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CHARACTERIZATIONS OF RANDOM APPROXIMATIONS

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Abstract. Some characterizations of random approximations are obtained in a locally convex space through duality theory.

1. Introduction and preliminaries

Random approximation theory has received much attention after the publication of survey paper by Bharucha-Reid [3] in 1976. The interested reader is referred to recent papers in normed space framework by Tan and Yaun [11], Sehgal and Singh [10], Papageorgiou [7], Lin [5], Beg and Shahzad [2] and Beg [1]. The interplay between random approximation and random fixed point results is interesting and valuable (see for example [5], [7] and [11]). The applications of this closely related concept to random differential equations and integral equations in the context of Banach spaces may be found in Itoh [4] and O'Regan [6] respectively. So random approximations are needed in the study of random equations. Recently, Beg [1] obtained a characterization of random approximations in a normed space by employing the Hahn-Banach separation theorem. Characterization theorems of best approximation in the locally convex space setting have been considered in [8]. In this paper, we establish the characterizations concerning existence of random approximation in locally convex spaces by using the Hahn Banach extension theorem and a result of Tukey [13] about separation of convex sets; in particular Theorem 1 provides a random version of Theorem 2.1 of Rao and Elumalai [8] and Theorem 2 sets an analogue for metrizable locally convex spaces of the theorem due to Beg [1].

We now fix our terminology. Let (Ω, Σ) be a measurable space where Σ is a sigma algebra of subsets of Ω and M a subset of a locally convex space E over the field F of real or complex numbers. A map $T : \Omega \times M \to E$ is called a random operator if for each fixed $x \in M$, the map $T(\cdot, x) : \Omega \to E$ is measurable. Let (E, d) be a metrizable locally convex space.

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(i) The ball with radius r and centre at x is defined as $B_r(x) = \{z \in E : d(z, x) \le r\}$; in particular the ball $B_r(0)$ has centre at 0.
(ii) $d(x, M) = \inf_{u \in M} d(x, u)$.
(iii) $P_M(x) = \{y \in M : d(x, y) = d(x, M)\}$ (set of best approximations of x from M).
(iv) For a ball $B_r(0)$ in (E, d), the set $\{z \in E : d(z, 0) = r\}$ is called metric boundary of $B_r(0)$. In general, the topological boundary of $B_r(0)$ is contained in its metric boundary. In case metric and topological boundaries of $B_r(0)$ coincide, we say $B_r(0)$ is bounding (cf. [12]).

In this note, cl, int, E^* and $E\backslash M$ denote the closure, interior, dual of E and difference of sets E and M, respectively.

2. Results

Theorem 1. *Let E be a separable locallay convex space with family P of seminorms and M a subspace of E. Suppose $T : \Omega \times M \to E$ is a random operator and $\xi : \Omega \to M$ a measurable map such that $T(\omega, \xi(\omega)) \in E\backslash M$. Then ξ is a random best approximation for T (i.e., $p(\xi(\omega) - T(\omega, \xi(\omega))) = d_p(T(\omega, \xi(\omega)), M)$ for each $p \in P$) if and only if for every $p \in P$ there exists $f^p \in E^*$ such that*
(a) *$f^p(g) = 0$ for all $g \in M$.*
(b) *$|f^p(T(\omega, \xi(\omega)) - \xi(\omega))| = p(T(\omega, \xi(\omega)) - \xi(\omega))$.*
(c) *$|f^p(T(\omega, \xi(\omega)) - g| \le p(T(\omega, \xi(\omega)) - g)$ for all $g \in M$.*

Proof. Suppose that ξ is a random approximation for T. Then for each $p \in P$ and $g \in M$,

$$p(T(\omega, \xi(\omega)) - \xi(\omega)) \le p(T(\omega, \xi(\omega)) - g)\,.$$

In particular, for any $0 \ne \alpha \in F$ and $g \in M$,

$$\text{(i)} \qquad p(T(\omega, \xi(\omega)) - \xi(\omega)) \le p\left(T(\omega, \xi(\omega)) - \left(\xi(\omega) - \frac{g}{\alpha}\right)\right)\,.$$

Let $B = \{g + \alpha(T(\omega, \xi(\omega)) - \xi(\omega)) : \alpha \in F\}$.

Define f_0^p on B by $f_0^p(g + \alpha[T(\omega, \xi(\omega)) - \xi(\omega)]) = \alpha p(T(\omega, \xi(\omega)) - \xi(\omega))$ for all $g \in M$. Then $f_0^p(g) = 0$ for all $g \in M$ and

$$f_0^p(T(\omega, \xi(\omega)) - \xi(\omega)) = p(T(\omega, \xi(\omega)) - \xi(\omega))\,.$$

For any $\alpha \ne 0$ and $g \in M$, we have

$$\begin{aligned} |f_0^p(g + \alpha[T(\omega, \xi(\omega)) - \xi(\omega)])| &= |\alpha| p(T(\omega, \xi(\omega)) - \xi(\omega)) \\ &\le |\alpha| p\left(T(\omega, \xi(\omega)) - \xi(\omega) + \frac{g}{\alpha}\right) \qquad \text{(by (i))} \\ &= p(g + \alpha[T(\omega, \xi(\omega)) - \xi(\omega)])\,. \end{aligned}$$

For $\alpha = 0$ and $g \in M$ this inequality obviously holds.

Hence for each $z \in B$ and for each $p \in P$,

$$|f_0^p(z)| \le p(z)\,.$$

Thus by the Hahn-Banach theorem, f_0^p can be extended to a continuous linear functional f^p on E such that $|f^p(x)| \leq p(x)$ for every $x \in E$ and

$$|f^p(z)| = |f_0^p(z)| \quad \text{for each} \quad z \in M.$$

The results (a)–(c) are now evident.

Conversely let the conditions (a)–(c) be satisfied. Then from (b) we get for all $p \in P$ and $g \in M$,

$$\begin{aligned} p(T(\omega,\xi(\omega)) - \xi(\omega)) &= |f^p(T(\omega,\xi(\omega)) - \xi(\omega))| \\ &= |f^p(T(\omega,\xi(\omega)) - g) + f^p(g - \xi(\omega))| \\ &= |f^p(T(\omega,\xi(\omega)) - g)| \qquad \text{(by (a))} \\ &\leq p(T(\omega,\xi(\omega)) - g) \qquad \text{(by (c))}. \end{aligned}$$

Hence $p(T(\omega,\xi(\omega)) - \xi(\omega)) = d_p(T(\omega,\xi(\omega)), M)$ for all $p \in P$. □

We shall follow the argument used in the proof of Theorem 2.3 of Thaheem [12] to prove the following:

Theorem 2. *Let (E, d) be a separable metrizable locally convex space with d as invariant metric. Assume that the ball $B_r(0)$ is convex and bounding and M a convex subset of E. Let $T : \Omega \times M \to E$ be a random operator and $\xi : \Omega \to M$ a measurable map such that $T(\omega,\xi(\omega)) \notin \mathrm{cl}(M)$. Then ξ is a random best approximation for T if and only if there exists a real continuous linear functional $f \in E^*_{\mathbf{R}}$ (**R** is the set of real numbers) such that*

(a) $f(T(\omega,\xi(\omega)) - \xi(\omega)) = d(T(\omega,\xi(\omega)), \xi(\omega)) = r(w) = r$ *(say; for notational simplicity).*
(b) $f(y - \xi(\omega)) \leq 0$ *for all y in M.*
(c) $\|f\|_r = \sup\{|f(z)| : z \in B_r(0)\} = r$.

Proof. Assume that $d(\xi(\omega), T(\omega,\xi(\omega))) = d(T(\omega,\xi(\omega)), M)$. Then M and $\mathrm{int}(B_r(T(\omega,\xi(\omega))))$, where $r = d(T(\omega,\xi(\omega)), M)$, are disjoint convex sets. By a result of Tukey [13] (see also Rudin [9]), there is a nonzero continuous real linear functional $f_{\xi(\omega)} \in E^*_R$ and a real number c such that

(ii) $$f_{\xi(\omega)}(T(\omega,\xi(\omega)) - y) \geq c \quad \text{for all} \quad y \in M\,,$$

and

$$f_{\xi(\omega)}(T(\omega,\xi(\omega)) - z) < c \quad \text{for all} \quad z \in \mathrm{int}(B_r(T(\omega,\xi(\omega))))\,.$$

The continuity of $f_{\xi(\omega)}$ implies that

$$f_{\xi(\omega)}(T(\omega,\xi(\omega)) - z) \leq c \quad \text{for all} \ \ z \in B_r(T(\omega,\xi(\omega)))\,.$$

Since $\xi(\omega) \in M \cap B_r(T(\omega,\xi(\omega)))$, it follows that

(iii) $$f_{\xi(\omega)}(T(\omega,\xi(\omega)) - \xi(\omega)) = c\,.$$

Obviously c is nonzero otherwise we get the contradiction that $f_{\xi(\omega)}$ is identically zero.

Put $f = (1/c)rf_{\xi(\omega)}$. This implies by (iii) that

$$
\begin{aligned}
f(T(\omega,\xi(\omega)) - \xi(\omega)) &= (1/c)rf_{\xi(\omega)}(T(\omega,\xi(\omega)) - \xi(\omega)) = r \\
f(y - \xi(\omega)) &= f(y - T(\omega,\xi(\omega))) + f(T(\omega,\xi(\omega)) - \xi(\omega)) \qquad (y \in M) \\
&= (1/c)rf_{\xi(\omega)}(y - T(\omega,\xi(\omega))) + (1/c)rf_{\xi(\omega)}(T(\omega,\xi(\omega)) - \xi(\omega)) \\
&\leq 0 \qquad \text{(by (ii) and (iii)).}
\end{aligned}
$$

It is easy to get by linearity of f that $\|f\|_r = r$.

Conversely suppose that there is a real continuous linear functional f satisfying the conditions (a)–(c).

If the conclusion is false, then for some x in M, we have

$$\text{(iv)} \qquad d(T(\omega,\xi(\omega)), x) < d(T(\omega,\xi(\omega)), \xi(\omega))\,.$$

The continuity of scalar multiplication implies that for any $\epsilon > 0$, there is $\beta > 0$ such that

$$\text{(v)} \qquad d(0, \beta T(\omega,\xi(\omega)) - \beta x) < \epsilon\,.$$

Consider

$$
\begin{aligned}
&d(0, (1+\beta)(T(\omega,\xi(\omega)) - x)) \\
&\quad\leq d(0, T(\omega,\xi(\omega)) - x) + d(T(\omega,\xi(\omega)) - x, (1+\beta)(T(\omega,\xi(\omega)) - x)) \\
&\quad= d(0, T(\omega,\xi(\omega)) - x) + d(0, \beta T(\omega,\xi(\omega)) - \beta x) \qquad \text{(by invariance of } d) \\
&\quad< d(0, T(\omega,\xi(\omega)) - x) + \epsilon \qquad \text{(by (v))} \\
&\quad\leq d(T(\omega,\xi(\omega)), \xi(\omega)) \qquad \text{(by (iv)).}
\end{aligned}
$$

The above inequality and the fact $f(\xi(\omega) - x) \geq 0$ lead to:

$$
\begin{aligned}
f((1+\beta)(T(\omega,\xi(\omega)) - x)) &= (1+\beta)f(T(\omega,\xi(\omega)) - x) \\
&\geq (1+\beta)f(T(\omega,\xi(\omega)) - \xi(\omega)).
\end{aligned}
$$

This implies that $f(T(\omega,\xi(\omega)) - \xi(\omega))$ is not the supremum of f over $B_r(0)$. This contradiction proves the result. $\square$

In case M is a subspace we have the following:

Corollary. *Let (E, d) be a separable metrizable locally convex space with invariant metric d and M a subspace of E. Assume that the ball $B_r(0)$ is convex and bounding. Suppose that $T : \Omega \times M \to E$ is a random operator and $\xi : \Omega \to M$ a measurable map such that $T(\omega,\xi(\omega)) \notin \mathrm{cl}(M)$. Then ξ is a random best approximation for T if and only if there exists a real continuous linear functional $f \in E_R^*$ such that*

(a) $f(T(\omega,\xi(\omega)) - \xi(\omega)) = d(T(\omega,\xi(\omega)), \xi(\omega)) = r(w) = r$ *(say).*
(b) $f(y) = 0$ *for all y in M.*
(c) $\|f\|_r = r$.

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