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THE MOVING FRAMES FOR DIFFERENTIAL EQUATIONS I. THE CHANGE OF INDEPENDENT VARIABLE

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Abstract. The article concerns the symmetries of differential equations with short digressions to the underdetermined case and the relevant differential equations with delay. It may be regarded as an introduction into the method of moving frames relieved of the geometrical aspects: the stress is made on the technique of calculations employing only the most fundamental properties of differential forms.

The present Part I is devoted to a single ordinary differential equation subjected to the change of the independent variable, the unknown function is preserved.

Preface

The E. Cartan's moving frames are eventually well-established in differential geometry, we may even refer to the recent systematical textbook [13]. This is however only a rather particular case of his approach to the "general equivalence method" based on the theory of pseudogroups [6] and namely the equivalence of differential equations and variational problem do not fit well into the narrow schema of differential geometry dealing only with the Lie groups. Recently several introductory articles and books devoted to the method of moving frames in the theory of differential equations were appearing, we can mention the beautiful booklet [9] and literature therein. Unfortunately, all they employ the actual geometrical language of G-structures and fibered spaces with many subtle and cumbersome concepts which are in principle needles in practice and even obscure the general principles of the method. So we return to the original E. Cartan's conception [6] which admits an alternative elementary exposition, see also [7]. It seems to be more transparent from the technical point of view: only the quite

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fundamental properties of differential forms, namely the exterior differential and the Frobenius theorem, are enough to resolve a large spectrum of problems.

In this Part I of the article, we deal with the equivalence and symmetries of the ordinary differential equations $y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),\ n \le 4$, with respect to the pseudogroup of transformations $\bar{x} = \varphi(x)$ of the independent variable if the unknown function $\bar{y} = y$ is preserved. Especially the instructive particular case $n = 2$ is thoroughly discussed and provides a succint self-contained introduction into the method of moving frames. The next subcase $n = 3$ provides a rather unexpected alternative way to the "dispersion theory" by O. Borůvka. Such favourable results should not be expected in $n > 3$ where the resulting formulae became rather complicated, so we briefly mention only the concluding case $n = 4$. (The case $n = 1$ is quite easy, of course, and therefore omitted here.)

For technical reasons, the exposition is carried out in C^∞-smooth and real-valued category. Every particular result can be nevertheless easily adapted to fulfill appropriately weakened smoothness assumptions.

The second order equation

1. The pseudogroup. In this Part I of the article, the pseudogroup under consideration will consist of all invertible transformations $\varphi : \mathcal{D}(\varphi) \to \mathcal{R}(\varphi)$, where $\mathcal{D}(\varphi), \mathcal{R}(\varphi) \subset \mathbf{R}$ are open subsets, see Remark 3 at the end of the article. In alternative classical notation,

$$\bar{x} = \varphi(x)\,,\ (x \in \mathcal{D}(\varphi))\,,\quad x = \varphi^{-1}(\bar{x})\ (\bar{x} \in \mathcal{R}(\varphi))\,, \tag{1}$$

where the definition domains will not be (as a rule) explicitly mentioned. The pseudogroup will be applied to curves $y = y(x)$ $(x \in \mathcal{D}(y))$ in such manner that the values are preserved: this curve turns into the transformed one $y = \bar{y}(\bar{x}) = y(\varphi(x))$, $(x \in \mathcal{D}(\varphi) \cap \mathcal{D}(y))$. In slightly abbreviated notation we have the formulae

$$\begin{gathered} y = \bar{y},\ y' = \bar{y}'\varphi',\ y'' = \bar{y}''\varphi'^2 + \bar{y}'\varphi'', \dots \\ \Big(y' = \frac{dy}{dx},\ \bar{y}' = \frac{d\bar{y}}{d\bar{x}},\ y'' = \frac{d^2y}{dx^2},\ \bar{y}'' = \frac{d^2\bar{y}}{d\bar{x}^2}\ \dots\Big)\,, \end{gathered} \tag{2}$$

for the transformed derivatives. (In alternative terms, formulae (2) provide a *prolongation* of the transformations (1) of the pseudogroup on the infinite-dimensional space $x, y, y', y'', \dots$ including all derivatives.)

2. Setting the problem. The pseudogroup will be applied to differential equations. Our main task is the *equivalence problem*: whether a given equation $y'' = f(x, y, y')$ could be transformed into another given equation $\bar{y}'' = \bar{f}(\bar{x}, \bar{y}, \bar{y}')$ by a substitution (1), (2). (One can observe that this is the case if and only if $\bar{f} = f/\varphi'^2 - y'\varphi''/\varphi'^3$ but this is of a little use and *we shall not directly employ* such explicit formula as the most important condition.) Then the *symmetry problem* concerning the transformation of an equation $y'' = f$ into itself (when $f = \bar{f}$) may be regarded as a particular subcase. Roughly saying, the strategy rests on the study of various differential forms (the *moving coframes*) that are preserved after applying (1), (2). They will depend on certain parameters but a *reduction procedure* may lead to parameter-free (the *Maurer–Cartan*) forms and even to many *invariant functions* (and a generalized *Frenet coframe*) which explicitly resolve the equivalence problem.

3. Moving (co)frames. The pseudogroup (1) is (trivially) characterized by the property of preserving the form $\omega_0 = Adx$, where $A \neq 0$ is a new variable. In more detail, if the transformed variables $\bar{x}, \bar{y}, \bar{y}', \ldots$ depend on the original variables $x, y, y', \ldots$ and the identity

$$\omega_0 = A\,dx = \bar{A}\,d\bar{x} = \bar{\omega}_0$$

holds true, then necessarily $\bar{x} = \varphi(x)$ is a function of x. Moreover $A \neq 0$, $\bar{A} = A/\varphi'$ ensures the invertibility of φ and the invariance of ω_0.

On the other hand, the equation $y'' = f$ can be represented by the Pfaffian system

$$\vartheta_1 = dy - y'\,dx = 0\,, \quad \vartheta_2 = dy' - f\,dx = 0\,, \tag{3}$$

the equation $\bar{y}'' = \bar{f}$ by the analogous system

$$\bar{\vartheta}_1 = d\bar{y} - \bar{y}'\,d\bar{x} = 0\,, \quad \bar{\vartheta}_2 = d\bar{y}' - \bar{f}\,d\bar{x} = 0\,. \tag{4}$$

In the equivalence transformation, ϑ_i *need not* be transformed into $\bar{\vartheta}_i$ individually but the *system* (3) should be transformed into the *system* (4). More explicitly, the equivalence of equations $y'' = f$ and $\bar{y}'' = \bar{f}$ takes place if and only if

$$\bar{\vartheta}_1 = B\vartheta_1 + C\vartheta_2\,, \quad \bar{\vartheta}_2 = D\vartheta_1 + E\vartheta_2 \quad (BE \neq CD)$$

with certain coefficients B, C, D, E after applying the equivalence transformation (1), (2).

Still better and more symmetrically: we shall introduce the differential forms

$$\omega_1 = B\vartheta_1 + C\vartheta_2\,, \quad \omega_2 = D\vartheta_1 + E\vartheta_2\,, \quad \bar{\omega}_1 = \bar{B}\bar{\vartheta}_1 + \bar{C}\bar{\vartheta}_2\,, \quad \bar{\omega}_2 = \bar{D}\bar{\vartheta}_1 + \bar{E}\bar{\vartheta}_2$$

with new variables $B, \ldots, \bar{E}$ (where $BE \neq CD$, $\bar{B}\bar{E} \neq \bar{C}\bar{D}$ is assumed) and then the equalities $\omega_1 = \bar{\omega}_1$, $\omega_2 = \bar{\omega}_2$ with appropriately transformed parameters ensure the equivalence of systems (3) and (4).

Conclusion. The equivalence problem is alternatively expressed by the invariance requirements

$$\omega_i = \bar{\omega}_i \quad (i = 0, 1, 2),\ y = \bar{y} \tag{5}$$

for the desired equivalence transformation (1). One can observe that then the equality of differentials

$$d\omega_i = d\bar{\omega}_i \quad (i = 0, 1, 2),\ dy = d\bar{y} \tag{6}$$

follows, too. The reduction procedure (specifying the values A, B, C, $\bar{A}$, $\bar{B}$, $\bar{C}$) will employ certain interrelations between (5) and (6).

4. The reduction of parameters. In the family of all forms ω_1 (B, C variable) there exists a unique form which is a linear combination of a form ω_0 (with A appropriately chosen) and dy. This is the form $dy - y'\,dx$. So choosing $B = 1$, $C = 0$, $A = y'$ we have performed the first step of the reduction: we have the simplified forms

$$\omega_0 = y'\,dx, \quad \omega_1 = dy - y'\,dx = dy - \omega_0$$

which are necessarilly transformed (in view of the uniqueness) into the dashed counterparts by the equivalence transformation:

$$\omega_0 = \bar\omega_0 = \bar y'\,d\bar x\,, \quad \omega_1 = \bar\omega_1 = d\bar y - \bar\omega_0\,.$$

(The equalities $\omega_1 = \vartheta_1$, $\bar\omega_1 = \bar\vartheta_1$ are a mere lucky accident.) Continuing in this way,

$$d\omega_1 = dx \wedge dy' = \omega_0 \wedge \frac{1}{y'}(dy' - f dx)\,,$$

where the factor $\frac{1}{y'}(dy' - f\,dx)$ is uniquely determined under the requirement that it should be a linear combination of ϑ_1 and ϑ_2 (equivalently: of ω_1 and ω_2). It follows that we have the simplified form ω_2 which turns into the dashed counterpart by the equivalence transformation:

$$\omega_2 = \frac{1}{y'}(dy' - f\,dx) = \frac{1}{\bar y'}(d\bar y' - \bar f\,d\bar x) = \bar\omega_2\,.$$

In other words, we have performed the reduction $D = 0$, $E = 1/y'$ (and $\bar D = 0$, $\bar E = 1/\bar y'$). The reduction procedure is done.

5. The concluding step. The differentials $d\omega_1 = -d\omega_0 = \omega_0 \wedge \omega_2$ already were employed and the equations $d\omega_i = d\bar\omega_i$ $(i = 0, 1)$ do not bring any novelty. However

$$d\omega_2 = dx \wedge d\left(\frac{f}{y'}\right) = \omega_0 \wedge (I\,dy + J\omega_2)\,, \quad I = \frac{f_y}{y'^2}\,, \quad J = \left(\frac{f}{y'}\right)_{y'} \tag{7}$$

by direct verification (see also the formula (8) below) and analogously for $d\bar\omega_2$. Then the equation $d\omega_2 = d\bar\omega_2$ implies the invariance of the corresponding coefficients: $I = \bar I$, $J = \bar J$. In more details,

$$F(x, y, y') = \bar F(\varphi, y, \frac{y'}{\varphi'}) \tag{8}$$

holds true with $F = I, J$ for the equivalence transformation.

On this occasion, two notes of general nature are in order. First, the differential of any function $F = F(x, y, y')$ admits the unique developments

$$dF = F_x\,dx + F_y\,dy + F_{y'}\,dy' = \frac{\partial F}{\partial\omega_0}\omega_0 + \frac{\partial F}{\partial dy}\,dy + \frac{\partial F}{\partial\omega_2}\omega_2\,, \tag{9}$$

where we have introduced the *covariant derivatives*

$$\frac{\partial F}{\partial\omega_0} = \frac{1}{y'}(F_x + fF_{y'})\,, \quad \frac{\partial F}{\partial\,dy} = F_y\,, \quad \frac{\partial F}{\partial\omega_2} = y'F_{y'}\,. \tag{10}$$

If this F is an invariant (i.e., $F = \bar{F}$ is preserved in the equivalences and therefore $dF = d\bar{F}$) then all the covariant derivatives are clearly invariants, too. Second, the identity

$$0 = d^2\omega_2 = d(\omega_0 \wedge (I\,dy + J\omega_2)) = \omega_0 \wedge (\omega_2 \wedge I\,dy - dI \wedge dy - dJ \wedge \omega_2)$$

yields the important *Bianchi relation*

$$J_y = y' I_y + I \tag{11}$$

if the developments (9) of differentials dI, dJ are inserted.

6. Digression: the implementation result. *Arbitrary functions $I = I(x, y, y')$, $J = J(x, y, y')$ satisfying* (11) *can be realized as invariants of a differential equation $y'' = f(x, y, y')$.*

Proof. Asssuming I, J for known, the function f is a solution of the overdetermined system

$$f_y = Iy'^2\,, \quad f_{y'} = Jy' + \frac{f}{y'} \tag{12}$$

with the only compatibility condition (11). We may choose $M = M(x, y, y')$ satisfying $I = M_y$ and then (11) reads $J_y = y' M_{yy'} + M_y$, hence

$$J = y' M_{y'} + M + N = (y'M)_{y'} + N$$

for appropriate function $N = N(x, y')$. On the other hand, $f = My'^2 + L$ with appropriate $L = L(x, y')$ in virtue of (12_1). Then (12_2) turns into the condition $N = (L/y')_{y'}$ for the function L. It follows that

$$f = My'^2 + L\,, \quad L = y' \int N dy' (+y'Q(x)) \tag{13}$$

where the last summand is the integration constant. □

7. The equivalence problem. Before passing to the problem proper, let us recall our results. *The equivalence of differential equations $y'' = f$ and $\bar{y}'' = \bar{f}$ is expressed by the invariance requirements*

$$\omega_0 = \bar{\omega}_0\,, \quad \omega_2 = \bar{\omega}_2\,, \quad y = \bar{y} \tag{14}$$

(*the equality $\omega_1 = \bar{\omega}_1$ may be omitted*)*. We moreover have the invariants $I = \bar{I}$ and $J = \bar{J}$ and many other invariants arising from them by repeated covariant derivations.*

(ι) *The general case.* Assume that there exist three functionally independent invariants, e.g., the invariants y, I, J. We know that *the equivalence transformation* (1) (if it exists) *satisfies* (8) *with F equal to any of the functions*

$$I,\ J,\ \frac{\partial I}{\partial \omega_0}\,,\ \frac{\partial J}{\partial \omega_0}\,,\ \frac{\partial I}{\partial\, dy}\,,\ \frac{\partial J}{\partial\, dy}\,,\ \frac{\partial I}{\partial \omega_2}\,,\ \frac{\partial J}{\partial \omega_2}\,. \tag{15}$$

More interesting is the converse assertion that *the equivalence transformations are characterized by the latter property.* In particular, they can be determined by using the invariants (15).

Proof. Let (15) be invariants for a transformation (1). Then $I = \bar{I}$, $J = \bar{J}$ (cf. $(15_{1,2})$) ensures $dI = d\bar{I}, dJ = d\bar{J}$ and, by using the developments (9) and the identities $F = \bar{F}$ with F equal to either of the remaining functions (15_{3-8}), we obtain

$$\frac{\partial F}{\partial \omega_0}(\omega_0 - \bar{\omega}_0) + \frac{\partial F}{\partial\, dy}(dy - d\bar{y}) + \frac{\partial F}{\partial \omega_2}(\omega_2 - \bar{\omega}_2) = 0 \quad (F = I, J)\,.$$

However $y = \bar{y}$ hence $dy = d\bar{y}$ and it follows that (14) is satisfied. □

In fact the proof gives even the stronger result: *if the overdetermined implicit system* (8) *with F equal to any of the functions* (15) *has a solution $\varphi = \varphi(x, y, y')$, $\varphi' = \varphi'(x, y, y')$, then the functions φ, φ' do not depend on the variables y, y', moreover $\varphi' = d\varphi/dx$ is the true derivative, and the equivalence of equations $y'' = f$ and $\bar{y}'' = \bar{f}$ is realized.*

($\iota\iota$) *The lower symmetry case.* Let us assume the existence of exactly two functionally independent invariants, i.e., let all invariants be of the kind $g(y, F)$ where F is a fixed invariant (depending on y'). In particular $\partial F/\partial \omega_2 = y'F_{y'} = g(y, F)$. This may be regarded as a differential equation for F and it follows that $F = G(y, c(x, y)y')$ with appropriate G, c. Clearly $K = c(x, y)y'$ is an invariant (and $c \neq 0$). We may choose K instead of F. Then the analogous argument applied to $\partial K/\partial\, dy = c_y y' = g(y, cy')$ implies that necessarily $c(x, y) = a(x)b(y)$. So we have an invariant $K = a(x)y'$ since the invariant factor $b(y)$ may be omitted.

With this preliminary result, let us determine the function f. Assuming $I = g(y, K) = M_y(y, K,)$ then $f = My'^2 + N$ where $N = N(x, y')$ cannot be arbitrarily chosen since the covariant derivative

$$\frac{\partial K}{\partial \omega_0} = a' + (My'^2 + N)\frac{a}{y'} = MK + \frac{1}{y'}\left(\frac{a'}{a} + \frac{N}{y'}\right)K$$

must be a function of only y and K. It follows that $a'/a + N/y' = 0$. So we deal with the differential equation

$$y'' = M(y, a(x)y')y'^2 - \frac{a'(x)}{a(x)}y'\,. \tag{16}$$

The equivalence transformation (1) on the counterpart equation

$$\bar{y}'' = \bar{M}(y, \bar{a}(\bar{x})\bar{y}')\bar{y}'^2 - \frac{\bar{a}'(\bar{x})}{\bar{a}(\bar{x})}\bar{y}'$$

is given by the first order equation $\bar{a}(\varphi)\varphi' = a(x)$ for the function φ (direct verification). In particular, we have a one-parameter pseudogroup of symmetries (1) of the equation (16) satisfying $a(\varphi)\varphi' = a(x)$.

($\iota\iota\iota$) *The higher symmetry case.* Let $I = I(y)$, $J = J(y)$ be functions of mere y. Then $J_y = I$ and we may choose $M = J$, $N = 0$ in Section 6 to obtain the differential equation

$$y'' = J(y)y'^2 + Q(x)y' \tag{17}$$

(see also articles [10], [1], [14]). The equivalent equation necessarily is of the kind $\bar{y}'' = J(\bar{y})\bar{y}'^2 + \bar{Q}(\bar{x})\bar{y}'$ and then the transformation rules (2) give the differential equation

$$\varphi'' + \bar{Q}(\varphi)\varphi'^2 = Q(x)\varphi' \tag{18}$$

for the equivalence transformation φ (direct verification). In particular, there is a two-parameter pseudogroup of symmetries (1) of the equation (17) satisfying $\varphi'' + Q(\varphi)\varphi'^2 = Q(x)\varphi'$.

($\iota\nu$) *Continuation.* The results of the preceding point ($\iota\iota\iota$) admit certain not self-evident interpretation. For this aim, let us introduce the composition function $G(y)$. Then the equation (17) leads to the symmetrical relation

$$G'^2(z'' - Q(x)z') = z'^2(G'' - J(y)G') \tag{19}$$

between functions $G = G(y)$ and $z = z(x) = G(y(x))$. One can observe that $z'(x) \neq 0$ ($G'(y) \neq 0$) on every nonconstant solution of the equation $z'' = Q(x)z'$ ($G'' = J(y)G'$). It follows that every nonconstant solution $y = y(x)$ of (17) provides a bijective correspondence between solutions of equations $z'' = Q(x)z'$ and $G'' = J(y)G'$. In other terms, the equation (17) provides the equivalence transformations between the mentioned equations.

The third order equations

8. Moving frames. We are passing to the equivalence problem with respect to the pseudogroup (1), (2) for the equations $y''' = f(x, y, y', y'')$. Analogously as above, there is the form $\omega_0 = y'\,dx$ with the invariance property $\omega_0 = y'dx = \bar{y}'d\bar{x} = \bar{\omega}_0$. On the other hand, the primary differential equation $y''' = f$ can be represented by the Pfaffian system

$$\vartheta_1 = dy - y'\,dx = 0\,, \quad \vartheta_2 = dy' - y''\,dx = 0\,, \quad \vartheta_3 = dy'' - f\,dx = 0\,. \tag{20}$$

It is transformed into the system

$$\bar{\vartheta}_1 = d\bar{y} - \bar{y}'d\bar{x} = 0\,, \quad \bar{\vartheta}_2 = d\bar{y}' - \bar{y}''\,d\bar{x} = 0\,, \quad \bar{\vartheta}_3 = d\bar{y}'' - \bar{f}\,d\bar{x} = 0 \tag{21}$$

corresponding to the equivalent equation

$$\bar{y}''' = \bar{f}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')\,, \quad \bar{f} = \frac{f}{\varphi'^3} - 3\frac{\varphi''}{\varphi''^4}y'' - \frac{1}{\varphi'^2}\left(\frac{\varphi'''}{\varphi'} - \frac{3}{2}\frac{\varphi''^2}{\varphi'^2}\right)y'\,. \tag{22}$$

Recall that the forms ϑ_i *need not* be transformed into each $\bar{\vartheta}_i$ individually, however, appropriate linear combinations of ϑ_i will have this favourable property. We shall not explicitly introduce the relevant linear combinations of forms ϑ_i with uncertain parameters (like in Section 3) since the simple method of Section 4 can be closely simulated.

First of all, we again have the invariant form $\omega_1 = dy - y'dx = dy - \omega_0$. Then the exterior derivative

$$d\omega_1 = \omega_0 \wedge \frac{1}{y'}(dy' - y''\,dx) = \bar{\omega}_0 \wedge \frac{1}{\bar{y}'}(d\bar{y}' - \bar{y}''\,d\bar{x}) = d\bar{\omega}_1$$

of the equality $\omega_1 = \bar\omega_1$ leads to the next invariant form $\omega_2 = \frac{1}{y'}(dy' - y''\,dx)$ quite analogously as in Section 4. Continuing in this way, clearly

$$d\omega_2 = \omega_0 \wedge \omega_3\,, \quad \omega_3 = \frac{1}{y'^2}\left(dy'' - f\,dx - \frac{y''}{y'}(dy' - y''\,dx)\right)\,.$$

The factor ω_3 is unique (under the additional requirement that it should be a linear combination of forms (20)), hence invariant for the equivalence transformation:

$$\omega_3 = \bar\omega_3 = \frac{1}{\bar y'^2}\left(d\bar y'' - \bar f\,d\bar x - \frac{\bar y''}{\bar y'}\,(d\bar y' - \bar y''\,d\bar x)\right)\,.$$

The reduction is done and the sought forms $\omega_i \quad (i = 0, \dots, 3)$ for the equivalence transformation are determined.

9. The concluding step. The differentials $d\omega_i$ $(i = 0, \dots, 2)$ do not provide any useful information at this place, however, the differential

$$d\omega_3 = \omega_0 \wedge (I\,dy + J\omega_2 + K\omega_3) - \omega_2 \wedge \omega_3$$

where

$$I = \frac{f_y}{y'^3}\,, \quad J = \frac{1}{y'^3}(y' f_{y'} + y'' f_{y''} - f)\,, \quad K = \frac{f_{y''}}{y'} - 3\frac{y''}{y'^2}$$

(direct verification) provides the invariants for the equivalence transformation. Then the developments of differentials

$$dF = F_x\,dx + F_y\,dy + F_{y'}\,dy' + F_{y''}\,dy'' = \frac{\partial F}{\partial\omega_0}\omega_0 + \frac{\partial F}{\partial\,dy}\,dy + \frac{\partial F}{\partial\omega_2}\omega_2 + \frac{\partial F}{\partial\omega_3}\omega_3$$

with coefficients of the covariant derivatives

$$\frac{\partial F}{\partial\omega_0} = \frac{1}{y'}(F_x + y'' F_{y'} + f F_{y''})\,, \quad \frac{\partial F}{\partial\,dy} = F_y\,,$$
$$\frac{\partial F}{\partial\omega_2} = y' F_{y'} + y'' F_{y''}\,, \quad \frac{\partial F}{\partial\omega_3} = y'^2 F_{y''}$$

(direct verification) still provide the new invariants when applied to an invariant function $F = \bar F$. Moreover the Bianchi relations

$$J_y = 2I + y' I_{y'} + y'' I_{y''}\,, \quad K_y = y'^2 I_{y''}\,, \quad y'^2 J_{y''} = K + y' K_{y'} + y'' K_{y''}$$

easily follow from the identity $d^2\omega_3 = 0$ when the developments of differentials dI, dJ, dK in terms of covariant derivatives are inserted.

10. Notes to the equivalence problem. To pleasure the reader, we shall follow an alternative way. Omiting the implementation result, let us directly deal with the equivalence of equations $y''' = f$ and $\bar y''' = \bar f$. *It is expressed by the invariance requirements*

$$\omega_i = \bar\omega_i \quad (i = 0, 2, 3)\,, \quad y = \bar y\,. \tag{23}$$

We have moreover obtained the additional equations

$$F(x, y, y', y'') = \bar F\left(\varphi, y, \frac{y'}{\varphi'}, \frac{y''}{\varphi'^2} - \frac{\varphi''}{\varphi'^3}\right) \tag{24}$$

for all invariants F.

(ι) *The general case.* Assume that there exist four functionally independent invariants, e.g., the invariants y, I, J, K. *Then the equivalence transformations are alternatively characterized by the overdetermined implicit system* (24) *where* F *runs over invariants* y, I, J, K *and their first order covariant derivatives.* The proof may be omitted.

($\iota\iota$) *The lower symmetry case.* Assume that there exist exactly three functionally independent invariants denoted $y, F = F(x, y, y', y''), G = G(x, y, y', y'')$ of the equation $y''' = f$. Then this equation has a one-parameter pseudogroup of symmetries. (Hint: apply the Frobenius theorem to equations (23) constrained moreover by $F = \bar{F}$ and $G = \bar{G}$.) By a change of independent variable $\bar{x} = \psi(x)$, the pseudogroup will consist of the transformations $\bar{x} \to \bar{x} + \mathrm{const}.$ and it follows that the transformed equation is of the kind $\bar{y}''' = h(\bar{y}, \bar{y}', \bar{y}'')$ with the right hand side independent of $\bar{x}$. Passing to the original equation, it reads $y''' = f(x, y, y', y'')$ where

$$f = h\left(y, \frac{y'}{\psi'}, \frac{y''}{\psi'^2} - y'\frac{\psi''}{\psi'^3}\right)\psi'^3 + 3\frac{\psi''}{\psi'}y'' + \left(\frac{\psi'''}{\psi'} - 3\frac{\psi''^2}{\psi'^2}\right)y' \tag{25}$$

(use formulae (2) with φ replaced by ψ). One can verify that the result is quite correct: the differential invariants I, J, K and their covariant derivatives can be represented as the composed functions of the kind

$$g(y, F, G) \quad \left(F = \frac{y'}{\psi'}\,,\; G = \frac{y''}{\psi'^2} - y'\frac{\psi''}{\psi'^3}\right) \tag{26}$$

where F, G itself are invariants. (One can also obtain formula like (16) if $1/\psi' = a$, $-\psi''/\psi'^2 = a'$, $-\psi'''/\psi'^2 + 2\psi''^2/\psi'^3 = a''$ is substituted into (25).) The pseudogroup symmetries (1) of the equation are given by the invariance of F: $\bar{y}'/\psi'(\varphi) = y'/\psi'$ where $\bar{y}'\varphi' = y'$, that is,

$$a(\varphi)\varphi' = a(x) \quad (a = 1/\psi')\,, \tag{27}$$

in well accordance with ($\iota\iota$) of Section 7.

The more general equivalence problem is quite clear. The equation $y''' = f$ with f given by (25) can be transformed only into equation $\bar{y}''' = \bar{f}$ where $\bar{f}$ is given analogously as (25). Then the equivalences (2) satisfy $\bar{a}(\varphi)\varphi' = a(x)$ ($a = 1/\psi'$, $\bar{a} = 1/\bar{\psi}'$) like in Section 7.

($\iota\iota\iota$) *The middle symmetry case.* Assume that there exist exactly two functionally independent invariants. Then the equation admit a two-parameter symmetry pseudogroup which consist of transformations $\bar{x} \to \mathrm{Const.} \cdot x + \mathrm{const.}$ ($\mathrm{Const.} \neq 0$) after appropriate change $\bar{x} = \psi(x)$ of the independent variable. It follows easily that the transformed equation is of the kind $\bar{y}''' = h(\bar{y}, \bar{y}''/y'^2)y'^3$ (we assume $y' \neq 0$ for simplicity here). Passing to the original variables, it reads $y''' = f(x, y, y', y'')$ where

$$f = h\left(y, \frac{y''}{y'^2} - \frac{\psi''}{\psi' y'}\right)y'^3 + 3\frac{\psi''}{\psi'}y'' + \left(\frac{\psi'''}{\psi'} - 3\frac{\psi''^2}{\psi'^2}\right)y'. \tag{28}$$

We obtain the unexpected invariant

$$F=\frac{y''}{y'^2}-\frac{\psi''}{\psi' y'}$$

such that all invariants of the equation can be expressed in terms of y and F. The pseudogroup of all symmetries (1) is defined by the equality $F=\bar{F}$ which simplifies to the condition

$$\varphi''+\frac{\psi''(\varphi)}{\psi'(\varphi)}\varphi'^2=\frac{\psi''(x)}{\psi'(x)}\varphi'\,. \tag{29}$$

One can observe that this is the equation (18) with $Q=\bar{Q}=\psi''/\psi'$ inserted. The more general equivalence problem leads just to the equation (18) and need not any comments.

($\iota\nu$) *The higher symmetry case.* If all invariants are depending only on the variable y, the equation $y'''=f$ admits a three-parameter pseudogroup of symmetries. It is not difficult to explicitly find the shape of the function f:

$$y'''=\frac{1}{2}J(y)y'^3+\frac{3}{2}\frac{y''^2}{y'}+Q(x)y'\,. \tag{30}$$

The equivalences (1) with analogous equations $\bar{y}'''=\bar{f}$ (with $J=\bar{J}$ but another function $\bar{Q}$) are defined by the equation

$$\frac{\varphi'''}{\varphi'}-\frac{3}{2}\frac{\varphi''^2}{\varphi'^2}+\bar{Q}(\varphi)\varphi'^2=Q(x) \tag{31}$$

(direct verification using (21) and transformation formulae (2) for the derivatives).

(ν) *Continuation.* We mention a certain alternative interpretation of the latter result. For this aim, let us introduce the composition function $G(y)$. Then the equation (30) leads to the symmetrical relation

$$G'^2(\{z,x\}-Q(x))=z'^2\left(\{G,y\}-\frac{1}{2}J(y)\right) \tag{32}$$

between functions $G=G(y)$ and $z=z(x)=G(y(x))$, where

$$\{y,x\}=\frac{y'''}{y'}-\frac{3}{2}\frac{y''^2}{y'^2}$$

denotes the familiar Schwarz derivatives. (Hint: Insert the inversion $y=F(G)$ of $G=G(y)$ into (30) which is better rewritten as $\{y,x\}=\frac{1}{2}Jy'^2+Q$ to obtain

$$\{G,x\}+G'^2\{F,G\}=\{F(G),x\}=\{y,x\}=\frac{1}{2}Jy'^2+Q$$

by using the familiar identity for the derivatives of composed functions. However, $\{F,G\}+F'^2\{G,y\}=\{G(F),y\}=\{y,y\}=0$ for the mutually inverse functions $y=F(G)$, $G=G(y)$ and (32) easily follows since $z'F'=y'=z'/G'$.) It follows that nonconstant solutions $y=y(x)$ of equation (30) provides a bijective correspondence between solutions of equations $\{z,x\}=Q(x)$ and $\{G,y\}=\frac{1}{2}J(y)$. There moreover exists a close interrelation between equations $\{u,x\}=q(x)$ and $v''=q(x)v$ (which need not be recalled here) and it follows that solutions $y=y(x)$

of the equation (30) also concerns the equivalence transformations between linear differential equations $z'' = Q(x)z$ and $G'' = \frac{1}{2}J(y)G$.

The fourth order equation

11. Survey of results. Dealing with the equivalence theory for the equations $y^{(4)} = f(x, y, y', y'', y''')$ with respect to the pseudogroup (1), (2), one can obtain the invariant forms

$$\omega_0 = y'\,dx\,, \quad \omega_1 = dy - \omega_0\,, \quad \omega_2 = \frac{1}{y'}(dy' - y''\,dx)\,,$$
$$\omega_3 = \frac{1}{y'^2}(dy'' - y'''\,dx - y''\omega_2)\,, \quad \omega_4 = \frac{1}{y'^3}(dy''' - f\,dx - y'''\omega_2 - 3y'y''\omega_3)$$

(and dy). Recall that the equation $y^{(4)} = f$ is equivalent to the Pfaffian system $\omega_i = 0$ $(i = 1, \dots, 4)$ consisting of invariant equations. The formulae

$$-d\omega_0 = d\omega_1 = \omega_0 \wedge \omega_2\,, \quad d\omega_2 = \omega_0 \wedge \omega_3\,, \quad d\omega_3 = \omega_3 \wedge \omega_2 + \omega_0 \wedge \omega_4\,,$$
$$d\omega_4 = 2\omega_4 \wedge \omega_2 + \omega_0 \wedge (I\,dy + J\omega_2 + K\omega_3 + L\omega_4)$$

hold true where

$$I = \frac{f_y}{y'^4}\,, \quad J = \frac{1}{y'^4}(y'f_{y'} + y''f_{y''} + y'''f_{y'''} - f)\,,$$
$$K = \frac{1}{y'^2}\left(f_{y''} + 3\frac{y''}{y'}f_{y'''} - 4\frac{y'''}{y'} - 3\frac{y''^2}{y'^2}\right)\,, \quad L = \frac{1}{y'}\left(f_{y'''} - 6\frac{y''}{y'}\right)$$

are invariant functions. Other invariant functions can be obtained by using covariant derivations

$$\frac{\partial F}{\partial \omega_0} = \frac{1}{y'}\left(F_x + y''F_{y'} + y'''F_{y''} + fF_{y'''}\right)\,, \quad \frac{\partial F}{\partial\, dy} = f_y\,,$$
$$\frac{\partial F}{\partial \omega_2} = y'F_{y'} + y''F_{y''} + y'''F_{y'''}\,, \quad \frac{\partial F}{\partial \omega_3} = y'^2\left(F_{y''} + 3\frac{y''}{y'}F_{y'''}\right)\,,$$
$$\frac{\partial F}{\partial \omega_4} = y'^3F_{y'''}$$

of any invariant F. The Bianchi relations written in terms of them are shorter:

$$\begin{aligned} &\frac{\partial J}{\partial\, dy} = \frac{\partial I}{\partial \omega_2} + 3I\,, \quad \frac{\partial J}{\partial \omega_3} = \frac{\partial K}{\partial \omega_2} + 2K\,, \quad \frac{\partial J}{\partial \omega_4} = \frac{\partial L}{\partial \omega_2} + L\,,\\ &\frac{\partial K}{\partial \omega_4} = \frac{\partial L}{\partial \omega_3} + 2\,, \quad \frac{\partial K}{\partial\, dy} = \frac{\partial I}{\partial \omega_3}\,, \quad \frac{\partial L}{\partial\, dy} = \frac{\partial I}{\partial \omega_4} \end{aligned} \tag{33}$$

and they conclude the short surwey of necessary results. For the convenience, we state the transformations of third order derivatives

$$y''' = \bar{y}'''\varphi'^3 + 3\bar{y}''\varphi'\varphi'' + \bar{y}'\varphi'''$$

to complete the formulae (2) and the transformed equation $\bar{y}^{(4)} = \bar{f}$ where $\bar{f} = \bar{f}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'', \bar{y}''')$ can be calculated from the equation

$$f = \bar{f}\varphi'^4 + 6\bar{y}'''\varphi'^2\varphi'' + \bar{y}'(3\varphi''^2 + 4\varphi'\varphi''') + \bar{y}'\varphi'^4$$

for the fourth order derivatives.

12. Few notes to the equivalence problem. The equivalences between equations $y^{(4)} = f$ and $\bar{y}^{(4)} = \bar{f}$ are described by the invariance requirements $\omega_i = \bar{\omega}_i$ $(i = 0, 2, 3, 4)$, $y = \bar{y}$. They can be (at least partly) replaced by simpler invariance conditions $F = \bar{F}$ with various invariants F other than y. Such invariants F always exist (see below). Analogously as above several possibilities are to be distiguished.

(ι) *The general case.* If there exist five functionally independent invariants then the equivalence problem can be resolved by means of them by resolving a certain overdetermined implicit system.

($\iota\iota$) *The lower symmetry case.* If exactly four independent invariants exist, the one-parameter families of equivalences and symmetries appear. They are given by certain first order differential equation (like (27)) for the function φ. Explicit formulae for the relevant function f (resembling (25)) can be easily obtained.

($\iota\iota\iota$) *The middle symmetry case.* If there are exactly three functionaly independent invariants, the equivalence and symmetries φ depend on two parameters and satisfy certain second order differential equation (analogous to (18) or (29)).

($\iota\nu$) *The higher symmetry case.* If there exists besides y only one functionally independent invariant F, we obtain three-parameter families of equivalences and symmetries. The mentioned invariant

$$F = \frac{1}{y'^2}\left(\frac{y'''}{y'} - \frac{3}{2}\frac{y''^2}{y'^2} - Q(x)\right) = \frac{1}{y'^2}\big(\{y, x\} - Q(x)\big)$$

can be obtained by a lengthy analysis of the covariant derivatives (see also ($\iota\iota$) Section 7) together with the (not very simple) formula

$$f = h(y, F)y'^4 + 6\frac{y''}{y'}\left(y''' - \frac{y''^2}{y'}\right) - 2Qy'' + Q'y'$$

for the differential equations under consideration. The equivalences φ are given by the third order equation with $F = \bar{F}$ which reads

$$\{\varphi, x\} + \bar{Q}(\varphi)\varphi'^2 = Q(x) \tag{34}$$

(direct verification) in terms of Schwarz derivatives.

(ν) *The impossible case.* The assumption that all invariants are functions of only y contradics the Bianchi relation $\partial K/\partial\omega_4 = \partial L/\partial\omega_3 + 2$.

Remarks

Remark 1. The symmetry equivalence problem for ordinary differential equations of the order $n + 1$ $(n \geq 1)$ and transformations (1), (2) is solved in [14] by means of functional equation under the restricted condition

$$\varphi^{(n+1)} = g(x, \varphi, \dots, \varphi^{(n)})\,.$$

The problem is resolved by a function

$$f(x,y,y',y'') = a(y)y'^3 + p_1(x)y' + p_2(x)y'' \tag{35}$$

with conditions

$$3\varphi'' = p_2(x)\varphi' - p_2(\varphi)\varphi'^2\,, \quad \varphi''' = p_1(x)\varphi' + p_2(x)\varphi'' - p_1(\varphi)\varphi'^3\,, \tag{36}$$

for the third order equations and

$$f(x,y,y',y'',y''') = b(y)y'^4 + p_1(x)y' + p_2(x)y'' + p_3(x)y''' \tag{37}$$

together with conditions

$$\begin{aligned} 6\varphi'' &= p_3(x)\varphi' - p_3(\varphi)\varphi'^2\,, \\ 3\varphi''^2 + 4\varphi'\varphi''' &= p_2(x)\varphi'^2 + 3p_3(x)\varphi'\varphi'' - p_2(\varphi)\varphi'^4\,, \\ \varphi^{(4)} &= p_1(x)\varphi' + p_2(x)\varphi'' + p_3(x)\varphi''' - p_1(\varphi)\varphi'^4 \end{aligned} \tag{38}$$

for the fourth order equations, respectively.

(ι) *Results* (35), (36) *are equivalent to the higher symmetry case* (Section 10 ($\iota\nu$)) *of the equivalence problem for the third order differential equations.*

Proof. Indeed, assuming

$$f(x,y,y',y'') = a(y)y'^3 + p_1(x)y' + p_2(x)y'' = \frac{1}{2}J(y)y'^3 + \frac{3}{2}\frac{y''^2}{y'} + Q(x)y' \tag{39}$$

in accordance with (30), (35), we obtain

$$a(y) = \frac{1}{2}J(y)\,, \quad p_2(x) = 3\frac{y''}{y'}\,, \quad Q(x) = p_1(x) + \frac{1}{6}p_2(x)^2 \tag{40}$$

by means of (39) and invariants

$$I = \frac{1}{y'^3}f_y\,, \quad J = \frac{1}{y'^3}(y'f_{y'} + y''f_{y''} - f)\,, \quad K = \frac{1}{y'}\left(f_{y''} - 3\frac{y''}{y'}\right)$$

(Section 9). We get

$$\bar{p}_2(\varphi) = 3\frac{\bar{y}''(\varphi)}{\bar{y}'(\varphi)} = \frac{3}{\varphi'}(\frac{y''}{y'} - \frac{\varphi''}{\varphi'}) = \frac{1}{\varphi'}(p_2(x) - 3\frac{\varphi''}{\varphi'})\,, \tag{41}$$

i.e., (36_1) in accordance with (2). Moreover

$$\frac{\varphi'''}{\varphi'} - \frac{3}{2}\frac{\varphi''^2}{\varphi'^2} + \bar{Q}(\varphi)\varphi'^2 - Q = \frac{\varphi'''}{\varphi'} + \bar{p}_1(\varphi)\varphi'^2 - \frac{\varphi''}{\varphi'}p_2 - p_1 = 0$$

and (36_2) is satisfied by means of (31), (41). Thus the conditions (31) and (36) are equivalent and the assertion (ι) is proved. □

($\iota\iota$) *For* $h(y,F) = b(y)$, *results* (37), (38) *are equivalent to the higher symmetry case* (Section 12 ($\iota\nu$)) *of the equivalence problem for the fourth order differential equations.*

Proof. Assuming

$$
\begin{aligned}
f &= b(y)y'^4 + p_3(x)y''' + p_2(x)y'' + p_1(x)y' \\
&= h(y,F)y'^4 + 6\frac{y''}{y'}(y''' - \frac{y''^2}{y'}) - 2Qy'' + Q'y' \qquad (42)
\end{aligned}
$$

(according to Section 12 $(\iota\nu)$ and (37)) we get

$$
b'(y) = h_y(y,F)\,, \quad b(y) = h(y,F)\,, \quad \frac{1}{y'}(p_3(x) - 6\frac{y''}{y'}) = h_F(y,F) \qquad (43)
$$

for

$$
F = \frac{1}{y'^2}(\{y,x\} - Q(x)) = \frac{1}{y'^2}\Big(\frac{y'''}{y'} - \frac{3}{2}\frac{y''^2}{y'^2} - Q(x)\Big)
$$

by means of invariants

$$
\begin{aligned}
I &= \frac{f_y}{y'^4}\,, \quad J = \frac{1}{y'^4}(y'f_{y'} + y''f_{y''} + y'''f_{y'''} - f)\,, \\
K &= \frac{1}{y'^2}\left(f_{y''} + 3\frac{y''}{y'}f_{y'''} - 4\frac{y'''}{y'} - 3\frac{y''^2}{y'^2}\right)\,, \quad L = \frac{1}{y'}\left(f_{y'''} - 6\frac{y''}{y'}\right)\,.
\end{aligned}
$$

(Section 11). By solving (43) we have

$$
\begin{aligned}
& h(y,F) = b(y) + kF^{3/2}\,, \quad p_3(x) = \frac{3}{2}ky'F^{1/2} + 6\frac{y''}{y'}\,, \\
& p_2(x) = 2Fy'^2 - \frac{9}{2}ky''F^{1/2} - 15\frac{y''^2}{y'^2} + 4\frac{y'''}{y'}\,,
\end{aligned} \qquad (44)
$$

$k \in \mathbf{R}$ being constant. It holds

$$
\begin{aligned}
\bar{p}_3(\varphi) &= \frac{3}{2}\frac{y'}{\varphi'}F^{1/2} + \frac{6}{\varphi'}\left(\frac{y''}{y'} - \frac{\varphi''}{\varphi'}\right)\,, \\
\bar{p}_2(\varphi) &= 2F\frac{y'^2}{\varphi'^2} - \frac{9}{2}kF^{1/2}\frac{1}{\varphi'^2}\left(y'' - y'\frac{\varphi''}{\varphi'}\right) \\
&\quad + \frac{1}{\varphi'^2}\left(4\frac{y'''}{y'} + 18\frac{y''}{y'}\frac{\varphi''}{\varphi'} - 4\frac{\varphi'''}{\varphi'} - 15\frac{y''^2}{y'^2} - 3\frac{\varphi''^2}{\varphi'^2}\right)
\end{aligned} \qquad (45)
$$

by using the transformed derivatives. Hence $(38)_{1,2}$ are identities. The relationship between coefficients p_1, p_2, p_3 of a function f and the condition $(38)_3$ we express only for $h(y,F) = b(y)$ $(k = 0)$ for simplicity. In such a case,

$$
p_3(x) = 6\frac{y''}{y'}\,, \quad p_2(x) = 2Fy'^2 - 15\frac{y''^2}{y'^2} + 4\frac{y'''}{y'}\,. \qquad (46)
$$

by using (44). We get

$$
\frac{y'''}{y'} = \frac{1}{6}p_3' + \frac{1}{36}p_3^2\,, \quad Fy'^2 = \{y,x\} - Q = \frac{1}{3}p_3' - \frac{1}{72}p_3^2 - Q\,,
$$

thus

$$
p_2 = p_3' - 2Q - \frac{1}{3}p_3^2 \qquad (47)
$$

by means of $(46)_{1,2}$ and the invariant F. In a similar way we obtain

$$p_1 = Q' - \frac{1}{6}p_3p_3' - \frac{1}{36}p_3^2 \tag{48}$$

in accordance with (42) for $h(y,F) = b(y)$. Moreover

$$\bar{p}_3(\varphi)\varphi'^2 = 6\frac{\bar{y}''(\varphi)}{\bar{y}'(\varphi)}\varphi'^2 = \left(6\frac{y''}{y'} - 6\frac{\varphi''}{\varphi'}\right)\varphi' = p_3\varphi' - 6\varphi''$$

(see (2)) and

$$\bar{p}_3'(\varphi)\varphi'^3 = p_3'\varphi' - p_3\varphi'' - 6\varphi''' + 12\frac{\varphi''^2}{\varphi'}$$

through differentiation. Then

$$\begin{aligned}\bar{p}_1(\varphi)\varphi'^4 &= \bar{Q}'(\varphi)\varphi'^4 - \frac{1}{6}\bar{p}_3(\varphi)\varphi'\bar{p}_3'(\varphi)\varphi'^3 + \frac{1}{36}(\bar{p}_3(\varphi)\varphi')^3\varphi' \\ &= \bar{Q}'(\varphi)\varphi'^4 + \frac{1}{36}p_3^3\varphi' - \frac{1}{3}p_3^2\varphi'' + p_3\varphi''' - \frac{1}{6}p_3p_3'\varphi' \\ &\quad + p_3'\varphi'' - 6\frac{\varphi''\varphi'''}{\varphi'} + 6\frac{\varphi''^3}{\varphi'^2}\,.\end{aligned}$$

We derive the relation $(38)_3$. We get

$$\varphi^{(4)} = Q'\varphi' - \bar{Q}(\varphi)\varphi'^4 + 4Q\varphi'' - 6\bar{Q}(\varphi)\varphi'^2\varphi'' + 3\frac{\varphi''^3}{\varphi'^2}$$

through differentiation of (34). At the same time

$$\begin{aligned}\varphi^{(4)} &= p_1(x)\varphi' + p_2(x)\varphi'' + p_3(x)\varphi''' - p_1(\varphi)\varphi'^4 \\ &= \left(Q' - \frac{1}{6}p_3p_3' + \frac{1}{36}p_3^3\right)\varphi' + \left(p_3' - 2Q - \frac{1}{3}p_3^2\right)\varphi'' + p_3\varphi''' - p_1(\varphi)\varphi'^4 \\ &= Q'\varphi' - Q'(\varphi)\varphi'^4 - 2Q\varphi'' + 6\frac{\varphi''\varphi'''}{\varphi'} - 6\frac{\varphi''^2}{\varphi'^2} \\ &= Q'\varphi' - Q'(\varphi)\varphi'^4 + 4Q\varphi'' - 6Q(\varphi)\varphi'^2\varphi'' + 3\frac{\varphi''^3}{\varphi'^2}\,.\end{aligned}$$

We see that the relations $(38)_3$, (34) are equivalent and the assertion is proved. □

Remark 2. The criterion of global equivalence of the second order *linear* differential equations was published by O. Borůvka [3], of the third and higher order linear equations by F. Neuman [11]. Transformations $z(t) = y(\varphi(t))$ were studied in [12] as a "motion" for n-th order linear differential equations. A general form

$$y''(x) = a(y(x))y'(x)^2 + p(x)y'(x)\,,$$

where φ satisfies a differential equation $\varphi''(x) = p(x)\varphi'(x) - p(\varphi(x))\varphi'(x)^2$ and a, p are arbitrary functions, was derived by J. Aczél [1] for the second order differential equations (eliminating regularity conditions from Moór–Pintér [10]) by means of functional equations and this result is in full accordance with the higher symmetry case of the equivalence problem for the second order differential equations (Section 7 $(\iota\iota\iota)$). This general form allows the transformation $z(t) = y(\varphi(t))$ and transforms the equation into itself on the whole interval of definition. In [14], similarly to J. Aczél, a general form of ordinary differential equations of the order $n+1$ $(n \geq 1)$ which allows transformations $z(t) = y(\varphi(t))$, is derived (See Remark 1. of this article).

Remark 3. Let $\mathbf{M}$ be a topological space, Γ a family of homeomorphisms $\varphi : \mathcal{D}(\varphi) \to \mathcal{R}(\varphi)$, where $\mathcal{D}(\varphi), \mathcal{R}(\varphi)$ are open subspaces of $\mathbf{M}$. We speak of a *pseudogroup* Γ of (local) transformations on $\mathbf{M}$ if the following requirements are satisfied (see e.g. [2], p. 150):

(ι) the identity id : $\mathcal{D}(id) = \mathbf{M} \to \mathcal{R}(\mathrm{id}) = \mathbf{M}$ belongs to Γ;

$(\iota\iota)$ if $\varphi \in \Gamma$ and $\mathcal{D} \subset \mathbf{M}$ is an open subspace then the restriction of φ to the subspace $\mathcal{D}(\varphi) \cap \mathcal{D}$ belongs to Γ;

$(\iota\iota\iota)$ if $\varphi \in \Gamma$ then $\varphi^{-1} \in \Gamma$;

$(\iota\nu)$ if $\varphi, \psi \in \Gamma$ and $\mathcal{R}(\varphi) \cap \mathcal{D}(\psi) \neq 0$ then the composition

$$\psi \circ \varphi : \varphi^{-1}(\mathcal{R}(\varphi) \cap \mathcal{D}(\psi)) \to \psi(\mathcal{R}(\varphi) \cap \mathcal{D}(\psi))$$

belongs to Γ;

(ν) if $\chi : \mathcal{D} \to \mathcal{R}$ is a local homeomorphism between open subspaces of $\mathbf{M}$ such that χ *locally coincides* with mappings from Γ, then $\chi \in \Gamma$. (In more detail: we suppose that to every $P \in \mathcal{D}$ there exists $\varphi \in \Gamma$ such that $\mathcal{D}(\varphi)$ is a neighbourhood of P and $\chi = \varphi$ on $\mathcal{D}(\varphi)$.)

In particular case when $\mathcal{D}(\varphi) = \mathcal{R}(\varphi) = \mathbf{M}$ for all $\varphi \in \Gamma$, we have the common *transformation group* Γ on $\mathbf{M}$. In general, the pseudogroups were alternatively (and a little misleadingly) named *groups of local diffeomorphisms* in classical mathematics. For a long time, they belong to indispensable tools in nonlinear theories where the definition domains cause many difficulties. Only rather particular classes of pseudogroups are appearing in common applications, namely the so called *Lie-Cartan pseudogroups* where the transformations either are defined by a system of differential equations (the Lie's approach) or, alternatively, by the property of preserving a certain family of functions and differential forms (the E. Cartan's approach). We follow the second point of view here.

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