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HOW TO CHARACTERIZE COMMUTATIVITY EQUALITIES FOR DRAZIN INVERSES OF MATRICES

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Abstract. Necessary and sufficient conditions are presented for the commutativity equalities $A^*A^D = A^DA^*$, $A^\dagger A^D = A^DA^\dagger$, $A^\dagger AA^D = A^DAA^\dagger$, $AA^DA^* = A^*A^DA$ and so on to hold by using rank equalities of matrices. Some related topics are also examined.

1. Introduction

The Drazin inverse of a complex square matrix A is defined as a solution X of the following three equations

$$(1) \quad A^kXA = A^k\,, \qquad (2) \quad XAX = X\,, \qquad (3) \quad AX = XA\,,$$

which uniquely exists and is often denoted by $X = A^D$, where k is the index of A, i.e., the smallest nonnegative integer k such that $r(A^k) = r(A^{k+1})$. In particular, when $\operatorname{Ind} A = 1$, the Drazin inverse of matrix A is called the group inverse of A, and is often denoted by $A^\#$. The Moore-Penrose inverse $A^\dagger$ of a complex matrix A is defined by the four Penrose equations

$$(1)\ AA^\dagger A = A\,, \quad (2)\ A^\dagger AA^\dagger = A^\dagger\,, \quad (3)\ (AA^\dagger)^* = AA^\dagger\,, \quad (4)\ (A^\dagger A)^* = A^\dagger A\,,$$

where $(\cdot)^*$ denotes the conjugate transpose of a complex matrix. A well-known result asserts that if A is square, then $A^D = A^k(A^{2k+1})^\dagger A^k$ (see, e.g., [3]), which implies that all problems for Drazin inverses of square matrices can be transformed into the problems related to Moore-Penrose inverses of matrices.

The purpose of this paper is to examine commutativity of the Drazin inverse A^D of a matrix A with A^* and $A^\dagger$, such as, $A^*A^D = A^DA^*$, $A^\dagger A^D = A^DA^\dagger$, $AA^DA^* = A^*A^DA$ and so on. There have been many results in the literature related to commutativity of generalized inverses of matrices, one of the well-known results is concerning the commutativity equality $AA^\dagger = A^\dagger A$ for the Moore-Penrose inverse and EP matrix, see, e.g., [1–3, 6–8, 11, 12]. In addition, the commutativity equalities $A^*A^\dagger = A^\dagger A^*$, $AA^*A^\dagger A = AA^\dagger A^*A$, $A^kA^\dagger = A^\dagger A^k$ and

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so on were also studied, see [7, 11].

An effective method in the investigation of equalities for generalized inverses of matrices is to establish rank formulas associated with the corresponding matrix expressions. In [11], the author shows that

$$\begin{aligned} &\operatorname{rank}(AA^\dagger - A^\dagger A) = 2\operatorname{rank}[A,\ A^*] - 2\operatorname{rank}(A), \\ &\operatorname{rank}(A^kA^\dagger - A^\dagger A^k) = \operatorname{rank}\begin{bmatrix} A^k \\ A^* \end{bmatrix} + \operatorname{rank}[A^k,\ A^*] - 2\operatorname{rank}(A), \\ &\operatorname{rank}(A^*A^\dagger - A^\dagger A^*) = \operatorname{rank}(AA^*A^\dagger A - AA^\dagger A^*A) = \operatorname{rank}(AA^*A^2 - A^2A^*A) \end{aligned}$$

and so on. From the rank equalities one can immediately find necessary and sufficient conditions for the commutativity equalities $AA^\dagger = A^\dagger A$, $A^*A^\dagger = A^\dagger A^*$, $A^kA^\dagger = A^\dagger A^k$, $AA^*A^\dagger A = AA^\dagger A^*A$ and so on to hold. These results and the equality $A^D = A^k(A^{2k+1})^\dagger A^k$ motivate us to find various possible rank formulas for expressions that involve the Drazin inverse of a matrix and then use them to characterize the commutativity of the Drazin inverse of matrix A with A^*, $A^\dagger$ and so on.

The matrices considered in this paper are over the field $\mathbb{C}$ of complex numbers. For $A \in \mathbb{C}^{m\times n}$, we use A^*, $r(A)$ and $\mathcal{R}(A)$ to stand for the conjugate transpose, the rank and the range (column space) of A, respectively.

Lemma 1.1 [11]. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then*

$$r(D - CA^\dagger B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A)\,. \tag{1.1}$$

In particular, if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

$$r(D - CA^\dagger B) = r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A)\,. \tag{1.2}$$

Let

$$C = [C_1,\ C_2]\,, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Then (1.1) becomes

$$\begin{aligned} r(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2) = r&\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D \end{bmatrix} \\ &- r(A_1) - r(A_2)\,. \end{aligned} \tag{1.3}$$

In particular, if $\mathcal{R}(B_1) \subseteq \mathcal{R}(A_1)$, $\mathcal{R}(C_1^*) \subseteq \mathcal{R}(A_1^*)$, $\mathcal{R}(B_2) \subseteq \mathcal{R}(A_2)$ and $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(A_2^*)$, then

$$r(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2) = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} - r(A_1) - r(A_2)\,. \tag{1.4}$$

Lemma 1.2 [9] (Rank cancellation rules)**.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given, and suppose that $\mathcal{R}(AQ) = \mathcal{R}(A)$ and $\mathcal{R}[(PA)^*] = \mathcal{R}(A^*)$. Then*

$$r[AQ,\ B] = r[\,A,\ B\,], \qquad r\begin{bmatrix} PA \\ C \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix}.$$

In addition, we shall also use in the sequel the following several basic rank formulas.

Lemma 1.3 [11]**.** *Let $A \in \mathbb{C}^{m\times n}$ be given, $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices. Then $PA - AQ$ satisfies the rank equality*

$$r(PA - AQ) = r\begin{bmatrix} PA \\ Q \end{bmatrix} + r[AQ,\ P] - r(P) - r(Q)\,. \tag{1.6}$$

In particular, if P and Q are of the same size, then

$$r(P - Q) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[Q,\ P] - r(P) - r(Q)\,. \tag{1.7}$$

Notice that if a matrix A is idempotent, then so is A^*. Thus we also find from (1.6) and (1.7) that for an idempotent matrix A,

$$r(A - A^*) = r(AA^* - A^*A) = 2r[A,\ A^*] - 2r(A) \tag{1.8}$$

holds.

2. Main Results

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times m}$ with* $\mathrm{Ind}\,(A) = k$*. Then*

(a) $r(A^\dagger A^D - A^D A^\dagger) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k,\ A^*] - 2r(A).$

(b) $r(A^\dagger AA^D A - AA^D AA^\dagger) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k,\ A^*] - 2r(A).$

(c) $r(A^\dagger AA^D - A^D AA^\dagger) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k,\ A^*] - 2r(A).$

(d) $r(A^\dagger A^\# - A^\# A^\dagger) = 2r[A,\ A^*] - 2r(A)$, *if* $\mathrm{Ind}\,(A) = 1$.

(e) $r(A^\dagger AA^\# - A^\# AA^\dagger) = 2r[A,\ A^*] - 2r(A)$, *if* $\mathrm{Ind}\,(A) = 1$.

In particular,

(f) $A^\dagger A^D = A^D A^\dagger \Leftrightarrow A^\dagger(AA^D A) = (AA^D A)A^\dagger \Leftrightarrow A^\dagger(AA^D) = (A^D A)A^\dagger \Leftrightarrow \mathcal{R}(A^k) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{R}[(A^k)^*] \subseteq \mathcal{R}(A)$.

(g) $A^\dagger A^\# = A^\# A^\dagger \Leftrightarrow A^\dagger(AA^\#) = (AA^\#)A^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A)$, *i.e., A is EP.*

Proof. Applying (1.3), block Gaussian elimination, $\mathcal{R}(A^D) = \mathcal{R}(A^D A^*) = \mathcal{R}(A^k)$, and Lemma 1.2, we find that

$$\begin{aligned} r(A^\dagger A^D - A^D A^\dagger) &= r\begin{bmatrix} A^*AA^* & 0 & A^*A^D \\ 0 & -A^*AA^* & A^* \\ A^* & A^D A^* & 0 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} A^*AA^* & A^*A^D AA^* & A^*A^D \\ 0 & 0 & A^* \\ A^* & A^D A^* & 0 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} 0 & 0 & A^*A^D \\ 0 & 0 & A^* \\ A^* & A^D A^* & 0 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} A^D \\ A^* \end{bmatrix} + r[\, A^D, \ A^* \,] - 2r(A) \\ &= r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[\, A^k, \ A^* \,] - 2r(A), \end{aligned}$$

as required for (a). Note that $AA^D = A^D A$ and both $AA^\dagger$ and $A^\dagger A$ are idempotent. We get by (1.6), $\mathcal{R}(AA^D AA^\dagger) = \mathcal{R}(A^k)$, $\mathcal{R}(A^\dagger A) = \mathcal{R}(A^*)$, and Lemma 1.2 that

$$\begin{aligned} r(\, A^\dagger AA^D A - AA^D AA^\dagger \,) &= r\begin{bmatrix} A^\dagger AA^D A \\ AA^\dagger \end{bmatrix} + r[\, AA^D AA^\dagger, \ A^\dagger A \,] - 2r(A) \\ &= r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[\, A^k, \ A^* \,] - 2r(A). \end{aligned}$$

Similarly we can find (c). (d)–(g) are direct consequences of (a)–(c) of the theorem. □

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times m}$ with* $\operatorname{Ind}(A) = k$. *Then*

(a) $r[(AA^\dagger)A^D - A^D(AA^\dagger)] = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} - r(A)$.

(b) $r[(A^\dagger A)A^D - A^D(A^\dagger A)] = r[A^k, \ A^*] - r(A)$.

(c) $r(A^\dagger A^D - A^D A^\dagger) = r[(AA^\dagger)A^D - A^D(AA^\dagger)] + r[(A^\dagger A)A^D - A^D(A^\dagger A)]$.

(d) $r[(AA^\dagger)A^\# - A^\#(AA^\dagger)] = r[(A^\dagger A)A^\# - A^\#(A^\dagger A)] = r[A, \ A^*] - r(A)$, *if* $\operatorname{Ind}(A) = 1$.

(e) A^D *commutes with* $AA^\dagger \Leftrightarrow \mathcal{R}[(A^k)^*] \subseteq \mathcal{R}(A)$.

(f) A^D *commutes with* $A^\dagger A \Leftrightarrow \mathcal{R}(A^k) \subseteq \mathcal{R}(A^*)$.

(g) $A^\dagger A^D = A^D A^\dagger \Leftrightarrow (AA^\dagger)A^D = A^D(AA^\dagger)$ *and* $(A^\dagger A)A^D = A^D(A^\dagger A) \Leftrightarrow \mathcal{R}(A^k) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{R}(A^k) \subseteq \mathcal{R}(A^*)$.

(h) $A^\dagger A^\# = A^\# A^\dagger \Leftrightarrow A^\#(AA^\dagger) = (AA^\dagger)A^\# \Leftrightarrow A^\#(A^\dagger A) = (A^\dagger A)A^\# \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A)$, *i.e., A is EP.*

Proof. Note that both $AA^\dagger$ and $A^\dagger A$ are idempotent. Thus (a) and (b) can easily be established through (1.6). Contrasting (a) and (b) with Theorem 2.9(a) yields (c). (d)–(h) are direct consequences of (a), (b) and (c) of the theorem. □

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times m}$ with* $\operatorname{Ind}(A) = k$. *Then*

(a) $r(A^*A^D - A^DA^*) = r\begin{bmatrix} A^k(AA^* - A^*A)A^k & 0 & A^kA^* \\ 0 & 0 & A^k \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k).$

(b) $r(A^*A^\# - A^\#A^*) = r\begin{bmatrix} A(AA^* - A^*A)A & 0 & AA^* \\ 0 & 0 & A \\ A^*A & A & 0 \end{bmatrix} - 2r(A),$

if $\mathrm{Ind}\,(A) = 1$.

(c) $r(A^*A^D - A^DA^*) = r(A^{k+1}A^*A^k - A^kA^*A^{k+1})$, *if* $\mathcal{R}(A^*A^k) \subseteq \mathcal{R}(A^k)$ *and* $\mathcal{R}[A(A^k)^*] \subseteq \mathcal{R}[(A^k)^*]$.

(d) $r(A^*A^D - A^DA^*) = r\begin{bmatrix} A^kA^* \\ A^k \end{bmatrix} + r[A^k,\ A^*A^k] - 2r(A^k)$, *if* $A^{k+1}A^*A^k = A^kA^*A^{k+1}$.

(e) $A^*A^D = A^DA^* \Leftrightarrow \mathcal{R}(A^*A^k) \subseteq \mathcal{R}(A^k)$, $\mathcal{R}[A(A^k)^*] \subseteq \mathcal{R}[(A^k)^*]$ *and* $A^{k+1}A^*A^k = A^kA^*A^{k+1}$.

(f) $r(A^*A^\# - A^\#A^*) = r(A^2A^*A - AA^*A^2)$, *if* $\mathcal{R}(A^*) = \mathcal{R}(A)$.

(g) $A^*A^\# = A^\#A^* \Leftrightarrow A^2A^*A = AA^*A^2$ *and* $\mathcal{R}(A^*) = \mathcal{R}(A)$.

Proof. Applying (1.4) and then block Gaussian elimination to $A^*A^D - A^DA^*$ yields

$$\begin{aligned} r(A^*A^D - A^DA^*) &= r[A^*A^k(A^{2k+1})^\dagger A^k - A^k(A^{2k+1})^\dagger A^kA^*] \\ &= r\begin{bmatrix} -A^{2k+1} & 0 & A^k \\ 0 & A^{2k+1} & A^kA^* \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^{2k+1}) \\ &= r\begin{bmatrix} -A^{2k+1} & 0 & A^k \\ -A^{k+1}A^*A^k & 0 & A^kA^* \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k) \\ &= r\begin{bmatrix} 0 & 0 & A^k \\ A^kA^*A^{k+1} - A^{k+1}A^*A^k & 0 & A^kA^* \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k) \\ &= r\begin{bmatrix} A^k(AA^* - A^*A)A^k & 0 & A^kA^* \\ 0 & 0 & A^k \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k)\,, \end{aligned}$$

as required for (a) of the theorem. (b), (c) and (d) are special cases of (a). (e), (f) and (g) follow from (a) and (b) of the theorem. □

Similarly we can also establish the following four theorems, which proofs are omitted.

Theorem 2.4. *Let* $A \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = k$. *Then*

(a) $r(AA^*A^D - A^DA^*A) = r\begin{bmatrix} A^k(A^2A^* - A^*A^2)A^k & 0 & A^kA^*A \\ 0 & 0 & A^k \\ AA^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k).$

(b) $r(A^kA^*A^D - A^DA^*A^k) = r(A^{2k+1}A^*A^k - A^kA^*A^{2k+1})$.

(c) $r(AA^*A^\# - A^\#A^*A) = r(A^3A^*A - AA^*A^3)$, *if* $\mathrm{Ind}\,(A) = 1$.

(d) $AA^*A^D = A^DA^*A \Leftrightarrow \mathcal{R}(AA^*A^k) = \mathcal{R}(A^k),\ \mathcal{R}[(A^kA^*A)^*] = \mathcal{R}[(A^k)^*]$ *and* $A^{k+2}A^*A^k = A^kA^*A^{k+2}$.

(e) $A^kA^*A^D = A^DA^*A^k \Leftrightarrow A^{k+1}(A^kA^*A^k) = (A^kA^*A^k)A^{k+1}$.

(f) $AA^*A^{\#} = A^{\#}A^*A \Leftrightarrow A^3A^*A = AA^*A^3$.

Theorem 2.5. *Let* $A \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = k$. *Then*

(a) $r(AA^DA^* - A^*A^DA) = r\begin{bmatrix} A^k \\ A^kA^* \end{bmatrix} + r[A^k,\ A^*A^k] - 2r(A^k)$.

(b) $r(AA^{\#}A^* - A^*A^{\#}A) = 2r[A,\ A^*] - 2r(A)$, *if* $\mathrm{Ind}\,(A) = 1$.

(c) $AA^DA^* = A^*A^DA \Leftrightarrow A(A^k)^\dagger = (A^k)^\dagger A \Leftrightarrow \mathcal{R}(A^*A^k) = \mathcal{R}(A^k)$ *and* $\mathcal{R}[A(A^k)^*] = \mathcal{R}[(A^k)^*]$.

(d) $AA^{\#}A^* = A^*A^{\#}A \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A)$.

Theorem 2.6. *Let* $A \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = k$. *Then*

(a) $r[AA^D(A^*)^k - (A^*)^kA^DA] = 2r[A^k,\ (A^k)^*] - 2r(A^k)$.

(b) $AA^D(A^*)^k = (A^*)^kA^DA \Leftrightarrow \mathcal{R}[(A^k)^*] = \mathcal{R}(A^k)$.

Theorem 2.7. *Let* $A \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = 1$ *and* λ *is a nonzero complex number. Then*

(a) $r[AA^{\#}(AA^* + \lambda A^*A) - (AA^* + \lambda A^*A)A^{\#}A] = 2r[A,\ A^*] - 2r(A)$.

(b) $AA^{\#}$ *commutes with* $AA^* + \lambda A^*A \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A)$, *i.e.,* A *is EP.*

Theorem 2.8. *Let* $A \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = k$. *Then*

(a) $r[(AA^D)^*A^\dagger - A^\dagger(AA^D)^*] = r\begin{bmatrix} A^kA^*A \\ A^k \end{bmatrix} + r[AA^*A^k,\ A^k] - 2r(A^k)$.

(b) $(AA^D)^*A^\dagger = A^\dagger(AA^D)^* \Leftrightarrow \mathcal{R}(AA^*A^k) = \mathcal{R}(A^k)$ *and* $\mathcal{R}[(A^kA^*A)^*] = \mathcal{R}[(A^k)^*]$.

(c) $(AA^{\#})^*A^\dagger = A^\dagger(AA^{\#})^*$, *if* $\mathrm{Ind}\,(A) = 1$.

Proof. Apply (1.6) and Lemma 1.2 to $(AA^D)^*A^\dagger - A^\dagger(AA^D)^*$ to yield

$$\begin{aligned} r[(AA^D)^*A^\dagger - A^\dagger(AA^D)^*] &= r\begin{bmatrix} (AA^D)^*A^\dagger \\ (AA^D)^* \end{bmatrix} + r[A^\dagger(AA^D)^*, (AA^D)^*] - 2r(AA^D) \\ &= r\begin{bmatrix} (A^k)^*A^\dagger \\ (A^k)^* \end{bmatrix} + r[A^\dagger(A^k)^*, (A^k)^*] - 2r(A^k)\,. \end{aligned}$$

Next applying (1.1), we can also find that

$$r\begin{bmatrix} (A^k)^*A^\dagger \\ (A^k)^* \end{bmatrix} = r[AA^*A^k,\ A^k] \quad \text{and} \quad r[A^\dagger(A^k)^*,\ (A^k)^*] = r\begin{bmatrix} A^kA^*A \\ A^k \end{bmatrix}.$$

Thus we get (a) and then (b) and (c) of the theorem. □

Theorem 2.9. *Let* $A \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = k$. *Then*

(a) $r[AA^D - (AA^D)^*] = 2r[A^k,\ (A^k)^*] - 2r(A^k)$.

(b) $r[(AA^D)(AA^D)^* - (AA^D)^*(AA^D)] = 2r[A^k,\ (A^k)^*] - 2r(A^k)$.

(c) $r[AA^{\#} - (AA^{\#})^*] = r[(AA^{\#})(AA^{\#})^* - (AA^{\#})^*(AA^{\#})] = 2r[A,\ A^*] - 2r(A)$, *if* $\mathrm{Ind}\,(A) = 1$.

(d) $AA^D = (AA^D)^* \Leftrightarrow (AA^D)(AA^D)^* = (AA^D)^*(AA^D) \Leftrightarrow \mathcal{R}(A^k) = \mathcal{R}[(A^k)^*]$, *i.e.,* A^k *is EP.*

(e) $AA^{\#} = (AA^{\#})^* \Leftrightarrow (AA^{\#})(AA^{\#})^* = (AA^{\#})^*(AA^{\#}) \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(A)$, *i.e., A is EP.*

Proof. Note that both AA^D and $(AA^D)^*$ are idempotent. It follows from (1.7) that

$$\begin{aligned} r[AA^D - (AA^D)^*] &= r\begin{bmatrix} AA^D \\ (AA^D)^* \end{bmatrix} + r[AA^D,\ (AA^D)^*] - r(AA^D) - r[(AA^D)^*] \\ &= 2r[AA^D,\ (AA^D)^*] - 2r(A^D) \\ &= 2r[A^k,\ (A^k)^*] - 2r(A^k)\,, \end{aligned}$$

as required for (a). The rank equality in (b) follows from (a) and (1.8). The results in (c), (d) and (e) follow immediately from (a) of the theorem. □

Finally we present a rank equality for the difference of $AA^D - BB^D$.

Theorem 2.10. *Let* $A,\ B \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}\,(A) = k$ *and* $\mathrm{Ind}\,(B) = l$. *Then*

(a) $r(AA^D - BB^D) = r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[\ A^k,\ B^l] - r(A^k) - r(B^l)$.

(b) $r(AA^{\#} - BB^{\#}) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] - r(A) - r(B)$, *if* $\mathrm{Ind}\,(A) = \mathrm{Ind}\,(B) = 1$.

(c) $AA^D = BB^D \Leftrightarrow \mathcal{R}(A^k) = \mathcal{R}(B^l)$ *and* $\mathcal{R}[(A^k)^*] = \mathcal{R}[(B^l)^*]$.

(d) $AA^{\#} = BB^{\#} \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(B)$ *and* $\mathcal{R}(A^*) = \mathcal{R}(B^*)$.

(e) *In particular, if* $\mathrm{Ind}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = 1$, *then*

$$r\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{\#} - \begin{bmatrix} AA^{\#} & 0 \\ 0 & DD^{\#} \end{bmatrix}\right) = r[A,\ B] + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

Proof. Note that both AA^D and BB^D are idempotent, and $\mathcal{R}(AA^D) = \mathcal{R}(A^k)$, $\mathcal{R}[(AA^D)^*] = \mathcal{R}[(A^k)^*]$, $\mathcal{R}(BB^D) = \mathcal{R}(B^k)$ and $\mathcal{R}[(BB^D)^*] = \mathcal{R}[(B^k)^*]$. Then it follows from (1.7) that

$$\begin{aligned} r(\,AA^D - BB^D\,) &= r\begin{bmatrix} AA^D \\ BB^D \end{bmatrix} + r[AA^D,\ BB^D] - r(AA^D) - r(BB^D) \\ &= r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[A^k,\ B^l] - r(A^k) - r(B^l), \end{aligned}$$

as required for (a). The results in (b)–(e) follow immediately from (a) of the theorem. □

In a recent paper [5] by Castro, Koliha and Wei, some other equivalent statements for the equality $AA^D = BB^D$ to hold are presented, one of which is

$$AA^D = BB^D \Longleftrightarrow B^D - A^D = A^D(A-B)B^D\,. \tag{2.1}$$

In fact, noting that

$$\mathcal{R}(\,I_m - AA^D\,) \cap \mathcal{R}(A^D) = \{0\} \quad \text{and} \quad \mathcal{R}[(I_m - AA^D\,)^*] \cap \mathcal{R}[(A^D)^*] = \{0\},$$

and using the two rank formulas (cf. [9])

$$r[\,A,\ B\,] = r(A) + r[(I_m - AA^-)B] \quad\text{and}\quad r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r[C(I_m - A^-A)]\,,$$

we find that

$$\begin{aligned} r[B^D - A^D - A^D(A-B)B^D] &= r[(I_m - AA^D)B^D - A^D(I_m - BB^D)] \\ &= r[(I_m - AA^D)B^D] + r[A^D(I_m - BB^D)] \\ &= r\begin{bmatrix} A^D \\ B^D \end{bmatrix} + r[A^D,\ B^D] - r(A^D) - r(B^D) \\ &= r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[A^k,\ B^l] - r(A^k) - r(B^l) \\ &= r(AA^D - BB^D)\,. \end{aligned}$$

Thus the equivalence (2.1) follows.

Remarks. In this paper, we have presented a method for establishing rank formulas for matrix expressions that involve Drazin inverses of matrices. Using the rank formulas obtained, one can characterize various matrix equalities for Drazin inverses of matrices. Besides the results shown in the paper, one can also establish various rank formulas for the differences $(AB)^D - B^DA^D$, $(AB)^D - B^\dagger A^\dagger$, $(AB)^D - B^\dagger(A^\dagger ABB^\dagger)^DA^\dagger$, $(AB)^D - B^D(A^DABB^D)^DA^D$, $(ABC)^D - C^DB^DA^D$, $(ABC)^D - C^D(A^DABCC^D)^DA^D$, $(ABC)^D - C^\dagger B^DA^\dagger$ and so on, and then find from them necessary and sufficient conditions for the corresponding reverse order laws for products of Drazin inverses to hold. We shall present the corresponding results in another paper. In addition, it is also worth considering how to partially extend the work in the paper to Drazin inverses of bounded linear operators over a Banach space and elements in C*-algebras, some similar work can be found in [4, 5, 8].

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