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THE ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF THE N–TH ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to deduce oscillatory and asymptotic behavior of the solutions of the $n-$th order neutral differential equation

$$(x(t) - px(t-\tau))^{(n)} - q(t)x(\sigma(t)) = 0\,,$$

where $\sigma(t)$ is a delayed or advanced argument.

We consider the $n-$th order differential equation with a deviating argument of the form

$$(x(t) - px(t-\tau))^{(n)} - q_1(t)x(\sigma_1(t)) = 0\,, \tag{1}$$

where

(i) n is even,
(ii) p and τ are positive numbers,
(iii) $q_1(t), \sigma_1(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $q_1(t)$ is positive, $\lim\limits_{t\to\infty} \sigma_1(t) = \infty$.

By a solution of Eq. (1) we mean a function $x : [T_x, \infty) \to \mathbb{R}$ which satisfies (1) for all sufficiently large t. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

We introduce the notation

$$Q_j(t) = q_j(t)\sum_{i=0}^{m} p^i\,, \qquad \text{where } m \text{ is a positive integer}, \quad j = 1, 2\,. \tag{2}$$

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Lemma 1. *Let $z(t)$ be an n times differentiable function on $\mathbb{R}_+$ of constant sign, $z^{(n)}(t) \not\equiv 0$ on $[T_0, \infty)$ which satisfies $z^{(n)}(t)z(t) \geq 0$. Then there is an integer l, $0 \leq l \leq n$ and $t_1 \geq T_0$ such that $n + l$ is even and for all $t \geq t_1$*

$$\begin{aligned} z(t)z^{(i)}(t) > 0, &\qquad 0 \leq i \leq \ell\,, \\ (-1)^{i-\ell} z(t)z^{(i)}(t) > 0, &\qquad \ell \leq i \leq n\,. \end{aligned} \tag{3}$$

Lemma 1 is a well-known lemma of Kiguradze [5].

A function $z(t)$ satisfying (3) is said to be a function of degree l. The set of all functions of degree l is denoted by $\mathcal{N}_l$. If we denote by $\mathcal{N}$ the set of all functions satisfying $z^{(n)}(t)z(t) \geq 0$ then the set $\mathcal{N}$ has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n\,.$$

Lemma 2. *Let $y(t)$ be a positive function of degree ℓ, $\ell \geq 2$. Then*

$$y(t) \geq \int_{t_1}^{t} y^{(\ell-1)}(s) \frac{(t-s)^{\ell-2}}{(\ell-2)!}\, ds\,. \tag{4}$$

The proof of this lemma is immediate from integration the identity $y^{(l-1)}(t) = y^{(l-1)}(t)$.

Theorem 1. *Assume that m is a positive integer. Let*

$$\sigma_1(t) < t - \tau\,, \quad \sigma_1(t) \in C^1\,, \quad \sigma_1'(t) \geq 0\,. \tag{5}$$

Further assume that the differential equation

$$y^{(n)}(t) + \frac{1}{p} q_1(t) y(\sigma_1(t) + \tau) = 0 \tag{6}$$

is oscillatory and the differential inequality

$$z^{(n)}(t) - Q_1(t) z(\sigma_1(t)) \geq 0 \tag{7}$$

has no solution of degree 0. Then every nonoscillatory solution of Eq. (1) tends to ∞ as $t \to \infty$.

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (1) and define

$$z(t) = x(t) - p x(t - \tau)\,. \tag{8}$$

It is easy to see that

$$z(t) < x(t)\,. \tag{9}$$

From Eq. (1) we have $z^{(n)}(t) > 0$ for all large t, say $t \geq t_0$. Thus $z^{(i)}(t)$ are monotonous, $i = 0, 1, \ldots, n-1$. If $z(t) < 0$ eventually, then we set $u(t) = -z(t)$. In the view of (8)

$$x(t-\tau) > \frac{1}{p} u(t)\,,$$

that is

$$x(t) > \frac{1}{p} u(t+\tau)\,.$$

One gets that $u(t)$ is a positive solution of the inequality

$$u^{(n)}(t) + \frac{1}{p} q_1(t) u(\sigma_1(t) + \tau) \leq 0$$

and by Kusano and Naito [1] the corresponding equation

$$u^{(n)}(t) + \frac{1}{p} q_1(t) u(\sigma_1(t) + \tau) = 0$$

has a positive solution $u(t)$. This contradicts that (6) is oscillatory.

Therefore $z(t) > 0$. According to Lemma 1 we have two possibilities for $z'(t)$:

(a) $z'(t) > 0$, for $t \geq t_1 \geq t_0$,
(b) $z'(t) < 0$, for $t \geq t_1$.

For case (a) by Lemma 1 we obtain $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$. It implies that $\lim_{t\to\infty} z(t) = \infty$ and from (9) also $\lim_{t\to\infty} x(t) = \infty$.

For case (b) Eq. (1) can be written in the form

$$z^{(n)}(t) - q_1(t) x(\sigma_1(t)) = 0\,.$$

Using (8) we have

$$z^{(n)}(t) - q_1(t) z(\sigma_1(t)) - p q_1(t) x(\sigma_1(t) - \tau) = 0\,.$$

Repeating this procedure $m-$times we arrive at

$$z^{(n)}(t) - q_1(t) \sum_{i=0}^{m} p^i z(\sigma_1(t) - i\tau) - p^{m+1} q_1(t) x(\sigma_1(t) - (m+1)\tau) = 0\,.$$

Since $z(t)$ is decreasing, we get

$$z^{(n)}(t) - q_1(t) z(\sigma_1(t)) \sum_{i=0}^{m} p^i \geq 0\,.$$

In the view of (2) we have

$$z^{(n)}(t) - Q_1(t) z(\sigma_1(t)) \geq 0\,. \tag{10}$$

Hence $z(t)$ is a solution of degree 0 of the inequality (10). This is a contradiction. □

Corollary 1. *Let m be a positive integer. Further assume that* (5) *holds, differential equation* (6) *is oscillatory and there exists $k \in \{0, 1, \dots, n-1\}$ such that*

$$(11)\quad \limsup_{t\to\infty} \frac{1}{k!(n-k-1)!} \int\limits_{\sigma_1(t)}^{t} [s-\sigma_1(t)]^k \, [\sigma_1(t)-\sigma_1(s)]^{n-k-1} \, Q_1(s)\, ds > 1 \,.$$

Then every nonoscillatory solution of Eq. (1) *tends to ∞ as $t \to \infty$.*

Proof. By [2, Theorem 1] it follows from (11) that the differential inequality (7) has no solution of degree 0. Our assertion follows from Theorem 1. □

Let us consider the $n-$th order differential equation with an advanced argument of the form

$$(12)\qquad (x(t) - px(t-\tau))^{(n)} - q_2(t)x(\sigma_2(t)) = 0 \,,$$

where (i), (ii) holds and moreover

(iv) $q_2(t), \sigma_2(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, $q_2(t)$ is positive, $\lim\limits_{t\to\infty} \sigma_2(t) = \infty$.

We introduce the notation

$$(13)\quad A_\ell(t) = \int\limits_t^\infty q_2(s) \frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!} \times \left[\int\limits_t^{\sigma_2(s)} \frac{(t-u)^{\ell-2}}{(\ell-2)!} du \right] ds \,, \qquad \text{for} \quad \ell = 2, 4, \dots, n-2 \,.$$

Theorem 2. *Assume that m is a positive integer and*

$$(14)\qquad \sigma_2(t) - m\tau > t \,, \quad \sigma_2(t) \in C^1 \,, \quad \sigma_2'(t) \ge 0 \,, \qquad 0 < p < 1 \,.$$

Further assume that

$$(15)\qquad A_\ell(t)(t-t_1) > 1 \quad \textit{for} \quad \ell = 2, 4, \dots, n-2$$

and the differential inequality

$$(16)\qquad z^{(n)}(t) - Q_2(t) z(\sigma_2(t) - m\tau) \ge 0$$

has no solution of degree n. Then every nonoscillatory solution of Eg. (12) *is bounded.*

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (12) and define

$$(17)\qquad z(t) = x(t) - px(t-\tau) \,.$$

From Eq. (12) we have $z^{(n)}(t) > 0$ for all large t, say $t \geq t_0$. Thus $z^i(t)$ are monotonous, $i = 0, 1, \ldots, n-1$. If $z(t) < 0$ eventually, then

$$x(t) < px(t-\tau) < p^2 x(t-2\tau) < \cdots < p^k x(t-k\tau)$$

for all large t, which implies $\lim\limits_{t\to\infty} x(t) = 0$.

If $z(t) > 0$, then according to a Lemma 1 we have two possibilities for $z'(t)$:

(a) $z'(t) > 0$, for $t \geq t_1 \geq t_0$,
(b) $z'(t) < 0$, for $t \geq t_1$.

For case (a) we have two possibilities:

(i) $\exists\, \ell \in 2, 4, \ldots, n-2$, such that $z(t) \in \mathcal{N}_\ell$,
(ii) $\ell = n$, i.e. $z(t) \in \mathcal{N}_n$.

For case (i) Eq. (12) can be written in the form

$$z^{(n)}(t) = q_2(t)x(\sigma_2(t))\,.$$

Integrating this equation from t to ∞ $n-\ell$ times and taking Lemma 2 into account, one gets

$$\begin{aligned} z^{(\ell)}(t) &\geqslant \int_t^\infty q_2(s)x(\sigma_2(s))\frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!}\,ds \geqslant \int_t^\infty q_2(s)z(\sigma_2(s))\frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!}\,ds \\ &\geqslant \int_t^\infty q_2(s)\frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!} \times \left[\int_{t_1}^{\sigma_2(s)} z^{(\ell-1)}(u)\frac{(t-u)^{\ell-2}}{(\ell-2)!}\,du\right] ds \end{aligned}$$

Taking into account that $\sigma_2(t)$ is nondecreasing, $t \geq t_1$ and $z^{(\ell-1)}(t)$ is increasing, the above inequalities led to

$$z^{(\ell)}(t) \geq z^{(\ell-1)}(t)A_\ell(t)\,. \tag{18}$$

Integration of the identity $z^{(\ell)}(t) = z^{(\ell)}(t)$ from t_1 to t provides

$$z^{(\ell-1)}(t) \geqslant \int_{t_1}^t z^{(\ell)}(s)\,ds \geqslant z^{(\ell)}(t)(t-t_1), \qquad t \geq t_1\,,$$

which in the view of (18) implies

$$1 \geq (t-t_1)A_\ell(t)\,.$$

This contradicts (15).

For case (ii) Eq. (12) can be written in the form

$$z^{(n)}(t) - q_2(t)x(\sigma_2(t)) = 0\,.$$

Using (17) we have

$$z^{(n)}(t) - q_2(t)z(\sigma_2(t)) - pq_2(t)x(\sigma_2(t) - \tau) = 0\,.$$

Repeating this procedure $m-$times we arrive at

$$z^{(n)}(t) - q_2(t)\sum_{i=0}^{m} p^i z(\sigma_2(t) - i\tau) - p^{m+1}q_2(t)x(\sigma_2(t) - (m+1)\tau) = 0\,.$$

Since $z(t)$ is increasing, we get

$$z^{(n)}(t) - q_2(t)z(\sigma_2(t) - m\tau)\sum_{i=0}^{m} p^i \geq 0\,.$$

In the view of (2) we have

$$z^{(n)}(t) - Q_2(t)z(\sigma_2(t) - m\tau) \geq 0\,. \tag{19}$$

Hence $z(t)$ is a solution of degree n of the inequality (19). This is a contradiction.

For case (b) we have $z(t) > 0$, $z'(t) < 0$. Hence there exists

$$\lim_{t\to\infty} z(t) = c \geq 0\,. \tag{20}$$

If $x(t)$ is unbounded eventually, then we can define the sequence $\{t_n\}$ where $t_n \to \infty$ as $n \to \infty$ as follows. Let us choose t_m for every $m \in \mathbb{N}$ such that

$$x(t_m) = \max\{x(s), t_0 \leq s \leq t_m\}\,.$$

Since

$$x(t_m - \tau) = \max\{x(s), t_0 \leq s \leq t_m - \tau\} \leq \max\{x(s), t_0 \leq s \leq t_m\} = x(t_m)\,,$$

we have

$$z(t_m) = x(t_m) - px(t_m - \tau) \geq x(t_m) - px(t_m) = (1-p)x(t_m)\,.$$

This implies $\lim\limits_{t\to\infty} z(t) = \infty$. This contradicts (20). □

Corollary 2. *Let m be a positive integer. Further assume that (14) and (15) hold and there exists $k \in \{0, 1, \dots, n-1\}$ such that*

$$\limsup_{t\to\infty} \frac{1}{k!(n-k-1)!}\int_t^{\sigma_2(t)} [\sigma_2(s) - \sigma_2(t)]^k\,[\sigma_2(t) - s]^{n-k-1}\,Q_2(s)\,ds > 1\,. \tag{21}$$

Then every nonoscillatory solution of Eq. (12) is bounded.

Proof. By [2, Theorem 4] it follows from (21) that the differential inequality (16) has no solution of degree n. Our assertion follows from Theorem 2. □

Now we want to extend our previous results to more general differential equation. So let us consider the $n-$th order differential equation with both arguments (advanced and delayed) of the form

$$(x(t) - px(t-\tau))^{(n)} - q_1(t)x(\sigma_1(t)) - q_2(t)x(\sigma_2(t)) = 0\,, \tag{22}$$

where (i), (ii), (iii), (iv) hold.

Theorem 3. *Let m be a positive integer. Further assume that* (5), (14) *and* (15) *hold, differential equality* (6) *is oscillatory, differential inequality* (7) *has no solution of degree 0 and differential inequality* (16) *has no solution of degree n.*

Then every solution of Eq. (22) *is oscillatory.*

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (22). Then $x(t)$ is solution of the inequality

$$(x(t) - px(t-\tau))^{(n)} - q_1(t)x(\sigma_1(t) \geq 0\,.$$

Using the same arguments as in Theorem 1 we can prove that $x(t)$ tends to ∞ as $t \to \infty$.

On the other hand, $x(t)$ is also solution of the inequality

$$(x(t) - px(t-\tau))^{(n)} - q_2(t)x(\sigma_2(t) \geq 0\,.$$

Now arguing exactly as in the proof of Theorem 2 we get that $x(t)$ is bounded. This is a contradiction. □

In a paper [2, Theorem 7] Kusano has presented conditions when the functional differential equation

$$y^{(n)}(t) - q_1(t)y(\sigma_1(t)) - q_2(t)y(\sigma_2(t)) = 0$$

is oscillatory. We have extended these conditions also for the neutral differential equation of the form (22). In a paper [6] Džurina and Mihalíková have presented sufficient conditions for all bounded solutions of the second order neutral differential equation with a delayed argument to be oscillatory. We have extended these conditions also for the n-th order neutral differential equation involving both delayed and advanced arguments.

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