(t)$ for $t \geq t_0$, i.e. $\tilde{w}(t)$ is not the eventually minimal solution and thus the solution $x(t)$ is not principal, a contradiction.

If $p > 2$, the implication "x is the principal solution $\implies I(x) = \infty$" is given in Proposition 1, part (ii). Concerning the opposite implication, suppose that $I(x) = \infty$ and x is not the principal solution, i.e. there exists a solution w of (2) which is defined on some interval $[t_0, \infty)$ and satisfies there the inequality $w < \tilde{w} := r\Phi(\frac{x'}{x})$. If h is the same as in the first part of the proof, using a similar reasoning as above we have the inequality

$$\frac{h'(t)}{h^2(t)} > \frac{Kp}{r(t)x^2(t)|x'(t)|^{p-2}} \quad \text{for} \quad t \geq t_0$$

and integrating this inequality from t_0 to $t_1 > t_0$ we get

$$\frac{1}{h(t_0)} > \frac{1}{h(t_0)} - \frac{1}{h(t_1)} > Kp \int_{t_0}^{t_1} \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}}\,.$$

Letting $t_1 \to \infty$ we have the required contradiction. Hence x is the principal solution. □

Remark 1. It is shown in [3] that under the assumptions of the part (iii) of Proposition 1, every solution of (2) is eventually positive. This means that (1) is regular since $0 < w_1(t) < w_2(t)$ is equivalent to $0 < \frac{w_1(t)}{w_2(t)} < 1$. From this point of view, the previous theorem is a natural extension of the integral characterization of the principal solution of (1) given in Proposition 1 (iii).

In the next part of this section we deal with (1) under the restriction $r(t) \equiv 1$, i.e. we investigate the equation

$$(\Phi(x'))' + c(t)\Phi(x) = 0\,. \tag{18}$$

Using the transformation of the independent variable

$$x(t) = y(s), \quad s = \int^t r^{1-q}(\tau)\, d\tau$$

which transforms (1) into (18), the results can be extended to general equation (1). In the previous remark we have shown that (18) with positive c (and $\int^\infty c(t)\,dt$ convergent) is a typical example of the regular half-linear equation. Next we deal with the case when the function c is negative and we present sufficient conditions on the function c which guarantee that (18) is regular.

Theorem 2. *Suppose that*

$$\lim_{t\to\infty} c(t) = -\alpha^2 < 0\,. \tag{19}$$

Then equation (18) *is regular.*

Proof. According to the definition we have to find a positive constants T such that (14) holds. We use the following notation. Let

$$\tilde{c}(t) := -\frac{c(t)}{p-1} > 0 \quad \text{and} \quad \beta^q := \frac{\alpha^2}{p-1} > 0\,.$$

Then our assumption $\lim_{t\to\infty} c(t) = -\alpha^2$ is equivalent to $\lim_{t\to\infty} \tilde{c}(t) = \beta^q$. This means that for every $0 < \varepsilon < \beta$ there exists $t_0 \in \mathbb{R}$ such that

$$(\beta - \varepsilon)^q < \tilde{c}(t) < (\beta + \varepsilon)^q$$

for every $t \ge t_0$. Now, consider the equations (the first one is rewritten equation (18))

$$(\Phi(x'))' - (p-1)\tilde{c}(t)\Phi(x) = 0\,, \tag{20}$$

$$(\Phi(y'))' - (p-1)(\beta+\varepsilon)^q\Phi(y) = 0\,, \tag{21}$$

$$(\Phi(z'))' - (p-1)(\beta-\varepsilon)^q\Phi(z) = 0\,, \tag{22}$$

i.e. (21) is the Sturmian minorant to (20) and (22) is the Sturmian majorant to (20). The Riccati type equations associated with (20), (21) and (22) are

$$w'(t) - (p-1)[\tilde{c}(t) - |w(t)|^q] = 0\,, \tag{23}$$

$$w_y'(t) - (p-1)[(\beta+\varepsilon)^q - |w_y(t)|^q] = 0\,, \tag{24}$$

$$w_z'(t) - (p-1)[(\beta-\varepsilon)^q - |w_z(t)|^q] = 0\,. \tag{25}$$

Equation (24) has constant solutions $w_y = \pm(\beta+\varepsilon)$ and any other solution of this equation satisfies

$$\lim_{t\to\infty} w_y(t) = \beta+\varepsilon\,, \tag{26}$$

see [6]. Similarly, (25) has constant solutions $w_z = \pm(\beta-\varepsilon)$ and all other solutions satisfy $\lim_{t\to\infty} w_z(t) = \beta-\varepsilon$.

Let $\tilde{w}(t)$ be the minimal solution of (23). Then according to the Sturmian comparison theorem for eventually minimal solutions of Riccati equations (Lemma 1) we obtain

$$-(\beta+\varepsilon) \le \tilde{w}(t) \le -(\beta-\varepsilon) \tag{27}$$

where $-(\beta+\varepsilon) = \tilde{w}_y(t)$ is the minimal solution of (24) and $-(\beta-\varepsilon) = \tilde{w}_z(t)$ is the minimal solution of (25). Let $w(t)$ be a solution of (23) which is not minimal. Then there exists $t_0 \in \mathbb{R}$ such that

$$w(t) > \tilde{w}(t) \quad t \ge t_0$$

and using (27) we have

$$w(t) > -(\beta+\varepsilon) \quad \text{for} \quad t \ge t_0\,. \tag{28}$$

Now, let w_y, w_z be the solutions of (24) and (25) given by the initial condition $w_y(t_0) = w(t_0) = w_z(t_0)$. Then by Lemma 1

$$w_z(t) \le w(t) \le w_y(t) \tag{29}$$

for every $t \ge t_0$. We distinguish the following cases according to the value of the initial condition $w(t_0)$:

(a) $-(\beta+\varepsilon) < w(t_0) \le -(\beta-\varepsilon)$. Then

$$w(t) \ge w_z(t) > -(\beta+\varepsilon) \quad \text{for} \quad t \ge t_0 \tag{30}$$

what we already know from (28).

(b) $-(\beta-\varepsilon) < w(t_0) \le \beta-\varepsilon$. Then using (29) we have

$$w(t) > \beta-\varepsilon\,, \quad t \ge t_0\,. \tag{31}$$

(c) $\beta-\varepsilon < w(t_0) \le \beta+\varepsilon$. Then according to (29) we have

$$\beta-\varepsilon \le w(t) \le \beta+\varepsilon\,, \quad t \ge t_0\,. \tag{32}$$

(d) $w(t_0) > \beta+\varepsilon$. Then by (29) and the fact that for all solutions of (24) relation (26) holds, we obtain

$$\limsup_{t\to\infty} w(t) < \beta+2\varepsilon\,. \tag{33}$$

From (28) - (33) we conclude that

$$-(\beta+\varepsilon) < w(t) < \beta + 2\varepsilon\,, \quad t \ge t_0\,, \tag{34}$$

for any proper solution $w(t)$ of (23) which is not minimal, and hence using (27) and (34) we obtain that

$$-\frac{\beta+2\varepsilon}{\beta-\varepsilon} < \frac{w(t)}{\tilde{w}(t)} < \frac{\beta+2\varepsilon}{\beta-\varepsilon}\,, \quad t \ge t_0\,.$$

Since $\varepsilon > 0$ was arbitrary (sufficiently small), we have

$$-1 \le \liminf_{t\to\infty} \frac{w(t)}{\tilde{w}(t)} \le \limsup_{t\to\infty} \frac{w(t)}{\tilde{w}(t)} \le 1\,. \tag{35}$$

Now, let w_1, w_2 be any solutions of (2) which exist on the whole interval $[t_0,\infty)$ and for which $w_2(t) > w_1(t)$ $(\le \tilde{w}(t))$ in this interval. Then by the previous analysis

$$\limsup_{t\to\infty} \left|\frac{w_1(t)}{w_2(t)}\right| \le 1$$

and hence (18) is regular. □

Remark 2. It is easy to see that condition (19) can be replaced by a weaker condition.

$$-\infty < \liminf_{t\to\infty} c(t) \le \limsup_{t\to\infty} c(t) < 0\,. \tag{36}$$

In the next theorem equation (18) is compared with the below given Euler type equation (38).

Theorem 3. *Suppose that*

$$\lim_{t\to\infty} t^p c(t) = \gamma < 0\,. \tag{37}$$

Then (18) *is regular.*

Proof. Similarly as in the proof of Theorem 2 we will find a positive constants T satisfying (14). To this end, consider the Euler equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0\,. \tag{38}$$

The transformation $y(t) = x(e^t)$ transforms this equation into the equation (with constant coefficients)

$$(\Phi(y'))' - (p-1)\Phi(y') + \gamma\Phi(y) = 0 \tag{39}$$

The corresponding Riccati equation is

$$w' = -(p-1)|w|^q + (p-1)w - \gamma\,. \tag{40}$$

The same transformation transforms (18) into

$$(\Phi(y'))' - (p-1)\Phi(y') + \mathrm{e}^{pt}c(\mathrm{e}^{pt})\Phi(y) = 0 \tag{41}$$

and the Riccati equation associated with (41) is

$$w' = -(p-1)|w|^q + (p-1)w - \mathrm{e}^{pt}c(\mathrm{e}^{pt})\,. \tag{42}$$

Now, if (37) holds, then for every $\varepsilon > 0, \varepsilon < |\gamma|$, there exists $t_0 \in \mathbb{R}$ such that

$$\gamma - \varepsilon < \mathrm{e}^{pt} c(\mathrm{e}^{pt}) < \gamma + \varepsilon$$

for every $t \geq t_0$. Consider the equations

$$(\Phi(x'))' - (p-1)\Phi(x') + \gamma + \varepsilon = 0 \tag{43}$$

and

$$(\Phi(x'))' - (p-1)\Phi(x') + \gamma - \varepsilon = 0\,, \tag{44}$$

i.e., (43) is the Sturmian majorant to (41) and (44) is the Sturmian minorant to (41). The corresponding Riccati equations are

$$v' = -(p-1)|v|^q + (p-1)v - (\gamma + \varepsilon) \tag{45}$$

and

$$z' = -(p-1)|z|^q + (p-1)z - (\gamma - \varepsilon)\,. \tag{46}$$

Note that equations (43), (44) can be written in the form (1), so Lemma 1 applies also to (45) and (46). Let $\lambda_1(\varepsilon) < 0 < \lambda_2(\varepsilon)$, $\mu_1(\varepsilon) < 0 < \mu_2(\varepsilon)$ be the roots of the equation

$$(p-1)(|\lambda|^q - \lambda) + \gamma + \varepsilon = 0\,, \quad (p-1)(|\mu|^q - \mu) + \gamma - \varepsilon = 0\,,$$

respectively. Then (45) has the constant solutions $v_{1,2}(t) = \mu_{1,2}(\varepsilon)$, (46) has constant solutions $z_{1,2}(t) = \lambda_{1,2}(\varepsilon)$ and the constants $\lambda_{1,2}(\varepsilon)$, $\mu_{1,2}(\varepsilon)$ play the same role as the constants $\mp(\beta + \varepsilon)$, $\mp(\beta - \varepsilon)$ in the proof of Theorem 2. In particular, the solutions $v = \mu_1(\varepsilon)$, $z = \lambda_1(\varepsilon)$ are eventually minimal of (45) and (46) and any solutions v, z which are not eventually minimal satisfy

$$\lim_{t\to\infty} v(t) = \mu_2(\varepsilon)\,, \quad \lim_{t\to\infty} z(t) = \lambda_2(\varepsilon)\,.$$

Let $\tilde{w}(t)$ be the minimal solution of (42). Then by the Sturmian comparison theorem we obtain

$$\lambda_1(\varepsilon) \leq \tilde{w}(t) \leq \mu_1(\varepsilon)\,, \tag{47}$$

If w is any nonminimal solution of (42) then comparing this solution with those of (45) and (46) we have that this solution satisfies either the same inequality as the minimal solution $\tilde{w}$ in (47) or

$$\mu_2(\varepsilon) \leq \liminf_{t\to\infty} w(t) \leq \limsup_{t\to\infty} w(t) \leq \lambda_2(\varepsilon)\,.$$

Since $\varepsilon > 0$ was arbitrary and $\lim_{\varepsilon\to 0+} \lambda_{1,2}(\varepsilon) = \lim_{\varepsilon\to 0+} \mu_{1,2}(\varepsilon) =: \tilde{\lambda}_{1,2}$, where $\tilde{\lambda}_1, \tilde{\lambda}_2$ are the negative and the positive root of the equation $(p-1)(|\lambda|^p - p) + \gamma = 0$, respectively, and since $|\tilde{\lambda}_1| < \tilde{\lambda}_2$ as can be verified by a direct computation, using the same argument as used in the proof of the previous theorem we have

$$\limsup_{t\to\infty} \left| \frac{w_1(t)}{w_2(t)} \right| < \frac{\tilde{\lambda}_2}{|\tilde{\lambda}_1|}\,.$$

This proves that (41) is regular and hence (18) is also regular since the transformation of independent variable $t \longmapsto \mathrm{e}^t$ preserves regularity of transformed equations. $\square$

Remark 3. Similarly as in Remark 2 we can replace condition (37) by a weaker condition

$$-\infty < \liminf_{t\to\infty} t^p c(t) \le \limsup_{t\to\infty} t^p c(t) < 0\,.$$

4. Remarks

(i) A subject of the current discussion is whether (15) is really a good candidate for the integral characterization of the principal solution of (1). There are several supporting points, the most important of them is that the expression

$$G(t) := \int^t \frac{ds}{r(s)x^2(s)|x'(s)|^{p-2}}$$

plays in oscillation theory of (1) the same role as the term $\int^t r^{-1}x^{-2}(s)\,ds$ in the linear oscillation criteria, see [4, 9]. On the other hand, this integral characterization requires restrictions on the functions r, c which guarantee that solutions of (1) satisfy $x' \ne 0$ for large t as shows the example the equation (1) with $p \in (1,2)$, $c \equiv 0$ and r satisfying $\int^\infty r^{1-q}\,dt = \infty$. In this case $x(t) \equiv 1$ is the principal solution but in its integral characterization $I(x) = 0$, where $I(x)$ is defined by (6).

(ii) In our paper we investigate the principal solution *at* ∞, i.e. a solution having some special properties for large t. Let $b < \infty$ be a regular point of (1) (in the sense that the initial conditions $x(b) = A$, $x'(b) = B$ determines the unique solution of (1) for any $A, B \in \mathbb{R}$) and denote by x_b the solution given by the initial condition $x_b(b) = 0$, $x_b'(b) = 1$. This solution can be regarded as the *principal solution at* b (since principal solution at ∞ is the limit (as $b \to \infty$) of such solutions). In the linear case x_b is the only solution (up to a multiple by a nonzero real constant) for which $\int^b r^{-1}(t)x^{-2}(t)\,dt = \infty$. A natural question is whether the principal solution x_b of (1) is also the only one solution of this equation which satisfies

$$I_b(x) := \int^b \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty$$

since this is the case for linear equation (3). Another solution which could satisfy $I_b(x) = \infty$ in case $p \in (2,\infty)$ is the solution $\tilde{x}_b$ given by the initial condition $\tilde{x}_b(b) = 1$, $\tilde{x}_b'(b) = 0$. In this case, if $c(b) \ne 0$, then $I_b(\tilde{x}_b) < \infty$, i.e. the solution x_b is really the only solution satisfying $I_b(x) = \infty$. Indeed, let $w = r\Phi(\tilde{x}_b'/\tilde{x}_b)$ be the solution of (2) corresponding to $\tilde{x}_b$. Then $w(b) = 0$, $w'(b) = c(b) \ne 0$ and since $\tilde{x}_b(b) = 1$, $r(b) > 0$, we have $\Phi(x'(t)) \sim w(t) \sim (b-t)\operatorname{sgn} c(b)$ for $t \to b$. Hence $|x'(t)|^{p-2} \sim |t-b|^{\frac{p-2}{p-1}}$ and thus $I_b(\tilde{x}_b) < \infty$. However, if $c(b) = 0$, it may generally happen that $I_b(\tilde{x}_b) = \infty$, i.e. $I_b(x)$ is not equivalent integral characterization of the principal solution at a *finite* point. This problem is a subject of the present investigation.

(iii) As mentioned at the beginning of this paper, another important characterization of the principal solution $\tilde{x}$ in the linear case is the limit characterization $\lim_{t\to\infty}(\tilde{x}(t)/x(t)) = 0$ for any solution x linearly independent of $\tilde{x}$. Concerning nonoscillatory half-linear equation (1), since the solution space of (1) is no longer additive and Wronskian identity is lost, see [7], it is an open problem whether the same hold also in half-linear case. The ratio $\tilde{x}/x$ is a monotonic function (since $\tilde{x}'x - \tilde{x}x'$ has no zero, otherwise $\tilde{x}'/\tilde{x} = x'/x$ at some point which means that there exist two different solution of the associated Riccati equation (2) satisfying the same initial condition, a contradiction), hence there exists a (finite or infinite) limit $L := \lim_{t\to\infty}\tilde{x}(t)/x(t)$. Moreover, if solutions $x, \tilde{x}$ are eventually positive, normalized in the sense that $x(t_0) = 1 = \tilde{x}(t_0)$ for some t_0 (sufficiently large) and $\tilde{x}$ is principal, then $L \in [0, 1)$. We conjecture, based on all nonoscillatory equations which can be computed explicitly, that similar to the linear case $L = 0$. This conjecture is is equivalent to the following conjecture.

Conjecture 1. *Suppose that* (1) *is nonoscillatory, $w, \tilde{w}$ are solutions of the associated Riccati equation* (2)*, the solution $\tilde{w}$ is minimal. Then*

$$\int^{\infty} r^{1-q}(t)\,[\Phi_q(w(t)) - \Phi_q(\tilde{w}(t))]\;dt = \infty\,.$$

The proof (disprove) of this conjecture is a subject of the present investigation. Note also that in the very recent paper [2] it is proved that if $c(t) < 0$ for large t, the above conjecture is true.

(iv) We conclude the paper with an example illustrating the statements of the previous section. Consider the Euler-type equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0\,, \tag{48}$$

with $\gamma < 0$. This equation satisfies the assumptions of Theorem 3, hence there exists unique (up to a multiple by a nonzero real constant) solution for which $I(x) = \infty$. Here we compute this solution explicitly. By a direct computation one can verify that equation (48) possesses (among others) two linearly independent solutions $x_1(t) = t^{\lambda_1}$, $x_2(t) = t^{\lambda_2}$, where $\lambda_1 < 0 < \lambda_2$ are roots of the equation

$$R(\lambda) := |\lambda|^p - \Phi(\lambda) + \frac{\gamma}{p-1} = 0\,,$$

all other solutions are asymptotically (as $t \to \infty$) equivalent to $x_2(t)$ and $\lambda_{1,2}$ are the only real roots of (49), see [6]. We have $R(0) = \frac{\gamma}{p-1} < 0$, $R(\infty) = \infty$ and $R(\frac{1}{q}) < 0$, hence $\lambda_2 > \frac{1}{q}$. This means that

$$\int^{\infty} \frac{dt}{x_2^2 |x_2'(t)|^{p-2}} = \int^{\infty} \frac{dt}{t^{p(\lambda_2 - 1)+2}} < \infty$$

since $p(\lambda_2 - 1) + 2 > 1$. Consequently, $x_1(t)$ is really the only solution for which $I(x) = \infty$.

References

[1] Allegretto, W., Huang, Y. X., *A Picone's identity for the p-Laplacian and applications*, Nonlin. Anal. **32** (1998), 819–830.

[2] Cecchi, M., Došlá, Z., Marini, M., *Principal solutions and minimal set for quasilinear differential equations*, to appear in Dynam. Syst. Appl.

[3] Došlý, O., Elbert, Á., *Integral characterization of the principal solution of half-linear differential equations*, Studia Sci. Math. Hungar. **36** (2000), No. 3-4, 455–469.

[4] Došlý, O. and Lomtatidze, A., *Oscillation and nonoscillation criteria for half-linear second order differential equations*, submitted.

[5] Elbert, Á., *A half-linear second order differential equation*, Colloq. Math. Soc. János Bolyai **30** (1979), 158–180.

[6] Elbert, Á., *Asymptotic behaviour of autonomous half-linear differential systems on the plane*, Studia Sci. Math. Hungar. **19** (1984), 447–464.

[7] Elbert, Á., *The Wronskian and the half-linear differential equations*, Studia Sci. Math. Hungar. **15** (1980), 101–105.

[8] Elbert, Á. and Kusano, T., *Principal solutions of nonoscillatory half-linear differential equations*, Advances in Math. Sci. Appl. **18** (1998), 745–759.

[9] Elbert, Á. and Schneider, A., *Perturbation of the half-linear Euler differential equations*, Result. Math. **37** (2000), 56–83.

[10] Hartman, P., Ordinary Differential Equations, John Wiley, New York, 1964.

[11] Jaroš, J. and Kusano, T., *A Picone type identity for half-linear differential equations*, Acta Math. Univ. Comenianea **68** (1999), 137–151.

[12] Leighton, W., Morse, M., *Singular quadratic functionals*, Trans. Amer. Math. Soc. **40** (1936) 252–286.

[13] Lorch, L., Newman, J. D., *A supplement to the Sturm separation theorem, with applications*, Amer. Math. Monthly **72** (1965), 359–366, 390.

[14] Mirzov, J. D., *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **53** (1976), 418–426.

[15] Mirzov, J. D., *Principal and nonprincipal solutions of a nonoscillatory system*, Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy **31** (1988), 100–117.

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