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THE CANONICAL TENSOR FIELDS OF TYPE $(1,1)$ ON $(J^r(\odot^2T^*))^*$

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Abstract. We prove that every natural affinor on $(J^r(\odot^2T^*))^*(M)$ is proportional to the identity affinor if $\dim M \geq 3$.

0. Introduction

For every n-dimensional manifold M we have the vector bundle

$$J^r(\odot^2T^*)(M) = \{j^r_x\tau | \tau \text{ is a symmetric tensor field type } (0,2) \text{ on } M, x \in M\}.$$

Every local diffeomorphism $\varphi : M \to N$ between n-manifolds gives a vector bundle homomorphism $J^r(\odot^2T^*)(\varphi) : J^r(\odot^2T^*)(M) \to J^r(\odot^2T^*)(N)$, $j^r_x\tau \to j^r_{\varphi(x)}(\varphi_*\tau)$. Functor $J^r(\odot^2T^*) : \mathcal{M}f_n \to \mathcal{VB}$ is a vector natural bundle over n-manifolds in the sense of [5]. Let $(J^r(\odot^2T^*))^* : \mathcal{M}f_n \to \mathcal{VB}$ be the dual vector bundle, $(J^r(\odot^2T^*))^*(M) = (J^r(\odot^2T^*)(M))^*$, $(J^r(\odot^2T^*))^*(\varphi) = (J^r(\odot^2T^*)(\varphi^{-1}))^*$ for M and φ as above.

An affinor on a manifold M is a tensor field of type $(1,1)$ on M.

A natural affinor Q on $(J^r(\odot^2T^*))^*$ is a system of affinors

$$Q : T(J^r(\odot^2T^*))^*(M) \to T(J^r(\odot^2T^*))^*(M)$$

on $(J^r(\odot^2T^*))^*(M)$ for every n-manifold M satisfying the naturality condition $T(J^r(\odot^2T^*))^*(\varphi) \circ Q = Q \circ T(J^r(\odot^2T^*))^*(\varphi)$ for every local diffeomorphism $\varphi : M \to N$ between n-manifolds.

In this paper we prove, that every natural affinor Q on $(J^r(\odot^2T^*))^*$ over n-manifolds is proportional to the identity affinor if $n \geq 3$.

The proof of the classification theorem is based on the method from paper [7], where there are determined the natural affinors on $(J^r(\bigwedge^2 T^*))^*$. However the proof is different, because the tensor field $dx^1 \odot dx^1$ on $\mathbf{R}^n$ is non-zero, in contrast to $dx^1 \wedge dx^1$.

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Natural affinors on some natural bundle F can be used to study torsions $[Q,\Gamma]$ of a connection Γ of F. That is why, the natural affinors have been study in many papers, [1] … [11], e.t.c.

The usual coordinates on $\mathbf{R}^n$ are denoted by x^i. The canonical vector fields on $\mathbf{R}^n$ are denoted by $\partial_i = \frac{\partial}{\partial x^i}$.

All manifolds are assumed to be finite dimensional and smooth, i.e. of class C^∞. Mappings between manifolds are assumed to be smooth.

1. The linear natural transformations $T(J^r(\odot^2 T^*))^* \to (J^r(\odot^2 T^*))^*$

A natural transformation $T(J^r(\odot^2 T^*))^* \to (J^r(\odot^2 T^*))^*$ over n-manifolds is a system of fibred maps

$$A : T(J^r(\odot^2 T^*))^*(M) \to (J^r(\odot^2 T^*))^*(M)$$

over id_M for every n-manifold M such that

$$(J^r(\odot^2 T^*))^*(f) \circ A = A \circ T(J^r(\odot^2 T^*))^*(f)$$

for every local diffeomorphism $f : M \to N$ between n-manifolds.

A natural transformation $A : T(J^r(\odot^2 T^*))^* \to (J^r(\odot^2 T^*))^*$ is called linear if A gives a linear map $T_y(J^r(\odot^2 T^*))^*(M) \to ((J^r(\odot^2 T^*))^*(M))_x$ for any $y \in ((J^r(\odot^2 T^*))^*(M))_x$, $x \in M$.

Theorem 1. *If $n \geq 3$ and r are natural numbers, then every linear natural transformation $A : T(J^r(\odot^2 T^*))^* \to (J^r(\odot^2 T^*))^*$ over n-manifolds is equal to 0.*

The proof of Theorem 1 will occupy Sections 2 – 6.

2. The reducibility propositions

Every element from the fibre $\big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$ is a linear combination of all elements $(j^r_0(x^\alpha\, dx^i \odot dx^j))^*$, where $\alpha \in (\mathbf{N} \cup \{0\})^n$, $|\alpha| \leq r$, $i \leq j$, $i,j = 1,\dots,n$. The elements $(j^r_0(x^\alpha\, dx^i \odot dx^j))^*$ are dual basis to the basis $j^r_0(x^\alpha\, dx^i \odot dx^j)$ of $\big(J^r(\odot^2 T^*)(\mathbf{R}^n)\big)_0$.

Consider a linear natural transformation $A : T(J^r(\odot^2 T^*))^* \to (J^r(\odot^2 T^*))^*$.

Lemma 1. *Suppose A satisfies*

$$\langle A(u),\, j^r_0(x^\alpha\, dx^i \odot dx^j)\rangle = 0$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $\alpha \in (\mathbf{N} \cup \{0\})^n$, $|\alpha| \leq r$, $i \leq j$, $i,j = 1,\dots,n$. Then $A = 0$.

Proof. If assumptions of Lemma 1 meets, then $A(u) = 0$ for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. Let $w \in (T(J^r(\odot^2 T^*))^*(M))_x$, $x \in M$. There exists a chart $\varphi : M \supset U \to \mathbf{R}^n$ such that $\varphi(x) = 0$ and U is open subset including x. Since A is invariant with respect to φ, we have $A(w) = T(J^r(\odot^2 T^*))^*\,(\varphi^{-1})(A(u))$, where $u = T(J^r(\odot^2 T^*))^*(\varphi)(w) \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. Then $A(w) = 0$, because $A(u) = 0$. That is why $A = 0$. The lemma is proved. $\square$

Lemma 2. *Suppose that*

$$\langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^1)\rangle = \langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^2)\rangle = 0$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $\alpha \in (\mathbf{N}\cup\{0\})^n$, $|\alpha| \le r$, $i \le j$, $i,j = 1,\dots,n$. *Then* $A = 0$.

Proof. Let $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $\alpha \in (\mathbf{N}\cup\{0\})^n$, $|\alpha| \le r$, $i \le j$, $i,j = 1,\dots,n$. It is enough to prove, that $\langle A(u),\, j^r_0(x^\alpha\, dx^i \odot dx^j)\rangle = 0$.

Consider two cases

a) $i = j$. Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism transforming x^i into x^1 and x^α into $x^{\tilde\alpha}$ for some $\tilde\alpha \in (\mathbf{N}\cup\{0\})^n$, $|\tilde\alpha| \le r$. From the invariance of A with respect to φ and the assumption of Lemma 2, we have $\langle A(u),\, j^r_0(x^\alpha\, dx^i \odot dx^i)\rangle = \langle A(\tilde u),\, j^r_0(x^{\tilde\alpha}\, dx^1 \odot dx^1)\rangle = 0$, where $\tilde u = T(J^r(\odot^2 T^*))^*(\varphi)(u)$

b) $i \ne j$. Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism transforming x^i in x^1, x^j in x^2 and x^α in $x^{\tilde\alpha}$ for some $\tilde\alpha \in (\mathbf{N}\cup\{0\})^n$, $|\tilde\alpha| \le r$. From invariance of A with respect to φ and the assumption of Lemma 2, we have $\langle A(u),\, j^r_0(x^\alpha\, dx^i \odot dx^j)\rangle = \langle A(\tilde u),\, j^r_0(x^{\tilde\alpha}\, dx^1 \odot dx^2)\rangle = 0$, where $\tilde u = T(J^r(\odot^2 T^*))^*(\varphi)(u)$. □

Lemma 3. *Suppose A satisfies*

$$\begin{aligned}\langle A(u),\, j^r_0(dx^1 \odot dx^1)\rangle &= \langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^1)\rangle\\ = \langle A(u),\, j^r_0(dx^1 \odot dx^2)\rangle &= \langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle = 0\end{aligned}$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $\alpha \in (\mathbf{N}\cup\{0\})^n$, $|\alpha| \le r$, $i \le j$, $i,j = 1,\dots,n$. *Then* $A = 0$.

Proof. Let $\alpha \in (\mathbf{N}\cup\{0\})^n$, $|\alpha| \le r$, $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $\alpha \ne e_3 = (0,0,1,0,\dots,0) \in (\mathbf{N}\cup\{0\})^n$.
On the strength of Lemma 2 it is enough to prove that

$$\langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^1)\rangle = \langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^2)\rangle = 0.$$

We can set that $\alpha \ne 0$. Let $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism transforming x^1 in x^1, x^2 in x^2 and $x^3 + x^\alpha$ in x^3. From the invariance of A with respect to φ and the assumption of Lemma 3, we have

$$\begin{aligned}\langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^1)\rangle &= \langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^1)\rangle + \langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^1)\rangle\\ &= \langle A(u),\, j^r_0((x^3 + x^\alpha)\, dx^1 \odot dx^1)\rangle\\ &= \langle A(\tilde u),\, j^r_0(x^3\, dx^1 \odot dx^1)\rangle = 0\end{aligned}$$

where $\tilde u = T(J^r(\odot^2 T^*))^*(\varphi)(u)$.
Similarly $\langle A(u),\, j^r_0(x^\alpha\, dx^1 \odot dx^2)\rangle = 0$. □

Lemma 4. *Suppose that*

$$\langle A(u),\, dx^1 \odot dx^2\rangle \;=\; \langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle \;=\; 0$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. *Then* $A = 0$.

Proof. By Lemma 3 it is sufficient to show that

$$\langle A(u),\, dx^1 \odot dx^1\rangle = \langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^1)\rangle = 0$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$.

Let $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. Consider a diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ transforming x^1 in x^1, x^2 in $x^1 + x^2$ and x^3 in x^3. Then from the invariance of A with respect to φ and the assumption of lemma, we have

$$\begin{aligned} 0 &= \langle A(\tilde u),\, j^r_0(dx^1 \odot dx^2)\rangle \\ &= \langle A(u),\, j^r_0(dx^1 \odot (dx^1 + dx^2))\rangle \\ &= \langle A(u),\, j^r_0(dx^1 \odot dx^1)\rangle + \langle A(u),\, j^r_0(dx^1 \odot dx^2)\rangle\ , \end{aligned}$$

where $\tilde u = T(J^r(\odot^2 T^*))^*(\varphi^{-1})(u)$. So $\langle A(u),\, j^r_0(dx^1 \odot dx^1)\rangle = 0$.
Similarly $\langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^1)\rangle = 0$. □

Using Lemma 4 we see that Theorem 1 will be proved after proving the following two propositions.

Proposition 1. *We have*

$$\langle A(u),\, j^r_0(dx^1 \odot dx^2)\rangle = 0$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$.

Proposition 2. *We have*

$$\langle A(u),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle = 0$$

for every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$.

3. Some notations

We have the obvious trivialization

$$\big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0 \cong \mathbf{R}^n \times \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0 \times \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$$

given by $(u_1, u_2, u_3) \to (\tilde u_1)^C(u_2) + \frac{d}{dt}_{|t=0}(u_2 + t u_3)$, where $\tilde u_1$ is the constant vector field on $\mathbf{R}^n$ such that $\tilde u_{1_{|0}} = u_1 \in \mathbf{R}^n \cong T_0\mathbf{R}^n$ and $(\tilde u_1)^C$ is the complete lift of $\tilde u_1$ to $(J^r(\odot^2 T^*))^*$.
Each $u_\tau \in \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $\tau = 2, 3$ can be expressed in the form

$$u_\tau = \sum u_{\tau,\alpha,i,j}\big(j^r_0(x^\alpha\, dx^i \odot dx^j)\big)^*\ ,$$

where the sum is over all $\alpha \in (\mathbf{N} \cup \{0\})^n,\ |\alpha| \le r,\ i \le j,\ i, j = 1, \dots, n$.
It defines $u_{\tau,\alpha,i,j}$ for each u_τ as above.

4. Proof of Proposition 1

We start with the following lemma.

Lemma 5. *There exists the number* $\lambda \in \mathbf{R}$ *such that*

$$\langle A(u),\, j^r_0(\, dx^1 \odot dx^2)\rangle = \lambda u_{3,(0),1,2}$$

for every $u = (u_1, u_2, u_3) \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$.

Proof. Let $\Phi : \mathbf{R}^n \times \big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0 \times \big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0 \to \mathbf{R}$ be such that

$$\Phi(u_1,u_2,u_3) = \langle A(u),\, j^r_0(\,dx^1 \odot dx^2)\rangle\,,$$

where $u=(u_1,u_2,u_3)$, $u_1=(u_1^\iota)\in\mathbf{R}^n$, $\iota=1,\dots,n$, $u_2\in\big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$, $u_3\in\big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$.
The invariance of A with respect to the homotheties $a_t=(t^1x^1,\dots,t^nx^n)$ for $t=(t^1,\dots,t^n)\in\mathbf{R}^n_+$ gives the homogeneous condition

$$\Phi(T(J^r(\odot^2T^*))^*(a_t)(u)) = t^1t^2\Phi(u)\,.$$

Then from the homogeneous function theorem, [5], it follows that $\Phi(u)$ is the linear combination of monomials in u_1^ι, $u_{\tau,\alpha,i,j}$ of weight t^1t^2. Moreover $\Phi(u_1,u_2,u_3)$ is linear in u_1,u_3 for u_2, since A is linear. It implies the lemma. $\square$

In particular from Lemma 5 it follows that

$$(*)\qquad \langle A(\partial^C_{1|w}),\, j^r_0(\,dx^1\odot dx^2)\rangle = \langle A(e_1,w,0),\, j^r_0(\,dx^1\odot dx^2)\rangle = 0$$

for every $w\in\big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$, where $\partial_1=\frac{\partial}{\partial x^1}$ and $()^C$ is the complete lift to $(J^r(\odot^2T^*))^*$.

We are now in position to prove Proposition 1. Let λ be from Lemma 5. It is enough to prove that λ is equal to 0.

We see that $\lambda=\langle A(0,0,(j^r_0(\,dx^1\odot dx^2))^*),\, j^r_0(\,dx^1\odot dx^2)\rangle$.

We have

$$\begin{aligned}
0 &= \langle A((x^1)^{r+1}\partial_1)^C_{|w},\, j^r_0(\,dx^1\odot dx^2)\rangle\\
(**)\qquad &= (r+1)\langle A(0,w,(j^r_0(\,dx^1\odot dx^2))^*+\dots),\, j^r_0(\,dx^1\odot dx^2)\rangle\\
&= (r+1)\langle A(0,0,(j^r_0(\,dx^1\odot dx^2))^*),\, j^r_0(\,dx^1\odot dx^2)\rangle,
\end{aligned}$$

where $w=(j^r_0((x^1)^r\,dx^1\odot dx^2))^*$ and the dots is a linear combination of the $(j^r_0(x^\alpha\,dx^i\odot dx^j))^*$ with $(j^r_0(x^\alpha\,dx^i\odot dx^j))^*\neq(j^r_0(\,dx^1\odot dx^2))^*$.
It remains to explain $(**)$.

At first we show the second equality in $(**)$. Let φ_t be the flow of $(x^1)^{r+1}\partial_1$. We have the following sequences of equalities

$$\begin{aligned}
\langle (x^1)^{r+1}\partial_1)^C_{|w},\, j^r_0(\,dx^1\odot dx^2)\rangle &= \langle \frac{d}{dt}_{|t=0}(J^r(\odot^2T^*))^*_0(\varphi_t)(w),\, j^r_0(\,dx^1\odot dx^2)\rangle\\
&= \frac{d}{dt}_{|t=0}\langle (J^r(\odot^2T^*))^*_0(\varphi_t)(w),\, j^r_0(\,dx^1\odot dx^2)\rangle\\
&= \frac{d}{dt}_{|t=0}\langle w,\, j^r_0((\varphi_{-t})_*\,dx^1\odot dx^2)\rangle\\
&= \langle w,\, j^r_0(\frac{d}{dt}_{t=0}(\varphi_{-t})_*\,dx^1\odot dx^2)\rangle\\
&= \langle w,\, j^r_0(L_{(x^1)^{r+1}\partial_1}(\,dx^1\odot dx^2))\rangle\\
&= (r+1)\langle w,\, j^r_0((x^1)^r\,dx^1\odot dx^2)\rangle = r+1\,.
\end{aligned}$$

Then $((x^1)^{r+1}\partial_1)^C_{|w} = (r+1)(j^r_0(\,dx^1 \odot dx^2))^* + \dots$ under the canonical isomorphism $V_w\big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big) \cong \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. So we have the second equality in $(**)$.

The last equality in $(**)$ is clear because of Lemma 5.

We can prove the first equality in $(**)$ as follows. Vector fields $\partial_1 + (x^1)^{r+1}\partial_1$ and ∂_1 have the same r-jets at $0 \in \mathbf{R}^n$. Then, by [12], there exists a diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ such that $j^{r+1}_0\varphi = \mathrm{id}$ and $\varphi_*\partial_1 = \partial_1 + (x^1)^{r+1}\partial_1$ in a certain neighborhood of 0. Obviously, φ preserves $j^r_0(\,dx^1 \odot dx^2)$ that is $j^r_0(\,dx^1 \odot dx^2) = J^r_0(\odot^2 T^*)(\varphi)\big(j^r_0(\,dx^1 \odot dx^2)\big)$ because $j^{r+1}_0\varphi = \mathrm{id}$. Then, using the invariance of A with respect to φ, from $(*)$ it follows that $\langle A(\partial_1 + (x^1)^{r+1}\partial_1)^C_{|w},\, j^r_0(\,dx^1 \odot dx^2)\rangle = \langle A(\partial^C_{1|w}),\, j^r_0(\,dx^1 \odot dx^2)\rangle = 0$ for every $w \in \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. Now, using the linearity of A, we end the proof of the first equality of $(**)$.

The proof of Proposition 1 is complete. □

5. Proof of Proposition 2

The proof of Proposition 2 is similar to the proof of Proposition 1. We start with the following lemma.

Lemma 6. *For every $u = (u^1, u^2, u^3) \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$ we have*

$$\begin{aligned}\langle A(u),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle &= a u^1_1 u_{2,(0),2,3} + b u^2_1 u_{2,(0),1,3} + c u^3_1 u_{2,(0),1,2}\\ &\quad + e u_{3,e_2,2,3} + f u_{3,e_2,1,3} + g u_{3,e_3,1,2}\end{aligned}$$

where $e_i = (0, 0, \dots, 1, 0, \dots, 0) \in (\mathbf{N} \cup \{0\})^n$, 1 in i-position.

Proof. We will use the similar arguments as in the proof of Lemma 5.

Let $\Phi : \mathbf{R}^n \times \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0 \times \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0 \to \mathbf{R}$ such that

$$\Phi(u_1, u_2, u_3) = \langle A(u),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle\,,$$

$u = (u_1, u_2, u_3)$, $u_1 = (u^\iota_1) \in \mathbf{R}^n$, $\iota = 1, \dots, n$, $u_2 \in \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$, $u_3 \in \big((J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$. The invariance of A with respect to the homotheties $a_t = (t^1x^1, \dots, t^nx^n)$ for $t = (t^1, \dots, t^n) \in \mathbf{R}^n_+$ gives the homogeneous condition

$$\Phi(T(J^r(\odot^2 T^*))^*(a_t)(u)) = t^1t^2t^3\Phi(u)\,.$$

Then from the homogeneous function theorem, [5], it follows that $\Phi(u)$ is the linear combination of monomials in u^ι_1, $u_{\tau,\alpha,i,j}$ of weight $t^1t^2t^3$. Moreover $\Phi(u_1, u_2, u_3)$ is linear in u_1 and u_3 for u_2, since A is linear. It implies the lemma. □

To prove Proposition 2 we have to show that $a = b = c = e = f = g = 0$. We need the following lemmas.

Lemma 7. *For every $u \in \big(T(J^r(\odot^2 T^*))^*(\mathbf{R}^n)\big)_0$ we have*

$$\langle A(u),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle = -\langle A(u'),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle\,,$$

where u' is the image of u by $(x^2, x^3, x^1) \times \mathrm{id}_{\mathbf{R}^{n-3}}$.

Proof. Consider $u \in \big(T(J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$. Let $\tilde{u}$ be the image of u by $\varphi = (x^1 + x^1x^3, x^2, \dots, x^n)$. From Proposition 1 we have

$$\langle A(\tilde{u}),\, j^r_0(\,dx^1 \odot dx^2)\rangle = \langle A(u),\, j^r_0(\,dx^1 \odot dx^2)\rangle = 0\,.$$

Using the invariance of A with respect to φ^{-1} we have

$$\begin{aligned} 0 &= \langle A(u),\, j^r_0(\,dx^1 \odot dx^2)\rangle \\ &= \langle A(u),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle + \langle A(u),\, j^r_0(x^1\,dx^2 \odot dx^3)\rangle \end{aligned}$$

because φ^{-1} preserves A, it transforms $\tilde{u}$ in u and $j^r_0(\,dx^1 \odot dx^2)$ in $j^r_0(\,dx^1 \odot dx^2) + j^r_0(x^3\,dx^1 \odot dx^2) + j^r_0(x^1\,dx^2 \odot dx^3)$. So, $\langle A(u),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle = -\langle A(u),\, j^r_0(x^1dx^2 \odot dx^3)\rangle$. Hence we have the lemma because $(x^2, x^3, x^1) \times \mathbf{R}^{n-3}$ sends u in u' and $j^r_0(x^1\,dx^2 \odot dx^3)$ in $j^r_0(x^3\,dx^1 \odot dx^2)$. □

Lemma 8. *We have $g = f = e = 0$.*

Proof. Obviously

$$g = \langle A(0,0,(j^r_0(x^3\,dx^1 \odot dx^2))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle$$

by Lemma 6. Similarly

$$\begin{aligned} f &= \langle A(0,0,(j^r_0(x^2\,dx^1 \odot dx^3))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle\,, \\ e &= \langle A(0,0,(j^r_0(x^1\,dx^2 \odot dx^3))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle\,. \end{aligned}$$

So, to prove Lemma 8 we have to show

$$\begin{aligned} &\langle A(0,0,(j^r_0(x^3\,dx^1 \odot dx^2))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle \\ &\quad = \langle A(0,0,(j^r_0(x^2\,dx^1 \odot dx^3))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle \\ &\quad = \langle A(0,0,(j^r_0(x^1\,dx^2 \odot dx^3))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle = 0\,. \end{aligned}$$

We can see that $(x^2,x^3,x^1) \times \mathrm{id}_{\mathbf{R}^{n-3}}$ sends $(j^r_0(x^3\,dx^1 \odot dx^2))^*$ in $(j^r_0(x^2\,dx^1 \odot dx^3))^*$ and $(j^r_0(x^2\,dx^1 \odot dx^3))^*$ in $(j^r_0(x^1\,dx^2 \odot dx^3))^*$. Then using Lemma 7 it is enough to verify that $\langle A(0,0,(j^r_0(x^3\,dx^1 \odot dx^2))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle = 0$. So, it is enough to prove the sequence of equalities:

$$\begin{aligned} 0 &= \langle A(((x^1)^r\partial_1)^C_{|w},\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle \\ &= r\langle A(0,w,(j^r_0(x^3\,dx^1 \odot dx^2))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle \\ &= r\langle A(0,0,(j^r_0(x^3\,dx^1 \odot dx^2))^*),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle\ , \end{aligned} \tag{$***$}$$

where $w = (j^r_0(x^3(x^1)^{r-1}\,dx^1 \odot dx^2))^* \in \big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$.

The third equality in $(***)$ is clear on the basis of Lemma 6.

Let us explain the first equality in $(***)$. Vector fields $\partial_1 + (x^1)^r\partial_1$ and ∂_1 have the same $(r-1)$-jets at $0 \in \mathbf{R}^n$. Then, by [12] there exist a diffeomorphism $\varphi = \varphi_1 \times \mathrm{id}_{\mathbf{R}^{n-1}} : \mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1} \to \mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$ such that $\varphi_1 : \mathbf{R} \to \mathbf{R}$, $j^r_0\varphi = \mathrm{id}$ and $\varphi_*\partial_1 = \partial_1 + (x^1)^r\partial_1$ in a certain neighborhood of $0 \in \mathbf{R}^n$. Let φ^{-1} sends ω in $\tilde{\omega}$. Then $\tilde{\omega}$ is a linear combination of the elements $(j^r_0(x^\alpha\,dx^i \odot dx^j))^* \in \big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$ for $r \ge |\alpha| \ge 1$, $i,j = 1,\dots n$, $i \le j$. (For $\langle \tilde{\omega},\, j^r_0(\,dx^i \odot dx^j)\rangle = \langle \omega,\, j^r_0(d(x^i \circ \varphi^{-1}) \odot d(x^i \circ \varphi^{-1}))\rangle = 0$.) Then, by Lemma 6, $\langle A(\partial^C_{1|\tilde{\omega}}),\, j^r_0(x^3\,dx^1 \odot dx^2)\rangle = \langle A(e_1,\tilde{\omega},0),\, j^r_0(x^3\,dx^1 \odot$

$dx^2)\rangle = 0$ (as $j^r_0\varphi = \mathrm{id}$). Then from naturality of A with respect to φ we obtain $\langle A((\partial_1 + (x^1)^r\partial_1)^C_{|\omega}),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle = 0$. Now, using the linearity of A we have $\langle A(((x^1)^r\partial_1)^C_{|\omega}),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle = 0$. This ends the proof of the first equality in $(***)$.

Let us explain the second equality in $(***)$. Analysing the flow of vector field $(x^1)^r\partial_1$ and taking $\omega = (j^r_0(x^3(x^1)^{r-1}\, dx^1 \odot dx^2))^* \in \big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$ we have (similarly as in the justification of the second equality of $(**)$)

$$\begin{aligned}\langle((x^1)^r\partial_1)^C_{|\omega},\, j^r_0(\alpha\, dx^i \odot dx^j)\rangle &= \langle\omega,\, j^r_0(L_{(x^1)^r\partial^1}(x^\alpha dx^i \odot dx^j))\rangle\\ &= \langle\omega,\, \alpha_1 j^r_0((x^1)^{r-1}x^\alpha\, dx^i \odot dx^j)\rangle\\ &\quad + \langle\omega,\, j^r_0(x^\alpha\delta^i_1 r(x^1)^{r-1}\, dx^1 \odot dx^j)\rangle\,,\end{aligned}$$

where δ^i_1 is the Kronecker delta.
Since $\omega = (j^r_0(x^3(x^1)^{r-1}\, dx^1 \odot dx^2))^*$ the last sum is equal to r if $\alpha = e_3$ and $(i,j) = (1,2)$, and 0 in the other cases. Then $(x^1)^r\partial_1)^C_{|\omega} = r(j^r_0(x^3\, dx^1 \odot dx^2))^*$. This ends the proof of the second equality of $(***)$.
The proof of Lemma 8 is complete. □

Lemma 9. *We have $a = b = c = 0$.*

Proof. Using Lemma 7 (similarly as for $g = f = e$) it is sufficient to prove that $c = 0$, i.e. $\langle A(\partial^C_{3|(j^r_0(\, dx^1\odot dx^2))^*}),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle = 0$.
But we have

$$\begin{aligned}0 &= \langle A(\partial^C_{3|(j^r_0((x^1)^r\, dx^1\odot dx^2))^*}),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle\\ (****)\qquad &= \langle A(\partial^C_{3|(j^r_0(\, dx^1\odot dx^2))^*+\dots}),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle\\ &= \langle A(\partial^C_{3|(j^r_0(\, dx^1\odot dx^2))^*}),\, j^r_0(x^3\, dx^1 \odot dx^2)\rangle\ ,\end{aligned}$$

where the dots is the linear combination of elements $(j^r_0(x^\alpha\, dx^i \odot dx^j))^* \neq (j^r_0(\, dx^1 \odot dx^2))^*$, $\alpha \in (\mathbf{N} \cup \{0\})^n$, $|\alpha| \le r$, $i \le j$, $i,j = 1,\dots,n$.

Equalities first and third are clear because of Lemma 6.

Let us explain the second equality. Consider the local diffeomorphism $\varphi = (x^1 + \frac{1}{r+1}(x^1)^{r+1}, x^2, \dots, x^n)^{-1}$. We see that φ^{-1} preserves $j^r_0(x^3\, dx^1 \odot dx^2)$ and ∂_3. Moreover φ^{-1} sends $(j^r_0((x^1)^r\, dx^1 \odot dx^2))^*$ in $(j^r_0(\, dx^1 \odot dx^2))^* + \dots$, where the dots is as above. Now, by the invariance of A with respect to φ^{-1} we get the second equality in$(****)$.
The proof of Lemma 9 is complete. □
The proof of Proposition 2 is complete. □
The proof of Theorem 1 is complete. □

7. The natural affinors on $(J^r(\odot^2T^*))^*$ of vertical type

A natural affinor $Q : T(J^r(\odot^2T^*))^* \to T(J^r(\odot^2T^*))^*$ on $(J^r(\odot^2T^*))^*$ is of *vertical type* if the image of Q is in the vertical space $V(J^r(\odot^2T^*))^*(M)$ for every n-manifolds M.

We have the natural isomorphism

$$V(J^r(\odot^2T^*))^*(M) \cong (J^r(\odot^2T^*))^*(M) \times (J^r(\odot^2T^*))^*(M)$$

given by $(u,w) = \frac{d}{dt}_{|t=0}(u+tv)$, $u,v \in (J^r(\odot^2T^*))^*_x(M)$, $x \in M$, and the natural projection $pr_2 : V(J^r(\odot^2T^*))^*M \to (J^r(\odot^2T^*))^*M$ on the second factor.

Let $Q : T(J^r(\odot^2T^*))^* \to T(J^r(\odot^2T^*))^*$ on $(J^r(\odot^2T^*))^*$ be a natural affinor of vertical type. Composing Q with pr_2 we get a natural linear transformation $pr_2 \circ Q : T(J^r(\odot^2T^*))^* \to (J^r(\odot^2T^*))^*$ over n-manifolds. It is equal to 0 because of Theorem 1. So, we have the following corollary.

Corollary 1. *Let $n \geq 3$, r be natural numbers. Every natural affinor Q of vertical type on $(J^r(\odot^2T^*))^*$ over n-manifolds is equal to* 0.

8. The linear natural transformations $T(J^r(\odot^2T^*))^* \to T$

Let π be the projection of natural bundle $(J^r(\odot^2T^*))^*$. Then the tangent map $T\pi_M : T(J^r(\odot^2T^*))^*(M) \to TM$ defines a linear natural transformation $T\pi : T(J^r(\odot^2T^*))^* \to T$. (The definition of a linear natural transformation $T(J^r(\odot^2T^*))^* \to T$ over n-manifolds is similar to the one in Section 1.)

Theorem 2. *Let n and r be natural numbers. Every linear natural transformation $B : T(J^r(\odot^2T^*))^* \to T$ over n-manifolds is proportional to $T\pi$.*

9. Proof of Theorem 2

Consider a linear natural transformation $B : T(J^r(\odot^2T^*))^* \to T$. We have

Lemma 10. *If $\langle B(u), d_0x^1\rangle = 0$ for every $u \in \big(T(J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$ then $B = 0$.*

Proof. The proof of Lemma 10 is similar to the proofs of Lemmas $1-4$. From the invariance of B with respect to the coordinate permutation we see that $\langle B(u), d_0x^i\rangle = 0$ for $i = 1,\dots,n$ and $u \in \big(T(J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$. So $B(u) = 0$ for every $u \in \big(T(J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$. Then using the invariance of B with respect to the charts we obtain that $B = 0$. □

Lemma 11. *We have $\langle B(u), d_0x^1\rangle = \lambda u_1^1$ for some $\lambda \in \mathbf{R}$, where $u = (u_1,u_2,u_3)$, $u_1 = (u_1^\iota) \in \mathbf{R}^n$, $\iota = 1,\dots,n$, and $u_2,u_3 \in \big((J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$.*

Proof. The proof of Lemma 11 is similar to the proof of Lemma 5. □

Lemma 11 shows that $\langle (B-\lambda T\pi)(u), d_0x^1\rangle = 0$ for every $u \in \big(T(J^r(\odot^2T^*))^*(\mathbf{R}^n)\big)_0$. Then $B - \lambda T\pi = 0$ by Lemma 10, i.e. $B = \lambda T\pi$.

The proof of Theorem 2 is complete. □

10. The main result

The main result of the present paper is the following theorem.

Theorem 3. *Let* $n \geq 3$ *and* r *be natural numbers. Every natural affinor* $Q : T(J^r(\odot^2 T^*))^* \to T(J^r(\odot^2 T^*))^*$ *on* $(J^r(\odot^2 T^*))^*$ *over* n*-manifolds is proportional to the identity affinor.*

Proof. The composition $T\pi \circ Q : T(J^r(\odot^2 T^*))^* \to T$ is a linear natural transformation. Hence, by Theorem 2, $T\pi \circ Q = \lambda T\pi$ for some $\lambda \in \mathbf{R}$. Then $Q - \lambda\,\mathrm{id} : T(J^r(\odot^2 T^*))^* \to T(J^r(\odot^2 T^*))^*$ is a natural affinor of vertical type, because $T\pi \circ (Q - \lambda\,\mathrm{id}) = T\pi \circ Q - \lambda T\pi = 0$. From Corollary 1 we obtain that $Q - \lambda\,\mathrm{id} = 0$. Thus $Q = \lambda\,\mathrm{id}$. The proof of Theorem 3 is complete. □

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