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BANACH FUNCTION SPACES AND EXPONENTIAL INSTABILITY OF EVOLUTION FAMILIES

MIHAIL MEGAN, ADINA LUMINIŢA SASU AND BOGDAN SASU

Abstract. In this paper we give necessary and sufficient conditions for uniform exponential instability of evolution families in Banach spaces, in terms of Banach function spaces. Versions of some well-known theorems due to Datko, Neerven, Rolewicz and Zabczyk, are obtained for the case of uniform exponential instability of evolution families.

1. Introduction

In the last few years, a significant progress has been made in the theory of evolution equations with unbounded coefficients, in Banach spaces. A new and interesting idea has been presented by Neerven in [17], where he introduced the theory of Banach function spaces in the study of asymptotic behavior of C_0-semigroups. In fact, he proved that a C_0-semigroup is uniformly exponentially stable if and only if all its orbits lie in a certain Banach function space over $\mathbf{R}_+$. This approach is a natural reformulation of a very well-known result in the theory of differential equations, due to Datko (see [4]), which says that an evolution family $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$, on a Banach space X, is uniformly exponentially stable if and only if there is $p \in [1,\infty)$ such that for every $x \in X$ and every $s \geq 0$ the mapping $t \mapsto \Phi(t+s,s)x$ belongs to $L^p(\mathbf{R}_+, X)$ and all these orbits are uniformly bounded in $L^p(\mathbf{R}_+, X)$.

In this spirit, characterizations for uniform exponential stability of evolution families, have been presented in [6], where it is proved that an evolution family $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially stable if and only if the orbits $t \mapsto \Phi(t+s,s)x$ belong to a certain Banach function space in a uniform way, for every $x \in X$ and every $s \geq 0$. These results have been generalized in [9], for the more general case of linear skew-product semiflows, over locally compact spaces.

Recently, new concepts of exponential expansiveness and in particular, of exponential instability, have been introduced and characterized (see [7], [12], [13],

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[16]). The structure of the unstable space of a uniformly exponentially dichotomic linear skew-product flow allowed bounded input - bounded output characterizations for uniform exponential dichotomy in terms of the solvability of discrete and integral equations on certain Banach sequence spaces and Banach function spaces, respectively (see [1], [11], [14]).

The natural question arises whether the uniform exponential instability of evolution families can be also expressed in terms of Banach function spaces.

The purpose of this paper is to answer this question, having in view the idea of giving an unitary treatment for uniform exponential stability and uniform exponential instability, too. In the present paper, we shall give some theorems of characterization for uniform exponential instability of evolution families in Banach spaces, using Banach function spaces. It is important to point out that the Banach sequence spaces and Banach function spaces, which will be considered, satisfy the same properties as in our papers concerning uniform exponential stability. As particular cases for the results presented in what follows, we shall obtain the versions of some well-known theorems due to Datko, Neerven, Zabczyk and Rolewicz, for the case of uniform exponential instability.

2. Definitions and preliminary results

In this section we shall present some definitions, notations and results about Banach function spaces and evolution families.

2.1. **Banach function spaces.** Let (Ω, Σ, μ) be a positive σ - finite measure space. By $M(\mu)$ we denote the linear space of μ-measurable functions $f : \Omega \to \mathbf{C}$, identifying the functions which are equal μ - a.e.

Definition 2.1. *A Banach function norm* is a function $\varrho : M(\mu) \to [0, \infty]$ with the following properties:

(i) $\varrho(f) = 0$ if and only if $f = 0\, \mu$ - a.e.;
(ii) if $|f| \leq |g|\, \mu$ - a.e. then $\varrho(f) \leq \varrho(g)$;
(iii) $\varrho(af) = |a|\varrho(f)$, for all $a \in \mathbf{C}$ and all $f \in M(\mu)$ with $\varrho(f) < \infty$;
(iv) $\varrho(f + g) \leq \varrho(f) + \varrho(g)$, for all $f, g \in M(\mu)$.

Let $B = B_\varrho$ be the set defined by:

$$B := \{f \in M(\mu) : |f|_B := \varrho(f) < \infty\}.$$

It is easy to see that $(B, |\cdot|_B)$ is a linear space. If B is complete then B is called *Banach function space* over Ω.

Remark 2.1. B is an ideal in $M(\mu)$, i.e. if $|f| \leq |g|\, \mu$ - a.e. and $g \in B$ then also $f \in B$ and $|f|_B \leq |g|_B$.

Remark 2.2. If $f_n \to f$ in norm in B, then there exists a subsequence (f_{k_n}) converging to f pointwise (see [5]).

Let $(\Omega, \Sigma, \mu) = (\mathbf{R}_+, \mathcal{L}, m)$, where $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable sets $A \subset \mathbf{R}_+$ and m the Lebesgue measure. For a Banach function space over $\mathbf{R}_+$, we define

$$F_B : (0,\infty) \to \bar{\mathbf{R}}_+, \quad F_B(t) := \begin{cases} |\chi_{[0,t)}|_B\,, & \text{if } \chi_{[0,t)} \in B\,, \\ \infty\,, & \text{if } \chi_{[0,t)} \notin B\,. \end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of $[0,t)$. The function F_B is called *the fundamental function of the Banach function space B.*

In what follows, we shall denote by $\mathcal{B}(\mathbf{R}_+)$ the set of all Banach function spaces, with the property that $\lim\limits_{t\to\infty} F_B(t) = \infty$ and there exists $c > 0$ such that $|\chi_{[n,n+1)}|_B \geq c$, for all $n \in \mathbf{N}$.

A trivial example of Banach function space over $\mathbf{R}_+$ which belongs to $\mathcal{B}(\mathbf{R}_+)$ is $L^p(\mathbf{R}_+, \mathbf{C})$, with $1 \leq p < \infty$.

Similarly, let $(\Omega, \Sigma, \mu) = (\mathbf{N}, \mathcal{P}(\mathbf{N}), \mu_c)$, where $\mathbf{N}$ is the set of all natural numbers, μ_c is the countable measure and let B be a Banach function space over $\mathbf{N}$ (in this case B is called *Banach sequence space*). We define $F_B : \mathbf{N}^* = \mathbf{N} \setminus \{0\} \to \bar{\mathbf{R}}_+$ by

$$F_B(n) := \begin{cases} |\chi_{\{0,\dots,n-1\}}|_B\,, & \text{if } \chi_{\{0,\dots,n-1\}} \in B\,, \\ \infty\,, & \text{if } \chi_{\{0,\dots,n-1\}} \notin B\,. \end{cases}$$

F_B is called *the fundamental function of the Banach sequence space B.*

We shall denote by $\mathcal{B}(\mathbf{N})$ the set of all Banach sequence spaces B with the properties $\lim\limits_{n\to\infty} F_B(n) = \infty$ and there exists $c > 0$ such that $|\chi_{\{n\}}|_B \geq c$, for all $n \in \mathbf{N}$.

Remark 2.3. If B is a Banach function space over $\mathbf{R}_+$, which belongs to $\mathcal{B}(\mathbf{R}_+)$, then

$$S_B := \{(\alpha_n)_n : \sum_{n=0}^{\infty} \alpha_n \chi_{[n,n+1)} \in B\}\,,$$

with respect to the norm

$$|(\alpha_n)_n|_{S_B} := \Big|\sum_{n=0}^{\infty} \alpha_n \chi_{[n,n+1)}\Big|_B$$

is a Banach sequence space which belongs to $\mathcal{B}(\mathbf{N})$.

Indeed, this assertion follows by observing that

$$|\chi_{\{n\}}|_{S_B} = |\chi_{[n,n+1)}|_B \quad \text{and} \quad F_{S_B}(n+1) = F_B(n+1), \quad \forall n \in \mathbf{N}\,.$$

Example 2.1. If $p \in [1,\infty)$ then $B = l^p(\mathbf{N}, \mathbf{C})$ with

$$|s|_p = \left(\sum_{n=0}^{\infty} |s(n)|^p\right)^{\frac{1}{p}}\,,$$

is a Banach sequence space which belongs to $\mathcal{B}(\mathbf{N})$.

Example 2.2. (*Orlicz sequence spaces*) Let $g : \mathbf{R}_+ \to \bar{\mathbf{R}}_+$ be a non-decreasing, left continuous function, which is not identically 0 or ∞ on $(0,\infty)$. We define the function:

$$Y_g(t) = \int_0^t g(s)\,ds\,,$$

which is called *the Young function* associated to g.

For every $s : \mathbf{N} \to \mathbf{C}$ we consider

$$M_g(s) := \sum_{n=0}^{\infty} Y_g(|s(n)|)\,.$$

The set O_g of all sequences with the property that there exists $k > 0$ such that $M_g(ks) < \infty$, is easily checked to be a linear space. With respect to the norm

$$|s|_g := \inf\,\{k > 0 : M_g(\frac{1}{k}s) \leq 1\}\,,$$

it is a Banach sequence space called *Orlicz sequence space.* Trivial examples of Orlicz sequence spaces are $l^p(\mathbf{N}, \mathbf{C}), 1 \leq p \leq \infty$, which are obtained for

$$g(t) = p\,t^{p-1}\,,\ \text{for } 1 \leq p < \infty \ \text{ and }\ g(t) = \begin{cases} 0\,, & 0 \leq t \leq 1 \\ \infty\,, & t > 1 \end{cases}\ ,\ \text{ for }\ p = \infty\,.$$

Remark 2.4. If $g : \mathbf{R}_+ \to \mathbf{R}_+$ is a non-decreasing left continuous function with $g(t) > 0$, for all $t > 0$ and $g(0) = 0$, then the Orlicz sequence space O_g, associated to g, belongs to $\mathcal{B}(\mathbf{N})$.

2.2. **Evolution families.** Let X be a real or complex Banach space. The norm on X and on the space $\mathcal{L}(X)$ of all bounded linear operators on X will be denoted by $\|\cdot\|$.

Definition 2.2. A family $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on X is called *evolution family* if the following properties are satisfied:

(i) $\Phi(t,t) = I$, the identity operator on X;

(ii) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;

(iii) there exist $M \geq 1, \omega > 0$ such that

$$\|\Phi(t,t_0)\| \leq M e^{\omega(t-t_0)}, \quad \forall t \geq t_0 \geq 0\,. \tag{2.1}$$

Definition 2.3. An evolution family $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ is said to be *uniformly exponentially unstable* if there are $N, \nu > 0$ such that

$$\|\Phi(t,t_0)x\| \geq N e^{\nu(t-t_0)}\|x\|\,, \quad \forall t \geq t_0 \geq 0, \forall x \in X\,.$$

Definition 2.4. An evolution family $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ is said to be

(i) *strongly measurable* if the mapping $t \mapsto \Phi(t,t_0)x \in X$ is measurable, for all $t_0 \geq 0$ and all $x \in X$;

(ii) *strongly continuous* if the mapping $t \mapsto \Phi(t,t_0)x \in X$ is continuous, for all $t_0 \geq 0$ and all $x \in X$;

(iii) *injective* if $\Phi(t,t_0)$ is an injective operator, for all $t \geq t_0 \geq 0$.

Remark 2.5. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup on the Banach space X then $\Phi_{\mathbf{T}} = \{T(t-s)\}_{t\geq s\geq 0}$ is a strongly continuous evolution family called *the evolution family associated to* $\mathbf{T}$. In this context, $\mathbf{T}$ is uniformly exponentially unstable if there are $N, \nu > 0$ such that $\|T(t)x\| \geq Ne^{\nu t}\|x\|$, for all $t \geq 0$ and all $x \in X$. Similarly, $\mathbf{T}$ is injective if $T(t)$ is injective, for all $t \geq 0$.

Proposition 2.1. *Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be an evolution family on the Banach space X. If there are $t_0 > 0$ and $\delta > 1$ such that*

$$\|\Phi(s+t_0, s)x\| \geq \delta\|x\|\,, \quad \forall (s,x) \in \mathbf{R}_+ \times X\,,$$

then Φ is uniformly exponentially unstable.

Proof. Let $M \geq 1$, $\omega > 0$ given by (2.1) and $\nu > 0$ such that $\delta = e^{\nu t_0}$.

Let $t \geq 0$ and $k \in \mathbf{N}$, $r \in [0, t_0)$ such that $t = kt_0 + r$. If $x \in X$ and $s \geq 0$, then using the hypothesis, it follows that

$$\delta^{k+1}\|x\| \leq \|\Phi(s + (k+1)t_0, s)x\| \leq Me^{\omega t_0}\|\Phi(s+t,s)x\|\,.$$

Denoting by $N = 1/Me^{\omega t_0}$, we deduce from above that

$$\|\Phi(s+t,s)x\| \geq Ne^{\nu t}\|x\|\,, \quad \forall (s,t,x) \in \mathbf{R}_+^2 \times X\,.$$

So Φ is uniformly exponentially unstable. □

3. Uniform exponential instability in terms of Banach function spaces

In this section we shall give necessary and sufficient conditions for uniform exponential instability of evolution families and we shall present some consequences for the case of C_0-semigroups of linear operators.

Let X be a Banach space and $C = \{x \in X : \|x\| = 1\}$.

Theorem 3.1. *Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be an evolution family on the Banach space X. Then Φ is uniformly exponentially unstable if and only if it is injective and there are a Banach sequence space $B \in \mathcal{B}(\mathbf{N})$ and a constant $K > 0$ such that for every $(s,x) \in \mathbf{R}_+ \times C$ the sequence*

$$\varphi_{s,x} : \mathbf{N} \to \mathbf{R}_+, \quad \varphi_{s,x}(n) = \frac{1}{\|\Phi(s+n,s)x\|}$$

belongs to B and $|\varphi_{s,x}|_B \leq K$, for all $(s,x) \in \mathbf{R}_+ \times C$.

Proof. *Necessity.* It is immediate for $B = l^1(\mathbf{N}, \mathbf{C})$.
Sufficiency. Let $c > 0$ such that $|\chi_{\{n\}}|_B \geq c$, for all $n \in \mathbf{N}$. Since $\varphi_{s,x}(n)\chi_{\{n\}} \leq \varphi_{s,x}$, using the hypothesis, it follows that

$$\frac{1}{\|\Phi(s+n,s)x\|} \leq \frac{K}{c}\,, \quad \forall (s,x,n) \in \mathbf{R}_+ \times C \times \mathbf{N}\,. \tag{3.1}$$

Let $n_0 \in \mathbf{N}^*$ such that $F_B(n_0) > 2K^2/c$. For every $i \in \{0, \ldots, n_0 - 1\}$, using relation (3.1), we deduce that

$$\frac{\|\Phi(s+i,s)x\|}{\|\Phi(s+n_0, s+i)\Phi(s+i,s)x\|} = \frac{\|\Phi(s+i,s)x\|}{\|\Phi(s+n_0,s)x\|} \leq \frac{K}{c}\,,$$

so
$$\frac{1}{\|\Phi(s+n_0,s)x\|} \leq \frac{K}{c}\frac{1}{\|\Phi(s+i,s)x\|}\,,$$
for every $i \in \{0,\dots,n_0-1\}$ and every $(s,x) \in \mathbf{R}_+ \times C$. It follows that
$$\frac{1}{\|\Phi(s+n_0,s)x\|}\,\chi_{\{0,\dots,n_0-1\}} \leq \frac{K}{c}\,\varphi_{s,x}\,,$$
and hence
$$\frac{1}{\|\Phi(s+n_0,s)x\|}\,F_B(n_0) \leq \frac{K}{c}\,|\varphi_{s,x}|_B \leq \frac{K^2}{c}\,.$$
Taking into account the way how n_0 was chosen, we finally have
$$\|\Phi(s+n_0,s)x\| \geq 2\,, \quad \forall (s,x) \in \mathbf{R}_+ \times C\,.$$

So, from Proposition 2.1. we conclude that Φ is uniformly exponentially unstable. □

As a consequence of the theorem from above and Remark 2.5. we obtain:

Corollary 3.1. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on the Banach space X. Then $\mathbf{T}$ is uniformly exponentially unstable if and only if it is injective and there are a Banach sequence space $B \in \mathcal{B}(\mathbf{N})$ and a constant $K > 0$ such that for every $x \in C$ the sequence*
$$s_x : \mathbf{N} \to \mathbf{R}_+, \quad s_x(n) = \frac{1}{\|T(n)x\|}$$
belongs to B and $|s_x|_B \leq K$, for all $x \in C$.

A characterization for uniform exponential instability, in terms of Banach function spaces, is given by

Theorem 3.2. *Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be a strongly measurable evolution family on the Banach space X. Then Φ is uniformly exponentially unstable if and only if it is injective and there are a Banach function space $B \in \mathcal{B}(\mathbf{R}_+)$ and a constant $K > 0$ such that for every $(s,x) \in \mathbf{R}_+ \times C$ the function*
$$\psi_{s,x} : \mathbf{R}_+ \to \mathbf{R}_+\,, \quad \psi_{s,x}(t) = \frac{1}{\|\Phi(s+t,s)x\|}$$
belongs to B and $|\psi_{s,x}|_B \leq K$, for all $(s,x) \in \mathbf{R}_+ \times C$.

Proof. *Necessity.* It follows for $B = L^1(\mathbf{R}_+, \mathbf{C})$.
Sufficiency. Let S_B be the Banach sequence space associated to B according to Remark 2.3. Let $(s,x) \in \mathbf{R}_+ \times C$ and
$$\varphi_{s,x} : \mathbf{N} \to \mathbf{R}_+\,, \quad \varphi_{s,x}(n) = \frac{1}{\|\Phi(s+n,s)x\|}\,.$$
If M, ω are given by (2.1), then for every $n \in \mathbf{N}$ and every $t \in [n, n+1)$, we have
$$\|\Phi(s+t,s)x\| \leq Me^{\omega}\|\Phi(s+n,s)x\|\,,$$
so,
$$\varphi_{s,x}(n)\chi_{[n,n+1)}(t) \leq Me^{\omega}\psi_{s,x}(t)\,, \quad \forall n \in \mathbf{N}\,, \quad \forall t \in [n,n+1)\,.$$

Thus, we deduce that

$$\sum_{n=0}^{\infty} \varphi_{s,x}(n)\, \chi_{[n,n+1)} \le Me^{\omega} \psi_{s,x}\,.$$

Using the hypothesis, it follows that $\varphi_{s,x} \in S_B$ and

$$|\varphi_{s,x}|_{S_B} \le Me^{\omega} |\psi_{s,x}|_B \le Me^{\omega} K\,, \quad \forall (s,x) \in \mathbf{R}_+ \times C\,.$$

Hence, from Theorem 3.1. we conclude that Φ is uniformly exponentially unstable. □

As an immediate consequence of the theorem from above, we obtain a version of Datko's theorem (see [4]), for the case of uniform exponential instability, given by:

Corollary 3.2. *Let $p \in [1,\infty)$ and let $\Phi = \{\Phi(t,s)\}_{t\ge s\ge 0}$ be a strongly measurable evolution family on the Banach space X. Then Φ is uniformly exponentially unstable if and only if it is injective and there exists $K > 0$ such that*

$$\int_0^{\infty} \frac{1}{\|\Phi(s+t,s)x\|^p}\, dt \le K\,, \quad \forall (s,x) \in \mathbf{R}_+ \times C\,.$$

Proof. *Necessity* is trivial.
Sufficiency. It follows from Theorem 3.2., by taking $B = L^p(\mathbf{R}_+, \mathbf{C})$. □

A theorem of Neerven type for uniform exponential instability of C_0-semigroups can be formulated as follows:

Corollary 3.3. *Let $\mathbf{T} = \{T(t)\}_{t\ge 0}$ be a C_0-semigroup on the Banach space X. Then $\mathbf{T}$ is uniformly exponentially unstable if and only if it is injective and there are a Banach function space $B \in \mathcal{B}(\mathbf{R}_+)$ and a constant $K > 0$ such that for every $x \in C$ the mapping*

$$f_x : \mathbf{R}_+ \to \mathbf{R}_+\,, \quad f_x(t) = \frac{1}{\|T(t)x\|}$$

belongs to B and $|f_x|_B \le K$, for all $x \in C$.

Theorem 3.3. *Let $\Phi = \{\Phi(t,s)\}_{t\ge s\ge 0}$ be an evolution family on the Banach space X. Then Φ is uniformly exponentially unstable if and only if it is injective and there are a non-decreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$, and a constant $K > 0$ such that*

$$\sum_{n=0}^{\infty} N\left(\frac{1}{\|\Phi(s+n,s)x\|}\right) \le K\,, \quad \forall (s,x) \in \mathbf{R}_+ \times C\,.$$

Proof. *Necessity.* It follows for $N(t) = t$, for all $t \ge 0$.
Sufficiency. Let $n_0 \in \mathbf{N}^*$ such that $K < n_0 N(1)$ and let M, ω given by (2.1).
Let $(s,x,n) \in \mathbf{R}_+ \times C \times \mathbf{N}$ and $i \in \{1,\ldots,n_0\}$. Then, we have

$$\|\Phi(s+n+i,s)x\| \le Me^{\omega n_0} \|\Phi(s+n,s)x\|\,,$$

so,

$$n_0 N\left(\frac{1}{Me^{\omega n_0}}\cdot\frac{1}{\|\Phi(s+n,s)x\|}\right)\leq\sum_{i=1}^{n_0}N\left(\frac{1}{\|\Phi(s+n+i,s)x\|}\right)<K\,.$$

Taking into account the way how n_0 was chosen and the fact that N is non-decreasing, we obtain that

$$\frac{1}{\|\Phi(s+n,s)x\|}\leq Me^{\omega n_0}\,,\quad\forall(s,x,n)\in\mathbf{R}_+\times C\times\mathbf{N}\,.$$

Without loss of generality, we may assume that N is left continuous - if not we can consider $\tilde{N}(t):=\lim\limits_{s\nearrow t}N(s)$ and the proof is unchanged.

Let $(O_N,|\cdot|_N)$ be the Orlicz sequence space associated to N and Y_N, the corresponding Young function. For $(s,x)\in\mathbf{R}_+\times C$ we consider the sequence

$$\varphi_{s,x}:\mathbf{N}\to\mathbf{R}_+\,,\quad\varphi_{s,x}(n)=\frac{1}{\|\Phi(s+n,s)x\|}\,.$$

Let $L=M(K+1)e^{\omega n_0}$. Thus, we have

$$Y_N\left(\frac{1}{L}\,\varphi_{s,x}(n)\right)\leq\frac{1}{L}\,\varphi_{s,x}(n)N\left(\frac{1}{L}\,\varphi_{s,x}(n)\right)\leq\frac{1}{K+1}\,N\left(\varphi_{s,x}(n)\right)\,,\quad\forall n\in\mathbf{N}\,.$$

It follows that $M_N(\frac{1}{L}\,\varphi_{s,x})<1$, so, $\varphi_{s,x}\in O_N$ and $|\varphi_{s,x}|_N\leq L$, for all $(s,x)\in\mathbf{R}_+\times C$. From Theorem 3.1. we conclude that Φ is uniformly exponentially unstable. □

Remark 3.1. Theorem 3.3. is the version of a theorem of Zabczyk (see [20]), for uniform exponential instability of evolution families.

Theorem 3.4. *Let $\Phi=\{\Phi(t,s)\}_{t\geq s\geq 0}$ be a strongly measurable evolution family on the Banach space X. Then Φ is uniformly exponentially unstable if and only if it is injective and there are a non-decreasing function $N:\mathbf{R}_+\to\mathbf{R}_+$ with $N(0)=0$ and $N(t)>0$, for all $t>0$, and a constant $K>0$ such that*

$$\int_0^\infty N\left(\frac{1}{\|\Phi(s+t,s)x\|}\right)dt\leq K\,,\quad\forall(s,x)\in\mathbf{R}_+\times C\,.$$

Proof. *Necessity.* It follows for $N(t)=t$, for all $t\geq 0$.

Sufficiency. Let $(s,x)\in\mathbf{R}_+\times C$. If M,ω are given by (2.1), then we have that

$$\|\Phi(s+t,s)x\|\leq Me^{\omega}\|\Phi(s+n,s)x\|\,,$$

for all $t\in[n,n+1)$ and $n\in\mathbf{N}$. It follows that

$$\sum_{n=0}^{\infty}N\left(\frac{1}{Me^{\omega}}\cdot\frac{1}{\|\Phi(s+n,s)x\|}\right)\leq\int_0^\infty N\left(\frac{1}{\|\Phi(s+t,s)x\|}\right)dt\leq K\,.$$

Considering the function

$$\tilde{N}:\mathbf{R}_+\to\mathbf{R}_+\,,\quad\tilde{N}(t)=N\left(\frac{1}{Me^{\omega}}t\right)\,,$$

we have that $\tilde{N}$ satisfies the conditions from Theorem 3.3. and

$$\sum_{n=0}^{\infty} \tilde{N}\left(\frac{1}{\|\Phi(s+n,s)x\|}\right) \leq K\,, \quad \forall (s,x) \in \mathbf{R}_+ \times C\,.$$

Finally, from Theorem 3.3. we conclude that Φ is uniformly exponentially unstable. □

Remark 3.2. Theorem 3.4. is the version of a well-known theorem due to Rolewicz (see [19]), for the case of uniform exponential instability of evolution families.

We end this section with a consequence of the last two theorems for the case of uniform exponential instability of C_0-semigroups.

Corollary 3.4. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on the Banach space X. Then the following assertions are equivalent:*

(i) $\mathbf{T}$ *is uniformly exponentially unstable;*

(ii) *there are a non-decreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$, and a constant $K > 0$ such that*

$$\sum_{n=0}^{\infty} N\left(\frac{1}{\|T(n)x\|}\right) \leq K\,, \quad \forall x \in C\,;$$

(iii) *there are a non-decreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$, and a constant $K > 0$ such that*

$$\int_0^{\infty} N\left(\frac{1}{\|T(t)x\|}\right) dt \leq K\,, \quad \forall x \in C\,.$$

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Department of Mathematics, Faculty of Mathematics and Computer Science
West University of Timişoara
Bul. V. Pârvan, Nr. 4, 1900 - Timişoara, Romania
E-mail: megan@hilbert.math.uvt.ro, sasu@hilbert.math.uvt.ro