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AN IMPROVED EXPONENTIAL DECAY RESULT FOR SOME SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS

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Abstract. We prove exponential decay for the solution of an abstract integrodifferential equation. This equation involves coefficients of polynomial type, weakly singular kernels as well as different powers of the unknown in some norms.

1. Introduction

We consider the integrodifferential problem

$$(1)\qquad \begin{cases} x'(t) + Ax(t) = f(t, x(t)) + \displaystyle\int_0^t g\left(t, s, x(s), \int_0^s K(s, \tau, x(\tau))\, d\tau\right) ds\,, \\ x(0) = x_0\,, \end{cases}$$

where $x \in X$ a Banach space and $t \in [0, T]\,,\ T > 0$. The operator $-A$ is the infinitesimal generator of a linear semigroup $e^{-tA},\ t \geq 0$ on X and x_0 is a given initial value. The functions $f : I \times X \to X,\ g : Q \times X \times X \to X$ and $K : Q \times X \to X$ are given, where $Q = \{(t, s),\ 0 \leq s \leq t \leq T\}$.

A similar problem has been considered by Balasubramaniam and Chandrasekaran [1]. They considered an infinitesimal generator of a C_0-semigroup and a nonlocal boundary condition. The authors proved existence and uniqueness of mild and strong solutions provided that the functions f, g and K are continuous and satisfy some Lipschitz conditions.

In the present paper the functions f, g and K possess some interesting new features. Indeed, besides inserting coefficients of polynomial type and weakly singular kernels, we allow different powers of the unknown.

2. An exponential decay result

In this section we consider $X = \mathbf{L}^p(\Omega)$, $p > 1$ with Ω a bounded domain of $\mathbb{R}^n,\ n \geq 1$. The operator $-A$ is supposed to be sectorial (see [2]) with $\operatorname{Re}\sigma(A) >$

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$b>0$ where $\operatorname{Re}\sigma(A)$ denotes the real part of the spectrum of A. We may define the fractional operators A^α, $0\le\alpha\le 1$ in the usual way on $D(A^\alpha)=X^\alpha$. The space X^α endowed with the norm $\|x\|_\alpha=\|A^\alpha x\|$ is a Banach space.

The functions f, g and K are assumed to fulfil the following hypotheses for every $x,u,v\in\mathbf{L}^p(\Omega)$, $(t,s)\in Q$ and $\sigma_1,\sigma_2,\sigma_3,\sigma_4\ge 0$:

$(H1)$ $\|f(t,x)\|_p\le t^{\sigma_1}\varphi_1(t)\,\|A^\alpha x\|_p^{m_1}$, $t\ge 0,\ x\in\mathbf{L}^p(\Omega),\ (\sigma_1\ge 0)$,

$(H2)$ $\|g(t,s,u,v)\|_p\le l(t-s)s^{\sigma_2}\varphi_2(s)\,\|A^\alpha u\|_p^{m_2}+p(t-s)s^{\sigma_3}\,\|v\|_p^{m_3}$,

$(H3)$ $\|K(s,\tau,x(\tau))\|_p\le k(s-\tau)\tau^{\sigma_4}\varphi_4(s)\,\|A^\alpha x\|_p^{m_4}$,

where $l(t)=t^{-\beta_2}e^{-\gamma_2 t}$, $p(t)=t^{-\beta_3}e^{-\gamma_3 t}$ and $k(t)=t^{-\beta_4}e^{-\gamma_4 t}$, $\beta_i\in(0,1)$, $i=2,3,4$ and $\gamma_i>0$, $i=2,3,4$. The functions $\varphi_i(t)$, $i=1,2,4$ are assumed to be nonnegative and continuous.

Global existence of mild solutions of (1) under these assumptions may be proved by modifying, for instance, the proof of Theorem 2.1 in [8] , see also [9] as well as [4]. Imposing Lipschitz conditions on f, g and K one may obtain a uniqueness result. Our primary goal here in this paper is to prove an exponential decay result in the space $C^\nu(\bar\Omega)$ for some values of ν.

To prove our next theorem we will use the same technique as used in the second author's paper [9]. Our problem is yet different in nature and presents some new difficulties. The last part of the corresponding proof in [9] has to be modified accordingly. To this end we prove a modified version of a result in Medved' ([5], Theorem 5). It will be clear that our results may be used to generalize those in [9] as our powers m_i are not necessarily equal.

The following lemmas will be used in the proof of our Theorem.

Lemma 1. *If $0\le\alpha\le 1$, then $D(A^\alpha)\subset C^\nu(\bar\Omega)$ for $0\le\nu<2\alpha-\frac{n}{p}$.*

Lemma 2. *If $0\le\alpha\le 1$, then $\left\|A^\alpha e^{-tA}\right\|_p\le c_1t^{-\alpha}e^{-bt}$, $t>0$ for some positive constant c_1.*

Lemma 3. *If δ, ν, $\tau>0$ and $z>0$, then*

$$z^{1-\nu}\int_0^z(z-\zeta)^{\nu-1}\zeta^{\delta-1}e^{-\tau\zeta}d\zeta\le K(\nu,\delta,\tau)\,,$$

where $K(\nu,\delta,\tau)=\max(1,2^{1-\nu})\Gamma(\delta)(1+\frac{\delta}{\nu})\tau^{-\delta}$.

The proofs of Lemma 1 and Lemma 2 may be found in [2] while for the proof of Lemma 3 one can see [7] or [3].

In order to lighten the statement of our next theorem we set the following conditions on the powers in $(H1)-(H3)$ and a definition.

$(H4)$ $0<\beta_4<1-\frac{1}{q^*m_3}$,

$(H5)$ $1+q(\sigma_1-\alpha m_1)>0$, $1+q(\sigma_2-\alpha m_2)>0$,
$1+q(\sigma_3-\beta_4 m_3)>0$, $1+\frac{q^*m_3}{q^*m_3-1}(\sigma_4-\alpha m_4)>0$,

$(H6)$ $bm_2>\gamma_2$, $bm_4>\gamma_4$, $\gamma_4 m_3>\gamma_3$, $\gamma_2+\gamma_3>b$,

for some q^* to be determined later, m_1, m_2 and $m_3>1$.

We also need the following condition:

$(H7)$ $\int_0^\infty h(t)\,dt = H_0 < e^{\delta(1-\tilde{c}_1\|x_0\|_p^{q^*})}\delta^{\delta-1}$,

where $\delta = \min(m_1, m_2, m_3m_4)$, $H(t) = \int_0^\infty h(t)\,dt$ and

$$h(t) = m_1^{m_1}\tilde{c}_2\varphi_1^{q^*}(t) + m_2^{m_2}\tilde{c}_3\int_0^t \varphi_2^{q^*}(s)\,ds + (m_3m_4)^{m_3m_4}\tilde{c}_4\int_0^t\left(\int_0^u \varphi_4^{q^*m_3}(\tau)\,d\tau\right)du$$

for some constants $\tilde{c}_i$ to be identified in the proof of the Theorem.

Theorem 1. *Assume the hypotheses $(H1)-(H7)$. Let $z_1 = \dfrac{1-\alpha}{\alpha}$, $z_2 = \dfrac{1-\beta_2}{\beta_2}$, $z_3 = \dfrac{1-\beta_3}{\beta_3}$ and $\xi = \min\{z_i, i = 1,2,3,4\}$. If $\varphi_i \in \mathbf{L}^{q^*}(0,\infty)$, $i = 1,2, \varphi_4^{m_3} \in \mathbf{L}^{q^*}(0,\infty)$ where*

$$q^* = \begin{cases} \frac{1}{\xi}+2\,, & \text{if } \ 0<\xi\le 1 \\ 2 & \text{if } \ \xi>1 \end{cases}$$

then any mild solution $x(t)$ to problem (1) satisfies the estimate

$$\|A^\alpha x(t)\|_p \le ct^{-\alpha}e^{-bt}\,, \quad t>0$$

for some constant $c>0$.

Proof. Let $x(t)$ be a mild solution of (1). We have

$$x(t) = e^{-tA}x_0 + \int_0^t e^{-(t-s)A}f(s,x(s))\,ds + \int_0^t e^{-(t-s)A}\left(\int_0^s g\left(s,u,x(u),\int_0^u K(u,\tau,x(\tau))\,d\tau\right)du\right)ds. \tag{2}$$

Applying the operator A^α, $0<\alpha<1$ to both sides of (2) and using hypotheses $(H1)-(H3)$ and Lemma 2, we get at once

$$\begin{aligned}\|A^\alpha x(t)\|_p \le{}& c_1t^{-\alpha}e^{-bt}\|x_0\|_p + c_1\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}s^{\sigma_1}\varphi_1(s)\|A^\alpha x(s)\|_p^{m_1}\,ds \\ &+ c_1\int_0^t (t-s)^{-\alpha}e^{-b(t-s)}\left[\int_0^s l(s-u)u^{\sigma_2}\varphi_2(u)\|A^\alpha x(u)\|_p^{m_2}\,du\right. \\ &\left.+\int_0^s p(s-u)u^{\sigma_3}\left(\int_0^u k(u-\tau)\tau^{\sigma_4}\varphi_4(\tau)\|A^\alpha x(\tau)\|_p^{m_4}\,d\tau\right)^{m_3}\right]ds\,.\end{aligned}$$

Next, using the definitions of $l(t)$, $p(t)$ and $k(t)$, we obtain

$$
\begin{aligned}
\|A^\alpha x(t)\|_p \leq c_1 t^{-\alpha} e^{-bt} \|x_0\|_p + c_1 e^{-bt} \int_0^t (t-s)^{-\alpha} e^{bs} s^{\sigma_1} \varphi_1(s) \|A^\alpha x(s)\|_p^{m_1}\, ds \\
+ c_1 e^{-bt} \int_0^t (t-s)^{-\alpha} e^{bs} \left[\int_0^s (s-u)^{-\beta_2} e^{-\gamma_2(s-u)} u^{\sigma_2} \varphi_2(u) \right. \\
\times \|A^\alpha x(u)\|_p^{m_2}\, du + \int_0^s (s-u)^{-\beta_3} e^{-\gamma_3(s-u)} u^{\sigma_3} \\
\left. \times \left(\int_0^u (u-\tau)^{-\beta_4} e^{-\gamma_4(u-\tau)} \tau^{\sigma_4} \varphi_4(\tau) \|A^\alpha x(\tau)\|_p^{m_4}\, d\tau \right)^{m_3} \right] ds\,.
\end{aligned} \tag{3}
$$

Multiplying both sides of (3) by $t^\alpha e^{bt}$, then denoting the obtained right hand side by $U(t)$, we get

$$
\begin{aligned}
& t^\alpha e^{bt} \|A^\alpha x(t)\|_p \leq U(t) \leq c_1 \|x_0\|_p \\
& \quad + c_1 t^\alpha \int_0^t (t-s)^{-\alpha} e^{b(1-m_1)s} s^{\sigma_1 - \alpha m_1} \varphi_1(s) U^{m_1}(s)\, ds \\
& \quad + c_1 t^\alpha \int_0^t (t-s)^{-\alpha} e^{(b-\gamma_2-\gamma_3)s} \\
& \quad \times \left[\int_0^s (s-u)^{-\beta_2} e^{(\gamma_2 - bm_2)u} u^{\sigma_2 - \alpha m_2} \varphi_2(u) U^{m_2}(u)\, du \right. \\
& \quad + \int_0^s (s-u)^{-\beta_3} e^{(\gamma_3 - \gamma_4 m_3)u} u^{\sigma_3} \\
& \quad \left. \times \left(\int_0^u (u-\tau)^{-\beta_4} e^{(\gamma_4 - bm_4)\tau} \tau^{\sigma_4 - \alpha m_4} \varphi_4(\tau) U^{m_4}(\tau)\, d\tau \right)^{m_3} du \right] ds.
\end{aligned} \tag{4}
$$

We may write inequality (4) as follows

$$
U(t) \leq c_1 \|x_0\|_p + c_1 t^\alpha A(t,U) + c_1 t^\alpha B(t,U), \tag{5}
$$

where

$$
\begin{aligned}
A(t,U) &= \int_0^t (t-s)^{-\alpha} e^{b(1-m_1)s} s^{\sigma_1 - \alpha m_1} \varphi_1(s) U^{m_1}(s)\, ds \\
B(t,U) &= \int_0^t (t-s)^{-\alpha} e^{(b-\gamma_2-\gamma_3)s} \left[C(s,U) + D(s,U) \right] ds \\
C(s,U) &= \int_0^s (s-u)^{-\beta_2} e^{(\gamma_2 - bm_2)u} u^{\sigma_2 - \alpha m_2} \varphi_2(u) U^{m_2}(u)\, du \\
D(s,U) &= \int_0^s (s-u)^{-\beta_3} e^{(\gamma_3 - \gamma_4 m_3)u} u^{\sigma_3} E(u,U)\, du \\
E(u,U) &= \left(\int_0^u (u-\tau)^{-\beta_4} e^{(\gamma_4 - bm_4)\tau} \tau^{\sigma_4 - \alpha m_4} \varphi_4(\tau) U^{m_4}(\tau)\, d\tau \right)^{m_3}.
\end{aligned}
$$

We shall estimate below all these expressions separately.

Applying Hölder inequality to $A(t,U)$, we find

$$A(t,U) \leq \left(\int_0^t (t-s)^{-q\alpha} e^{qb(1-m_1)s} s^{q(\sigma_1-\alpha m_1)}\, ds\right)^{1/q} \times \left(\int_0^t \varphi_1^{q^*}(s) U^{q^* m_1}(s)\, ds\right)^{1/q^*}$$

where q^* is the conjugate exponent of q , that is $\frac{1}{q}+\frac{1}{q^*}=1$.

It can be seen by $(H5),(H6)$ and Lemma 3 that

$$A(t,U) \leq K_1^{\frac{1}{q}} t^{-\alpha} \left(\int_0^t \varphi_1^{q^*}(s) U^{q^* m_1}(s)\, ds\right)^{1/q^*} \tag{6}$$

where $K_1 = K(1-q\alpha, 1+q(\sigma_1-\alpha m_1), qb(m_1-1))$.

It is worth to observe that when $\xi > 1$ one has $0 < \alpha, \beta_2, \beta_3 < \frac{1}{2}$ and if $0 < \xi \leq 1$, then

$$\min\{1-q\alpha, 1-q\beta_2, 1-q\beta_3\} \geq \min\left\{\frac{1}{2(1+\xi)}, \frac{\xi^2}{1+\xi^2}\right\} > 0\,.$$

Next, estimating $C(s,U)$ in the same manner we obtain

$$C(s,U) \leq K_2^{\frac{1}{q}} s^{-\beta_2} \left(\int_0^s \varphi_2^{q^*}(u) U^{q^* m_2}(u)\, du\right)^{1/q^*}, \tag{7}$$

where $K_2 = K(1-q\beta_2, 1+q(\sigma_2-\alpha m_2), q(bm_2-\gamma_2))$.

Now we apply Hölder inequality to $E(u,U)$ with $\frac{1}{r}+\frac{1}{r^*}=1$ to get

$$E(u,U) \leq \left(\int_0^u (u-\tau)^{-r\beta_4} e^{r(\gamma_4-bm_4)\tau} \tau^{r(\sigma_4-\alpha m_4)}\, d\tau\right)^{\frac{m_3}{r}} \times \left(\int_0^u \varphi_4^{r^*}(\tau) U^{r^* m_4}(\tau)\, d\tau\right)^{\frac{m_3}{r^*}}.$$

We choose r^* so that $\frac{m_3}{r^*} = \frac{1}{q^*}$, that is, $r^* = q^* m_3$, we conclude as before by $(H5),(H6)$ and Lemma 3 that

$$E(u,U) \leq K_3^{m_3-\frac{1}{q^*}} u^{-\beta_4 m_3} \left(\int_0^u \varphi_4^{q^* m_3}(\tau) U^{q^* m_3 m_4}(\tau)\, d\tau\right)^{\frac{1}{q^*}}, \tag{8}$$

where $K_3 = K(1-\frac{q^* m_3 \beta_4}{q^* m_3 - 1}, 1+\frac{q^* m_3(\sigma_4-\alpha m_4)}{q^* m_3-1}, \frac{q^* m_3(bm_4-\gamma_4)}{q^* m_3-1})$.

If we apply once again Hölder inequality to $D(s,U)$, taking into account the estimate (8), we get

$$\begin{aligned} D(s,U) &\le K_3^{m_3-\frac{1}{q^*}} \int_0^s (s-u)^{-\beta_3} e^{(\gamma_3-\gamma_4 m_3)u} u^{\sigma_3-\beta_4 m_3} \\ &\quad \times \left(\int_0^u \varphi_4^{q^* m_3}(\tau) U^{q^* m_3 m_4}(\tau)\, d\tau \right)^{\frac{1}{q^*}} du \\ &\le K_3^{m_3-\frac{1}{q^*}} \left(\int_0^s (s-u)^{-q\beta_3} e^{q(\gamma_3-\gamma_4 m_3)u} u^{q(\sigma_3-\beta_4 m_3)} \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_0^s \left(\int_0^u \varphi_4^{q^* m_3}(\tau) U^{q^* m_3 m_4}(\tau)\, d\tau \right) du \right)^{\frac{1}{q^*}}. \end{aligned}$$

Therefore

$$D(s,U) \le K_3^{m_3-\frac{1}{q^*}} K_4^{\frac{1}{q}} s^{-\beta_3} \left(\int_0^s \left(\int_0^u \varphi_4^{q^* m_3}(\tau) U^{q^* m_3 m_4}(\tau)\, d\tau \right) du \right)^{\frac{1}{q^*}}, \tag{9}$$

where $K_4 = K(1-q\beta_3, 1+q(\sigma_3-\beta_4 m_3), q(\gamma_4 m_3-\gamma_3))$.

Inserting now the estimates (7) and (9) into the expression of $B(t,U)$ we find

$$\begin{aligned} B(t,U) \le & \int_0^t (t-s)^{-\alpha} e^{(b-\gamma_2-\gamma_3)s} \left[K_2^{\frac{1}{q}} s^{-\beta_2} \left(\int_0^s \varphi_2^{q^*}(u) U^{q^* m_2}(u)\, du \right)^{1/q^*} \right. \\ & \left. + K_3^{m_3-\frac{1}{q^*}} K_4^{\frac{1}{q}} s^{-\beta_3} \left(\int_0^s \left(\int_0^u \varphi_4^{q^* m_3}(\tau) U^{q^* m_3 m_4}(\tau)\, d\tau \right) du \right)^{\frac{1}{q^*}} \right] ds \\ := & F(t,U) + G(t,U). \end{aligned} \tag{10}$$

It is obvious that

$$\begin{aligned} F(t,U) &\le K_2^{\frac{1}{q}} \left(\int_0^t (t-s)^{-q\alpha} e^{q(b-\gamma_2-\gamma_3)s} s^{-q\beta_2}\, ds \right)^{1/q} \\ &\quad \times \left(\int_0^t \left(\int_0^s \varphi_2^{q^*}(u) U^{q^* m_2}(u)\, du \right) ds \right)^{1/q^*} \\ &\le (K_2 K_5)^{\frac{1}{q}} t^{-\alpha} \left(\int_0^t \left(\int_0^s \varphi_2^{q^*}(u) U^{q^* m_2}(u)\, du \right) ds \right)^{1/q^*}, \end{aligned} \tag{11}$$

with $K_5 = K(1-q\alpha, 1-q\beta_2, q(\gamma_2+\gamma_3-b))$ and

$$\begin{aligned} G(t,U) &\le K_3^{m_3-\frac{1}{q^*}} K_4^{\frac{1}{q}} \left(\int_0^t (t-s)^{-q\alpha} e^{q(b-\gamma_2-\gamma_3)s} s^{-q\beta_3}\, ds \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_0^t \left(\int_0^s \left(\int_0^u \varphi_4^{q^* m_3}(\tau) U^{q^* m_3 m_4}(\tau)\, d\tau \right) du \right) ds \right)^{\frac{1}{q^*}}. \end{aligned}$$

Applying again hypotheses $(H5), (H6)$ and Lemma 3 we obtain

$$G(t,U) \leq K_3^{m_3-\frac{1}{q^*}}(K_4K_6)^{\frac{1}{q}}t^{-\alpha}$$
$$\times\left(\int_0^t\left(\int_0^s\left(\int_0^u \varphi_4^{q^*m_3}(\tau)U^{q^*m_3m_4}(\tau)\,d\tau\right)du\right)ds\right)^{\frac{1}{q^*}}, \tag{12}$$

where $K_6 = K(1-q\alpha, 1-q\beta_3, q(\gamma_2+\gamma_3-b))$.

Now, if we substitute all the obtained estimates in (5), namely (6) and (10) -(12), we get the following

$$U(t) \leq c_1\|x_0\|_p + c_1K_1^{\frac{1}{q}}\left(\int_0^t \varphi_1^{q^*}(s)U^{q^*m_1}(s)\,ds\right)^{1/q^*}$$
$$+ c_1(K_2K_5)^{\frac{1}{q}}\left(\int_0^t\left(\int_0^s \varphi_2^{q^*}(u)U^{q^*m_2}(u)\,du\right)ds\right)^{1/q^*}$$
$$+ c_1K_3^{m_3-\frac{1}{q^*}}(K_4K_6)^{\frac{1}{q}}$$
$$\times\left(\int_0^t\left(\int_0^s\left(\int_0^u \varphi_4^{q^*m_3}(\tau)U^{q^*m_3m_4}(\tau)\,d\tau\right)du\right)ds\right)^{1/q^*}. \tag{13}$$

Applying the following algebraic inequality

$$\left(\sum_{i=1}^m a_i\right)^s \leq m^{s-1}\left(\sum_{i=1}^m a_i^s\right), \quad \forall m\in\mathbb{N}^*, \quad \forall s, a_1, ..., a_m \in \mathbb{R}^+, \tag{14}$$

to (13) with $s = q^*$ and $m = 4$, we infer

$$U^{q^*}(t) \leq 4^{q^*-1}c_1^{q^*}\left\{\|x_0\|_p^{q^*} + K_1^{\frac{q^*}{q}}\int_0^t \varphi_1^{q^*}(s)U^{q^*m_1}(s)\,ds\right.$$
$$+ (K_2K_5)^{\frac{q^*}{q}}\int_0^t\left(\int_0^s \varphi_2^{q^*}(u)U^{q^*m_2}(u)\,du\right)ds$$
$$\left.+ K_3^{q^*m_3-1}(K_4K_6)^{\frac{q^*}{q}}\int_0^t\left(\int_0^s\left(\int_0^u \varphi_4^{q^*m_3}(\tau)U^{q^*m_3m_4}(\tau)\,d\tau\right)du\right)ds\right\}.$$

Define the new constants

$$\tilde{c}_1 = 4^{q^*-1}c_1^{q^*}, \ \ \tilde{c}_2 = 4^{q^*-1}c_1^{q^*}K_1^{\frac{q^*}{q}},$$
$$\tilde{c}_3 = 4^{q^*-1}c_1^{q^*}(K_2K_5)^{\frac{q^*}{q}}, \ \ \tilde{c}_4 = 4^{q^*-1}c_1^{q^*}K_3^{q^*m_3-1}(K_4K_6)^{\frac{q^*}{q}}$$

and set $v(t) = U^{q^*}(t)$. It readily follows that the last inequality becomes

$$v(t) \leq \tilde{c}_1\|x_0\|_p^{q^*} + \tilde{c}_2\int_0^t \varphi_1^{q^*}(s)v^{m_1}(s)\,ds$$
$$+ \tilde{c}_3\int_0^t\left(\int_0^s \varphi_2^{q^*}(u)v^{m_2}(u)\,du\right)ds$$
$$+ \tilde{c}_4\int_0^t\left(\int_0^s\left(\int_0^u \varphi_4^{q^*m_3}(\tau)v^{m_3m_4}(\tau)\,d\tau\right)du\right)ds. \tag{15}$$

Denote the right hand side of (15) by $V(t)$, then $V(0) = \tilde{c}_1 \|x_0\|_p^{q^*}$ and $v(t) \le V(t)$. Next, differentiating the function $V(t)$, we get at once

$$V^{'}(t) = \tilde{c}_2 \varphi_1^{q^*}(t) v^{m_1}(t) + \tilde{c}_3 \int_0^t \varphi_2^{q^*}(s) v^{m_2}(s)\,ds + \tilde{c}_4 \int_0^t \left(\int_0^u \varphi_4^{q^* m_3}(\tau) v^{m_3 m_4}(\tau)\,d\tau \right) du\,.$$

Now since $V(t)$ is nondecreasing, it is then straightforward that

$$V'(t) \le \tilde{c}_2 \varphi_1^{q^*}(t) V^{m_1}(t) + \left(\tilde{c}_3 \int_0^t \varphi_2^{q^*}(s)\,ds \right) V^{m_2}(t) + \left(\tilde{c}_4 \int_0^t \left(\int_0^u \varphi_4^{q^* m_3}(\tau)\,d\tau \right) du \right) V^{m_3 m_4}(t)\,. \tag{16}$$

Let $\delta = \min(m_1, m_2, m_3 m_4)$, then multiplying both sides of inequality (16) by $e^{-\delta V}$ and making use of the algebraic inequality

$$y^a e^{-by} \le \left(\frac{a}{eb} \right)^a, \quad \forall a, b, y > 0$$

we infer that

$$V^{'} e^{-\delta V} \le e^{-\delta} \delta^{-\delta} h(t)\,, \tag{17}$$

where

$$h(t) = m_1^{m_1} \tilde{c}_2 \varphi_1^{q^*}(t) + m_2^{m_2} \tilde{c}_3 \int_0^t \varphi_2^{q^*}(s)\,ds + (m_3 m_4)^{m_3 m_4} \tilde{c}_4 \int_0^t \left(\int_0^u \varphi_4^{q^* m_3}(\tau)\,d\tau \right) du\,.$$

Integrating (17) from 0 to t, we find the following

$$\frac{1}{\delta} \left(e^{-\delta V(0)} - e^{-\delta V(t)} \right) \le e^{-\delta} \delta^{-\delta} \int_0^t h(s)\,ds \le e^{-\delta} \delta^{-\delta} H_0\,.$$

But since we have by assumption $H_0 < e^{\delta(1 - \tilde{c}_1 \|x_0\|_p^{q^*})} \delta^{\delta - 1}$, then $e^{\delta(1 - \tilde{c}_1 \|x_0\|_p^{q^*})} - \delta^{-\delta+1} H_0 > 0$ and therefore,

$$V(t) \le \frac{1}{\delta} \ln \left(\frac{(e\delta)^\delta}{(\delta e^{1 - \tilde{c}_1 \|x_0\|_p^{q^*}})^\delta - \delta H_0} \right), \quad \forall t > 0\,.$$

Accordingly, we have

$$t^{q^* \alpha} e^{q^* bt} \|A^\alpha x(t)\|_p^{q^*} \le U^{q^*}(t) = v(t) \le V(t) \le \frac{1}{\delta} \ln \left(\frac{(e\delta)^\delta}{(\delta e^{1 - \tilde{c}_1 \|x_0\|_p^{q^*}})^\delta - \delta H_0} \right), \quad \forall t > 0$$

from which we get the desired estimate

$$\|A^\alpha x(t)\|_p \le c t^{-\alpha} e^{-bt}, \quad \forall t > 0\,, \tag{18}$$

with $c = \dfrac{1}{\delta} \ln \left(\dfrac{(e\delta)^\delta}{(\delta e^{1-\tilde{c}_1 \|x_0\|_p^{q^*}})^\delta - \delta H_0} \right)$. This completes the proof of the Theorem. □

Corollary 1. *If* $0 \le \alpha \le 1$ *and* $0 \le \nu < 2\alpha - \frac{n}{p}$, *then*

$$|A^\alpha x(t)|_\nu \le ct^{-\alpha} e^{-bt}, \quad \forall t > 0,$$

(*where* $|.|_\nu$ *is the norm of the space* $C^\nu(\bar{\Omega})$).

Proof. It is a straightforward consequence of Lemma 1. □

Remark 1. It is fairly apparent that our inequality (5) is more complicated than inequality (29) stated in [5].

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