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THE COMPLEX GEOMETRY OF AN INTEGRABLE SYSTEM

AHMED LESFARI

Abstract. In this paper, a finite dimensional algebraic completely integrable system is considered. We show that the intersection of levels of integrals completes into an abelian surface (a two dimensional complex algebraic torus) of polarization $(2,8)$ and that the flow of the system can be linearized on it.

1. Introduction

Consider a hamiltonian vector fields

$$\dot{z} = J\frac{\partial H}{\partial z}, \qquad z \in \mathbb{R}^n \,, \tag{1}$$

where $J = J(z)$ is a skew-symmetric matrix with polynomial entries in z, for which the corresponding Poisson bracket

$$\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial z}\,,\; J\frac{\partial H_j}{\partial z} \right\rangle ,$$

satisfies the Jacobi identity.

The system (1) with polynomial right hand side will be called *algebraic complete integrable* when:

a) the system is completely integrable with polynomial invariants, to be precise, the system possesses k polynomial $H_1, \dots, H_k$ (functions whose gradients $\partial H_i/\partial z$ are null vectors of J) and $m = (n-k)/2$ polynomial first integrals $H_{k+1} = H, \dots, H_{k+m}$ in involution ($\{H_i, H_j\} = 0$), which give rise to m commuting vector fields X_i generated by (1) applied to H_{k+i}, $1 \le i \le m$; for generic c_i, the invariant manifolds

$$\bigcap_{i=1}^{k+m} \{z \in \mathbb{R}^n : H_i\,(z) = c_i\} \ ,$$

are compact and connected and therefore there exists a diffeomorphism

$$\bigcap_{i=1}^{k+m} \{z \in \mathbb{R}^n : H_i\,(z) = c_i\} \longrightarrow \mathbb{R}^m/\text{Lattice} \quad \text{(real tori)} \ ,$$

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according to the Arnold-Liouville theorem [3, 18].

b) The invariant manifolds, thought of as affine varieties in $\mathbb{C}^m$ (non-compact)

$$A = \bigcap_{i=1}^{k+m} \{z \in \mathbb{C}^n : H_i(z) = c_i\} ,$$

can be completed into complex algebraic tori $\widetilde{A}$ (abelian varieties) as follows

$$A = \widetilde{A} \setminus \mathcal{D} ,$$

where $\mathcal{D}$ is a divisor in $\widetilde{A}$. In the natural coordinates $(t_1, \ldots, t_m)$ of $\widetilde{A} = \mathbb{C}^m/\text{Lattice}$ coming from $\mathbb{C}^m$, the functions $z_i = z_i(t_1, \ldots, t_m)$ are meromorphic and in particular (1) defines straight line motion on $\widetilde{A}$.

Adler and van Moerbeke [1] have developed and used the following algebraic complete integrability criterion: If the hamiltonian system (1) is algebraic complete integrable, then each z_i blows up for some value of $t \in \mathbb{C}$ and whenever it blows up, the solution $z(t)$ behaves as a Laurent series

$$z_i = t^{-k_i}\left(z_i^{(0)} + z_i^{(1)}t + z_i^{(2)}t^2 + \cdots\right) , \quad k_i \in \mathbb{Z} , \quad \text{some} \quad k_i > 0 ,$$

which admits dim(phase space)$-1 = n - 1$ free parameters.

In this paper we consider the following system of differential equations in the unknowns $z_1, \ldots, z_4$:

$$\begin{aligned} \dot{z}_1 &= z_3 , \\ \dot{z}_2 &= z_4 , \\ \dot{z}_3 &= \Omega z_1 - \Omega_0 z_2 - z_1^2 z_2 , \\ \dot{z}_4 &= \Omega z_2 - \Omega_0 z_1 - z_1 z_2^2 , \end{aligned} \tag{2}$$

where Ω_0 and Ω are two arbitrary constants. If $\Omega = \Omega_0 = 0$, this system coincide with the Yang-Mills system (see Appendix 1) describing a homogeneous two-component field having the gauge group $SU(2)$. The solutions of this system can also be related to the coupled nonlinear Schrödinger equations (see Appendix 2). The following two quartics are first integrals for this system

$$H_1 = \frac{1}{4}\Omega_0\left(z_1^2 + z_2^2\right) - \frac{1}{2}\Omega z_1 z_2 + \frac{1}{4}z_1^2 z_2^2 + \frac{1}{2}z_3 z_4 , \tag{3}$$

$$\begin{aligned} H_2 = &-\frac{1}{2}\left(\Omega_0^2 - \Omega^2\right) z_1 z_2 - \frac{1}{2}\Omega z_3 z_4 - \frac{1}{4}\Omega_0\left(z_3^2 + z_4^2\right) - \frac{1}{8}\Omega_0 z_1 z_2\left(z_1^2 + z_2^2\right) \\ &- \frac{1}{4}\Omega z_1^2 z_2^2 - \frac{1}{16}z_1^2 z_4^2 + \frac{1}{8}z_1 z_4 z_2 z_3 - \frac{1}{16}z_2^2 z_3^2 . \end{aligned} \tag{4}$$

The system (2) can be written in the form (1) with $n = 4$, $m = 2$, $k = 0$; to be precise

$$\dot{z} = f(z) = J\frac{\partial H}{\partial z} , \qquad z = (z_1, z_2, z_3, z_4)^\top ,$$

where $H = H_1$ (3),

$$\frac{\partial H}{\partial z} = \left(\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3} \frac{\partial H}{\partial z_4}\right)^\top , \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} ,$$

The second flow commuting with the first is regulated by the equations

$$\dot{z}= J\frac{\partial H_2}{\partial z}, \qquad z = (z_1, z_2, z_3, z_4)^\top$$

with H_2 defined by (4) and is written explicitly as

$$\begin{aligned}
\dot{z}_1 &= \frac{1}{2}\Omega z_4 + \frac{1}{2}\Omega_0 z_3 - \frac{1}{8} z_2 \left(z_1 z_4 - z_2 z_3\right) ,\\
\dot{z}_2 &= \frac{1}{2}\Omega z_3 + \frac{1}{2}\Omega_0 z_4 + \frac{1}{8} z_1 \left(z_1 z_4 - z_2 z_3\right) ,\\
\dot{z}_3 &= \frac{1}{8} z_2 \left(4\Omega^2 - 4\Omega_0^2 - 3\Omega_0 z_1^2 - \Omega_0 z_2^2 - 4\Omega z_1 z_2\right) - \frac{1}{8} z_4 \left(z_1 z_4 - z_2 z_3\right) ,\\
\dot{z}_4 &= \frac{1}{8} z_1 \left(4\Omega^2 - 4\Omega_0^2 - \Omega_0 z_1^2 - 3\Omega_0 z_2^2 - 4\Omega z_1 z_2\right) + \frac{1}{8} z_3 \left(z_1 z_4 - z_2 z_3\right) .
\end{aligned} \tag{5}$$

The invariant (or level) variety

$$A = \bigcap_{i=1}^{2} \{H_i(z) = c_i\} \subset \mathbb{C}^4 , \tag{6}$$

is a smooth affine surface for generic $c = (c_1, c_2) \in \mathbb{C}^2$. So, the question I address is how does one find the compactification of A into an Abelian surface? Now granted the system (2) is integrable, how does one effectively integrate the problem? The proof of the Liouville theorem concerning integrals in involution and invariant tori is non-constructive: neither does it enable you to decide about its integrability, nor does it provide means for integrating the problem. The idea of the direct proof we shall give here is closely related to the geometric spirit of the (real) Arnold-Liouville theorem [3, 18]. Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n-dimensional complex torus $\mathbb{C}^n/\text{Lattice}$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the main problem will be to complete $A\,(6)$ into a non singular compact complex algebraic variety $\widetilde{A} = A \cup \mathcal{D}$ in such a way that the vector fields (2) and (5) extend holomorphically along the divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{A}$ is an algebraic complex torus (an abelian variety) and the coordinates z_i restricted to A are abelian functions. A naive guess would be to take the natural compactification $\overline{A}$ of A by projectivizing the equations:

$$\overline{A} = \bigcap_{i=1}^{2} \left\{H_i(Z) = c_i Z_0^4\right\} \subset \mathbb{CP}^4 .$$

Indeed, this can never work for a general reason: an abelian variety $\widetilde{A}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be descriped in some projective space $\mathbb{CP}^n$ by n-dim $\widetilde{A}$ global polynomial homogeneous equations. In other words, if A is to be the affine part of an abelian surface, $\overline{A}$ must have a singularity somewhere along the locus at infinity $A_\infty = \overline{A} \cap \{X_0 = 0\}$.

In fact, we shall show that the existence of meromorphic solutions to the differential equations (2) depending on 3 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor.

2. Laurent expansions solutions and curves

Consider points at infinity which are limit points of trajectories of the flow. If the hamiltonian flow (1) is algebraic complete integrable, it means that the variables z_i are meromorphic on the torus $\mathbb{C}^m/\text{Lattice}$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset \mathbb{C}^m/\text{Lattice}$. By the algebraic complete integrability definition, the flow (1) is a straight line motion in $\mathbb{C}^m/\text{Lattice}$ and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equation must admit Laurent expansions which depend on the $m-1$ parameters defining $\mathcal{D}$ and the $m+k$ constants c_i defining the torus $\mathbb{C}^m/\text{Lattice}$ the total count is therefore $n-1 = \dim$ (phase space)-1 parameters.

Proposition 1. *The pole solutions of the system* (2) *restricted to the surface* A (6) *is a curve* $\mathcal{D}$ (8) *of geometric genus* 7*. Its smooth version is a hyperelliptic curve* $\mathcal{C}$ (10) *of genus* 5*, which is a double cover ramified at* 4 *points of a genus* 2 *hyperelliptic curve* $\mathcal{C}_0$ (11)*.*

Proof. The first fact to observe is that if the system (2) is to have Laurent expansions solutions depending on dim(phase space)$-1 = 3$ free parameters, the solutions must blow up like:

$$
\begin{aligned}
z_1 &= \frac{z_1^{(0)}}{t} + z_1^{(1)} + z_1^{(2)}t + z_1^{(3)}t^2 + z_1^{(4)}t^3 + \cdots \\
z_2 &= \frac{z_2^{(0)}}{t} + z_2^{(1)} + z_2^{(2)}t + z_2^{(3)}t^2 + z_2^{(4)}t^3 + \cdots \\
z_3 &= -\frac{z_1^{(0)}}{t^2} + z_1^{(2)} + 2z_1^{(3)}t + 3z_1^{(4)}t^2 + \cdots \\
z_4 &= -\frac{z_2^{(0)}}{t^2} + z_2^{(2)} + 2z_2^{(3)}t + 3z_2^{(4)}t^2 + \cdots
\end{aligned} \tag{7}
$$

this follows immediately from the differential equations (2). Putting (7) into (2), solving inductively for the $z^{(k)}$, one finds at the 0^{th} step a non-linear equation, and at the k^{th} step, a linear system of equations. One free parameter α appear at the 0^{th} step and the two remaining ones β, γ at the k^{th} step, $k=3$, $k=4$. More precisely, we find

$$
\begin{aligned}
z_1^{(0)} &= \alpha\,, & z_1^{(0)} z_2^{(0)} &= -2\,,\\
z_1^{(1)} &= 0\,, & z_2^{(1)} &= 0\,,\\
z_1^{(2)} &= -\frac{1}{6}\Omega\alpha + \frac{1}{3}\Omega_0 z_2 + \frac{1}{12}\alpha^3\Omega_0\,, & z_2^{(2)} &= -\frac{1}{6}\Omega z_2 + \frac{1}{3}\Omega_0\alpha + \frac{1}{12}z_2^3\Omega_0\,,\\
z_1^{(3)} &= \frac{1}{2}\alpha^2\beta\,, & z_2^{(3)} &= \beta\,,
\end{aligned}
$$

$$
z_1^{(4)} = \frac{1}{12}\Omega\Omega_0 z_2 + \frac{1}{24}\Omega\alpha^3\Omega_0 - \frac{1}{24}z_2^3\Omega_0^2 - \frac{1}{48}\alpha^5\Omega_0^2 - \frac{1}{2}\alpha^2\gamma\,, \qquad z_2^{(4)} = \gamma\,,
$$

Substituting the solutions (7) into $H_1 = c_1$ and $H_2 = c_2$, and equating the t°-terms yields

$$
\begin{aligned}
H_1 = \; & -\frac{5}{72}\alpha^4\Omega_0^2 + \frac{1}{9}\Omega\alpha^2\Omega_0 - \frac{17}{36}\Omega_0^2 - \frac{7}{36}\Omega^2 + \frac{5}{36}z_2^4\Omega_0^2 - \frac{11}{36}\Omega z_2^2\Omega_0 - 5\alpha\gamma\,,\\
H_2 = \; & -\frac{9}{4}\alpha^2\beta^2 - \frac{19}{576}\alpha^6\Omega_0^3 + \frac{17}{144}\Omega\alpha^4\Omega_0^2 - \frac{1}{8}\Omega_0^3\alpha^2 - \frac{11}{144}\Omega^2\alpha^2\Omega_0 + \frac{7}{36}\Omega^3\\
& - \frac{2}{9}\Omega\Omega_0^2 + \frac{11}{576}z_2^6\Omega_0^3 - \frac{1}{9}\Omega\Omega_0^2 z_2^4 + \frac{1}{12}\Omega_0^3 z_2^2 + \frac{25}{144}\Omega^2\Omega_0 z_2^2 + \frac{9}{8}\Omega_0 z_2\gamma\\
& - \frac{5}{4}\Omega_0\alpha^3\gamma + 3\Omega\alpha\gamma\,,
\end{aligned}
$$

with $z_1^{(0)} z_2^{(0)} = -2$. Eliminating γ from these equations, leads to the followings curve of geometric genus 7,

$$
\mathcal{D}:\; \alpha^8\beta^2 - P(\alpha) = 0\,, \tag{8}
$$

where

$$
\begin{aligned}
P(\alpha) \equiv \; & -\frac{1}{144}\Omega_0^3\alpha^{12} + \frac{7}{324}\Omega\Omega_0^2\alpha^{10} + \frac{1}{3240}\Omega_0\left(35\Omega_0^2 + 360H_1 + 56\Omega^2\right)\alpha^8\\
& - \frac{1}{405}\left(-14\Omega^3 + 180H_2 + 108\Omega H_1 + 45\Omega\Omega_0^2\right)\alpha^6\\
& - \frac{1}{1620}\Omega_0\left(7\Omega_0^2 - 35\Omega^2 - 324H_1\right)\alpha^4 + \frac{19}{405}\Omega_0^2\Omega\alpha^2 + \frac{8}{81}\Omega_0^3\,.
\end{aligned} \tag{9}
$$

Its smooth version is the following hyperelliptic curve of genus 5,

$$
\mathcal{C}: u^2 - P(\alpha) = 0\,. \tag{10}
$$

It is a double ramified cover of the genus 2 hyperelliptic curve,

$$
\mathcal{C}_0: v^2 - P(\xi) = 0\,, \tag{11}
$$

ramified at the four points covering $\xi = 0$ and ∞. The polynome $P(\xi)$ of degree 6 is given by (9) with $\xi \equiv \alpha^2$. This concludes the proof of Proposition 1. $\square$

3. Invariant surfaces as affine part of an abelian surface

We now wish to give a direct geometric proof that the surface $A\,(6)$ is an affine part of an abelian surface. Let $\widetilde{A}$ be a smooth surface compactifying A. Consider a basis $1, f_1, \ldots, f_N$ of the vector space

$$L \equiv L\,(\mathcal{C}) = \left\{ f : f \text{ meromorphic on } \widetilde{A}, (f) \geq -C \right\} ,$$

of meromorphic functions on $\widetilde{A}$ with at worst a simple pole along $\mathcal{C}$ and the map

$$\widetilde{A} \to \mathbb{CP}^N,\ p \longmapsto [1, f_1(p), \ldots, f_N(p)] ,$$

considered projectively, because if at p some $f_i(p) = \infty$, we divide by f_i having the highest order pole near p, which makes every element finite. The Kodaira embedding theorem tells us that if the line bundle associated with the divisor is positive, then for $k \in \mathbb{N}$, the functions of $L\,(k\mathcal{C})$ embed smoothly $\widetilde{A}$ into $\mathbb{CP}^N$ and then by Chow's theorem, $\widetilde{A}$ can be realized as an algebraic variety, i.e.,

$$\widetilde{A} = \bigcap_i \left\{ z \in \mathbb{CP}^N : P_i\,(z) = 0 \right\} ,$$

where $P_i\,(z)$ are homogeneous polynomials. In fact we shall show that the diviseur $2\mathcal{C}$ provides a smooth embedding into $\mathbb{CP}^{15}$, via the meromorphic section of $L\,(2\mathcal{C})$. Put $\mathcal{S} \equiv 2C$ and let

$$\chi\,(\mathcal{S}) = \dim H^0(\widetilde{A}, \mathcal{O}(\mathcal{S})) - \dim H^1\big(\widetilde{A}, \mathcal{O}(\mathcal{S})\big) ,$$

be the Euler characteristic of S. The adjunction formula and the Riemann-Roch theorem for divisors on abelian surfaces imply that

$$\begin{aligned} g\,(\mathcal{S}) &= \frac{K_{\widetilde{A}} \cdot \mathcal{S} + \mathcal{S} \cdot \mathcal{S}}{2} + 1\,, \\ \chi\,(\mathcal{S}) &= p_a(\widetilde{A}) + 1 + \frac{1}{2}\big(\mathcal{S} \cdot (\mathcal{S} - K_{\widetilde{A}})\big) \ , \end{aligned}$$

where $g\,(\mathcal{S})$ is the geometric genus of $\mathcal{S}$ and $p_a(\widetilde{A})$ is the arithmetic genus of $\widetilde{A}$. Since $\widetilde{A}$ is an abelian surface $\big(K_{\widetilde{A}} = 0, p_a(\widetilde{A}) = -1\big)$,

$$\begin{aligned} g\ \ (\mathcal{S}) - 1 &= \frac{\mathcal{S} \cdot \mathcal{S}}{2}\,, \\ &= \chi\,(\mathcal{S})\ . \end{aligned}$$

Using Kodaira-Serre duality [6, p. 153], Kodaira-Nakano vanishing theorem [6, p. 154] and a theorem on theta-functions [6, p. 317], it is easy to see that

$$\begin{aligned} g\,(\mathcal{S}) - 1 &= \dim L(\mathcal{S}) \quad \big(\equiv h^0\,(L)\big)\ , \\ &= N + 1\,, \\ &= \delta_1\delta_2\,, \end{aligned} \tag{12}$$

where $\delta_1, \delta_2 \in \mathbb{N}$, are the elementary divisors of the polarization of $\widetilde{A}$.

Proposition 2. *The orbits of the vector field* (2) *running through* $\mathcal{S} \equiv 2\mathcal{C}$ *form a smooth surface* Δ_p *near* $\mathcal{S}$. *Near* $p \in \mathcal{S}$, *the surface strip* Δ_p *coincides with* $\widetilde{A}$ *and* $\Delta_p \setminus \mathcal{S} \subseteq A$; Δ_p *is the only part of* $\widetilde{A}$ *in a small neighbourhood of* p. *Moreover, the variety*

$$A \cup \Big(\bigcup_{p \in \mathcal{S}} \Delta_p \Big) = \widetilde{A} = A \cup \mathcal{S}\,,$$

is smooth, compact, connected and embeds smoothly into $\mathbb{CP}^{15}$.

Proof. Based on the above motivation, it is easy to find a set of polynomial functions $\{1, f_1, \ldots, f_N\}$ in $L(\mathcal{S})$ such that the embedding of $\mathcal{S} \equiv 2\mathcal{C}$ with those functions into $\mathbb{CP}^N$ yields a curve of genus $N+2$. Straightforward calculation, using asymptotic expansions, shows that the space $L(\mathcal{S})$ is spanned by the following functions

$$L(\mathcal{S}) = \{1, f_1, \ldots, f_{15}\}\,,$$

where

$$\begin{aligned}
f_1 &= z_1 = \frac{\alpha}{t} + o(t)\ ,\\
f_2 &= z_2 = \frac{z_2^{(0)}}{t} + o(t)\ ,\\
f_3 &= z_3 = -\frac{\alpha}{t^2} + o(t)\ ,\\
f_4 &= z_4 = -\frac{z_2^{(0)}}{t^2} + o(t)\ ,\\
f_5 &= z_1^2 = \frac{\alpha^2}{t^2} + o(t)\ ,\\
f_6 &= z_2^2 = \frac{4}{\alpha^2 t^2} + o(t)\ ,\\
f_7 &= z_1 z_2 = -\frac{2}{t^2} + o(t)\ ,\\
f_8 &= z_1 z_4 - z_2 z_3 = \Omega_0\left(\alpha^2 - 2\right)\frac{\alpha^2+2}{\alpha^2 t} + o(t)\ ,\\
f_9 &= \left(z_1 z_4 - z_2 z_3\right)^2 = \Omega_0^2\left(\alpha^2 - 2\right)^2\frac{\left(\alpha^2+2\right)^2}{\alpha^4 t^2} + o(t)\ ,\\
f_{10} &= z_1\left(z_1 z_4 - z_2 z_3\right) = \Omega_0\left(\alpha^2 - 2\right)\frac{\alpha^2+2}{\alpha t^2} + o(t)\ ,\\
f_{11} &= z_2\left(z_1 z_4 - z_2 z_3\right) = -2\Omega_0\left(\alpha^2 - 2\right)\frac{\alpha^2+2}{\alpha^3 t^2} + o(t)\ ,\\
f_{12} &= z_3\left(z_1 z_4 - z_2 z_3\right) + \Omega_0 z_1\left(z_1^2 - z_2^2\right) = -6\frac{\alpha^2}{t^2}\beta + o(t)\ ,\\
f_{13} &= z_4\left(z_1 z_4 - z_2 z_3\right) + \Omega_0 z_2\left(z_1^2 - z_2^2\right) = \frac{12}{t^2}\beta + o(t)\ ,
\end{aligned}$$

$$\begin{aligned}
f_{14} &= \left(z_1^2 - z_2^2\right) z_1 z_2 + 2\left(z_3^2 - z_4^2\right) ,\\
&= \frac{1}{2}\left(-\Omega_0\alpha^4 + 4\Omega\alpha^2 - 4\Omega_0\right)\left(\alpha^2+2\right)\frac{\alpha^2-2}{t^2\alpha^4} + o\left(t\right) ,\\
f_{15} &= z_1 z_2\left(z_1 z_4 - z_2 z_3\right) - 2\Omega_0\left(z_1 z_3 - z_2 z_4\right) - 2\Omega\left(z_1 z_4 - z_2 z_3\right) ,\\
&= -\frac{12}{t^2}\alpha\beta + o\left(t\right) .
\end{aligned}$$

The map

$$\begin{aligned}
&\mathcal{S} \to \mathbb{CP}^{15} ,\\
&p \mapsto \lim_{t\to 0} t^2\left(1, f_1(p), \ldots, f_{15}(p)\right) ,\\
&\qquad = \left(0,0,0,f_3^{(0)}(p),\ldots,f_7^{(0)}(p),0,f_9^{(0)}(p),\ldots,f_{15}^{(0)}(p)\right) ,
\end{aligned}$$

maps the curve $\mathcal{S}$ into $\widetilde{\mathcal{S}} \subseteq \mathbb{CP}^{15}$ and the genus of $\mathcal{S}$ is 17, satisfying the requirement (12). Let $\Delta_p \subset \mathbb{CP}^{15}$ be the surface formed by the divisor $\mathcal{S}$ with fibres defined by the flow (for small t) departing from the points of $\mathcal{S}$ in a neighbourhood of the point $p \in \mathcal{S}$. Let $\mathcal{H} \subset \mathbb{CP}^{15}$ be a hyperplane transversal to the direction of the flow at p and consider the segment of the curve $\mathcal{S}'$ defined by $\mathcal{S}' = \mathcal{H} \cap \Delta_p$. If $\mathcal{S}'$ is smooth, then using the implicit function theorem the surface Δ_p is smooth. But if $\mathcal{S}'$ is singular at 0, then Δ_p would be singular along the trajectory (t-axis) which go immediately into the affine part A. Hence, A would be singular which is a contradiction because A is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ and so smooth for almost all the two constants of the motion c_i. Next, let $\overline{A}$ be the projective closure of A into $\mathbb{CP}^4$, let $Z = (Z_0, Z_1, Z_2, Z_3, Z_4) \in \mathbb{CP}^4$ and let $A_\infty = \overline{A} \cap \{Z_0 = 0\}$ be the locus at infinity. Consider the map

$$f : \overline{A} \subseteq \mathbb{CP}^4 \to \mathbb{CP}^{15},\ Z \mapsto f\left(Z\right) ,$$

where $f = (1, f_1, \ldots, f_{15}) \in L\left(\mathcal{S}\right)$ and let $\widetilde{A} = f(\overline{A})$. In a neighbourhood $V(p) \subseteq \mathbb{CP}^{15}$ of p, we have $\Delta_p = \widetilde{A}$ and $\Delta_p \setminus \mathcal{S} \subseteq A$; Δ_p is the only part of $\widetilde{A}$ in a small neighbourhood of p. Otherwise there would exist an element of surface $\Delta_p' \subseteq \widetilde{A}$ intersecting Δ_p at $p \in \mathcal{S}$. Let $g_{X_1}^t\left(p\right)$ be the orbit going with the vector field (2) through p. We have

$$\begin{aligned}
\text{orbit}\left\{g_{X_1}^t\left(p\right) ,\ 0 < |t| < \varepsilon\right\} &= \left(\Delta_p \cap \Delta_p'\right)\backslash A ,\\
\Delta_p \cap \Delta_p' &= t\text{-axis.}
\end{aligned}$$

Hence A would be singular along the t-axis which is impossible. Since the variety $\overline{A} \cap \{Z_0 \neq 0\}$ is irreductible and since the generic hyperplane section $\mathcal{H}_{\text{generic}}$ of $\overline{A}$ is also irreducible, all hyperplane sections are connected and hence A_∞ is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{CP}^4 \times \mathbb{CP}^{15}$ of the map f, which is irreducible together with $\overline{A}$. It follows from the irreducibility of A_∞ that a generic hyperplane section $\Gamma_f \cap \{\mathcal{H}_{\text{generic}} \times \mathbb{CP}^{15}\}$ is irreducible, hence the special hyperplane section $\Gamma_f \bigcap \{\{Z_0 = 0\} \times \mathbb{CP}^{15}\}$ is connected and therefore the projection map

$$\mathrm{proj}_{\mathbb{CP}^{15}}[\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{CP}^{15}\}] = f(A_\infty) \equiv \mathcal{S}\,,$$

is connected; a projection maintains connectivity by continuity. Hence, the variety

$$A \cup (\bigcup_{p\in\mathcal{S}} \Delta_p) = \widetilde{A} = A \cup \mathcal{S},$$

is compact, connected and embeds smoothly into $\mathbb{CP}^{15}$ via f. This completes the proof of Proposition 2. □

Proposition 3. *The variety $\widetilde{A}$ comes equipped with two everywhere independent commuting vector fields, which extend holomorphically on $\widetilde{A}$.*

Proof. Let g^{t_1} and g^{t_2} be the flows generated respectively by vector fields (2) and (5). For $p \in \mathcal{S}$ and for small $\varepsilon > 0$, $g^{t_1}(p)$, $\forall t_1$, $0 < |t_1| < \varepsilon$, is well defined and $g^{t_1}(p) \in A$. Then we may define g^{t_2} on A by

$$g^{t_2}(q) = g^{-t_1} g^{t_2} g^{t_1}(q),\ q \in U(p) = g^{-t_1}(U(g^{t_1}(p)))\,,$$

where $U(p)$ is a neighbourhood of p. By commutativity one can see that g^{t_2} is independent of t_1;

$$\begin{aligned} g^{-t_1-\varepsilon_1} g^{t_2} g^{t_1+\varepsilon_1}(q) &= g^{-t_1} g^{-\varepsilon_1} g^{t_2} g^{t_1} g^{\varepsilon_1}\,, \\ &= g^{-t_1} g^{t_2} g^{t_1}(q)\,. \end{aligned}$$

We affirm that $g^{t_2}(q)$ is holomorphic away from S. This because $g^{t_2}g^{t_1}(q)$ is holomorphic away from $\mathcal{S}$ and that g^{t_1} is holomorphic in $U(p)$ and maps biholomorphically $U(p)$ onto $U(g^{t_1}(p))$. This finishes Proposition 3. □

Proposition 4. *The affine surface A (6) defined by the intersection of two quartics completes into an abelian surface $\widetilde{A} \simeq$ a complex algebraic torus $\mathbb{C}^2$/period lattice. The system of differential equations (2) is algebraic complete integrable and the corresponding flow evolues on this torus.*

Proof. Since the flows g^{t_1} and g^{t_2} are holomorphic and independent on $\mathcal{S}$, we can show along the same lines as in the Arnold-Liouville theorem that $\widetilde{A}$ is a torus. And that will done, by considering the holomorphic map

$$\mathbb{C}^2 \longrightarrow \widetilde{A},\ (t_1, t_2) \longmapsto g^{t_1} g^{t_2}(p)\,,$$

for a fixed origin $p \in A$. Then

$$\text{lattice } = \{\ (t_1, t_2) \in \mathbb{C}^2 : g^{t_1} g^{t_2}(p) = p\}\,,$$

is a lattice of $\mathbb{C}^2$ and hence $\mathbb{C}^2/\text{lattice} \to \widetilde{A}$ is a biholomorphic diffeomorphism. Therefore $\widetilde{A} \subseteq \mathbb{CP}^{15}$ is conformal to a complex torus $\mathbb{C}^2$/lattice and an abelian surface as a consequence of Chow theorem. This establishes Proposition 4. □

4. Abelian surface of polarization (2,8), Prym and Jacobian varieties

We can go further and describe what abelian surfaces arise this way. The system (2) can then be solved by quadratures, that is to say their solutions can be expressed in terms of Abelian integrals. We make also a few comments about the relation between the abelian variety $\widetilde{A}$, the Prym variety $\operatorname{Prym}(\mathcal{C}/\mathcal{C}_0)$ and the Jacobian variety $\operatorname{Jac}(\mathcal{C})$.

Proposition 5. *The flow* (2) *evolues on abelian surfaces* $\widetilde{A} \subseteq \mathbb{CP}^{15}$ *of period matrix*

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 8 & c & b \end{pmatrix}, \qquad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0,$$

and it will be expressed in terms of abelian integrals, involving the differentials (15).

Proof. Note that the affine invariant surface A (6), has the following involution

$$\sigma : (z_1, z_2, z_3, z_4) \longmapsto (-z_1, -z_2, -z_3, -z_4),$$

which amounts to a reflection about some appropriately chosen origin on A. This map acts on the parameters of the Laurent solutions (7) as follows

$$(t, \alpha, \beta) \longmapsto (-t, -\alpha, \beta).$$

Since L is symmetric ($\sigma^* L \simeq L$), σ can be lifted to L as an involution $\widetilde{\sigma}$ in two ways differing in sign and for each section (theta-function) $s \in H^0(L)$, we therefore have $\widetilde{\sigma}s = \pm s$. Recall that a section $s \in H^0(L)$ is called even (resp. odd) if $\widetilde{\sigma}s = +s$ (resp. $\widetilde{\sigma}s = -s$). Under $\widetilde{\sigma}$ the vector space $H^0(L)$ splits into an even and odd subspace

$$H^0(L) = H^0(L)^{\text{even}} \oplus H^0(L)^{\text{odd}},$$

with $H^0(L)^{\text{even}}$ containing all the even sections and $H^0(L)^{\text{odd}}$ all odd ones. Using the inverse formula [21, p. 331], we see after a small computation that

$$\begin{aligned} h^0(L)^{\text{even}} &\equiv \dim H^0(L)^{\text{even}} = \begin{cases} \frac{\delta_1\delta_2}{2} + 2\left(1 + [\frac{\delta_2}{2}] - \frac{\delta_2}{2}\right) & \text{for even } \delta_1 \\ \frac{\delta_1\delta_2}{2} + \left(1 + [\frac{\delta_2}{2}] - \frac{\delta_2}{2}\right) & \text{for odd } \delta_1 \end{cases}, \\ h^0(L)^{\text{odd}} &\equiv \dim H^0(L)^{\text{odd}} = \begin{cases} \frac{\delta_1\delta_2}{2} - 2\left(1 + [\frac{\delta_2}{2}] - \frac{\delta_2}{2}\right) & \text{for even } \delta_1 \\ \frac{\delta_1\delta_2}{2} - \left(1 + [\frac{\delta_2}{2}] - \frac{\delta_2}{2}\right) & \text{for odd } \delta_1 \end{cases}. \end{aligned} \tag{13}$$

By the classification theory of ample line bundles on abelian varieties and Proposition 4, $\widetilde{A} \simeq \mathbb{C}^2/L_\Omega$ with period lattice given by the columns of the matrix

$$\Omega = \begin{pmatrix} \delta_1 & 0 & a & c \\ 0 & \delta_2 & c & b \end{pmatrix}, \qquad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} \rangle\ 0,$$

according to (12), with

$$\delta_1\delta_2 = h^0(L) = g\ (S) - 1 = 16, \qquad \delta_1 \mid \delta_2,\ \delta_i \in \mathbb{N}^*.$$

Hence we have the following possibilities: (i) $\delta_1 = 1$, $\delta_2 = 16$ and (ii) $\delta_1 = 2$, $\delta_2 = 8$. From formula (13), the corresponding line bundle L has in case (i), 9 even sections, 7 odd ones and in cases (ii) 10 even sections, 6 odd ones. Among the functions of L, there are 10 even and 6 odd functions for the involution σ, showing that cases (ii) is the only alternative and the period matrices have the form

$$\begin{pmatrix} 2 & 0 & a & c \\ 0 & 8 & c & b \end{pmatrix}, \qquad \operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0\,.$$

Consider on $\widetilde{A}$ the holomorphic 1-forms dt_1 and dt_2 defined by

$$dt_i\,(X_j) = \delta_{ij}\,,$$

where X_1 is the flow (2) and X_2 the other flow (5) commuting with the first. The holomorphic differentials on $\mathcal{C}$ can be spanned as well by

$$\omega_k = \frac{\alpha^{k-1} d\alpha}{\sqrt{P\,(\alpha)}}\,, \qquad 1 \le k \le 5\,. \tag{14}$$

The restriction of dt_1 and dt_2 to the curve $\mathcal{C}$ are easily computed:

$$\begin{aligned} dt_1\,|_{\mathcal{C}} &= \omega_2 = \frac{\alpha d\alpha}{\sqrt{P\,(\alpha)}}, \\ dt_2\,|_{\mathcal{C}} &= \omega_4 = \frac{\alpha^3 d\alpha}{\sqrt{P\,(\alpha)}}\,. \end{aligned} \tag{15}$$

This ends the proof of Proposition 5. □

Finally we make a few comments about the relation between the abelian variety $\widetilde{A}$, the Prym variety $\operatorname{Prym}(\mathcal{C}/\mathcal{C}_0)$ and the jacobian variety $\operatorname{Jac}(\mathcal{C})$. As pointed out before, $\mathcal{C}$ is a double ramified cover of a hyperelliptic curve $\mathcal{C}_0$ of genus 2, whose sheets are interchanged by the involution $(\alpha, u) \longmapsto (-\alpha, u)$. Hence

$$\operatorname{Jac}(\mathcal{C}) = \operatorname{Prym}(\mathcal{C}/\mathcal{C}_0) \oplus \operatorname{Jac}(\mathcal{C}_0)\ .$$

Since the space of holomorphic differentials splits according to odd and even differentials respectively (for that involution)

$$\Omega\,(\mathcal{C}) = \{\omega_1, \omega_3, \omega_5\} \oplus \{\omega_2, \omega_4\}\ ,$$

the flows evolve on the 3-dimensional $\operatorname{Prym}(\mathcal{C}/\mathcal{C}_0)$ and therefore

$$\widetilde{A} \subseteq \operatorname{Prym}(\mathcal{C}/\mathcal{C}_0)\ .$$

Its differentials restricted to the curve $\mathcal{C}$ are given by ω_1, ω_3 and ω_5. This shows that $\operatorname{Prym}(\mathcal{C}/\mathcal{C}_0)$ splits further, up to isogenies, into an elliptic curve $\mathcal{E}$ and the 3-dimensional invariant torus $\widetilde{A}$:

$$\operatorname{Prym}(\mathcal{C}/\mathcal{C}_0) = \widetilde{A} \oplus \mathcal{E}\,.$$

The torus $\widetilde{A}$ can also be regarded as a double cover of the Jacobi variety $\operatorname{Jac}(\mathcal{C}_0)$ and the system (2) can be integrated in terms of genus 2 hyperelliptic functions of time. The differentials dt_1 and dt_2, corresponding to the flows (2) and (5), restricted to the curve $\mathcal{C}$, go down to $\mathcal{C}_0$. Indeed, using (15)

$$dt_1 \mid_{\mathcal{C}} = \omega_2 = \frac{\alpha\, d\alpha}{\sqrt{P(\alpha)}} = \frac{d\xi}{\sqrt{P(\xi)}}\,,$$
$$dt_2 \mid_{C} = \omega_4 = \frac{\alpha^3\, d\alpha}{\sqrt{P(\alpha)}} = \frac{\xi\, d\xi}{\sqrt{P(\xi)}}\,,$$

yielding the two hyperelliptic differentials on $\mathcal{C}_0$.

Appendix 1. Consider the Yang-Mills system for a field with gauge group $SU(2)$:

$$D_k F_{kl} = \frac{\partial F_{kl}}{\partial x_k} + [A_k, F_{kl}] = 0\,,$$

where F_{kl}, $A_k \in T_e SU(2)$, $1 \leq k, l \leq 4$ and

$$F_{kl} = \frac{\partial A_l}{\partial x_k} - \frac{\partial A_k}{\partial x_l} + [A_k, A_l]\ .$$

In the case of homogeneous two-component field,

$$\frac{\partial A_l}{\partial x_k} = 0 \ \ (k \neq 1)\ ,\ A_1 = A_2 = 0\,,\ A_3 = n_1 U_1\,,\ A_4 = n_2 U_2\,,$$

where $n_{1,2}$ = constant, $n_1 = [n_2, [n_1, n_2]]$, $n_2 = [n_1, [n_2, n_1]]$ and the system becomes

$$\frac{\partial^2 U_1}{\partial t^2} + U_1 U_2^2 = 0\,,$$
$$\frac{\partial^2 U_2}{\partial t^2} + U_2 U_1^2 = 0\,,$$

with $t = x_1$. By setting

$$U_1 = \frac{1}{2}\left(\sqrt[4]{2}\right)^3 z_1\,, \qquad \frac{\partial U_1}{\partial t} = \frac{\sqrt{2}\,(1-i)}{4}\,(z_3 + i z_4)\ ,$$
$$U_2 = \frac{1}{2}\left(\sqrt[4]{2}\right)^3 z_2\,, \qquad \frac{\partial U_2}{\partial t} = \frac{\sqrt{2}\,(1+i)}{4}\,(z_3 - i z_4)\ .$$

Yang-Mills equations are reduced to hamiltonian system with the hamiltonian

$$H = \frac{1}{2} z_3 z_4 + \frac{1}{4} z_1^2 z_2^2\,,$$

which obviously coincides with (3) for $\Omega = \Omega_0 = 0$.

Appendix 2. It's well known [5] that the system of two coupled nonlinear Schrödinger equations is given by

$$i\frac{\partial A}{\partial Z} + \frac{1}{2}\frac{\partial^2 A}{\partial T^2} + \sigma B + \left(|A|^2 + \gamma\,|B|^2\right) A = 0\,,$$
$$i\frac{\partial B}{\partial Z} + \frac{1}{2}\frac{\partial^2 B}{\partial T^2} + \sigma A + \left(|B|^2 + \gamma\,|A|^2\right) B = 0\,.$$

Making the change of variables

$$A = \frac{\sqrt{2}}{2}(u+iv)\,, \qquad z = \frac{1}{2}(\gamma+1)\,Z\,,$$
$$B = \frac{\sqrt{2}}{2}(u-iv)\,, \qquad t = \sqrt{(\gamma+1)\,T}\,,$$

gives the system

$$i\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial t^2} + \Omega_0 u + \frac{2}{3}\left(|u|^2+|v|^2\right)u + \frac{1}{3}\left(u^2+v^2\right)\overline{u} = 0\,,$$
$$i\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial t^2} + \Omega_0 v + \frac{2}{3}\left(|u|^2+|v|^2\right)v + \frac{1}{3}\left(u^2+v^2\right)\overline{v} = 0\,,$$

where $\Omega_0 = \frac{2}{3}\sigma$. We seek solutions of these equations in the following form:

$$u(z,t) = \frac{1}{2}(z_1+z_2)(t)\exp(i\Omega z)\,,$$
$$v(z,t) = \frac{1}{2i}(z_1-z_2)\exp(i\Omega z)\,,$$

where $z_1(t)$ and $z_2(t)$ are two functions and Ω is an arbitrary constant. Then we obtain the system

$$\ddot{z}_1 = \Omega z_1 - \Omega_0 z_2 - z_1^2 z_2\,,$$
$$\ddot{z}_2 = \Omega z_2 - \Omega_0 z_1 - z_1 z_2^2\,,$$

with hamiltonian

$$H_1 = \frac{1}{4}\Omega_0\left(z_1^2+z_2^2\right) - \frac{1}{2}\Omega z_1 z_2 + \frac{1}{4}z_1^2 z_2^2 + \frac{1}{2}z_3 z_4\,,$$

which obviously coincides with (3).

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Department of Mathematics, Faculty of Sciences
University of Chouaïb Doukkali, B.P. 20, El-Jadida, Morocco
E-mail: lesfari@ucd.ac.ma lesfariahmed@yahoo.fr