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APPROXIMATION OF SOME REGULAR DISTRIBUTION IN $\mathcal{S}'(\mathcal{R})$ BY FINITE, CONVEX, LINEAR COMBINATIONS OF BLASCHKE DISTRIBUTIONS

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0. Introduction

0.1. Some background on Blaschke products and Marshall's theorem for functions on H^∞.

Let U be the open, unit disc in the plane, $T = \partial U$. $H^\infty(U)$ is the space of all bounded analytic functions $f(z)$ on U, for which the norm is defined by

$$\|f\|_{H^\infty} = \sup_{z\in U} |f(z)|\,.$$

If $f \in H^\infty(U)$, then the radial boundary function

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

is defined almost everywhere on T with respect to the Lebesque measure on T and $\log|f^*(e^{i\theta})| \in L^1(T)$.

Let $\{z_n\}$ be a sequence of points in U such that

$$\sum_{n=1}^{\infty}(1-|z_n|) < \infty\,. \tag{0.1.1}$$

Let m be the number of z_n equal to 0. Then the infinite product

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|}\,\frac{z-z_n}{1-\overline{z}_n z} \tag{0.1.2}$$

converges on U. The function $B(z)$ of the form (0.1.2) is called Blaschke product. $B(z)$ is in $H^\infty(U)$, and the zeros of $B(z)$ are precisely the points z_n, each zero

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having multiplicity equal the number of times it occurs in the sequence $\{z_n\}$. Moreover $|B(z)| \le 1$ and $|B^*(e^{i\theta})| = 1$ a.e.

For the needs of our subsequent work we will define the Blaschke product in the upper half plane Π^+. In the upper half plane Π^+, condition (0.1.1) is replaced by

$$\sum_{n=1}^{\infty} \frac{y_n}{1+|z_n|^2} < \infty\,, \quad z_n = x_n + iy_n \in \Pi^+ \tag{0.1.3}$$

and the Blaschke product with zeros z_n is

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{n=1}^{\infty} \frac{|z_n^2+1|}{z_n^2+1}\,\frac{z-z_n}{z-\overline{z}_n}\,. \tag{0.1.4}$$

Note. If the number of zeros z_n in (0.1.2) or (0.1.4) is finite, then we call $B(z)$ finite Blaschke product.

For the needs of our subsequent work we will state the Marshall's theorem for approximation of functions of $H^\infty(U)$ by finite, convex, linear combinations of Blaschke products. The theorem is given in [6].

Marshall's theorem. *Let $f \in H^\infty(U)$ and $\|f\|_{H^\infty} \le 1$. Then for every $\varepsilon > 0$, there are Blaschke products $B_1(z), B_2(z), \dots, B_n(z)$ and positive numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, $\sum\limits_{k=1}^{n} \lambda_k = 1$ such that*

$$\|f(z) - \sum_{k=1}^{n} \lambda_k B_k(z)\|_{H^\infty} < \varepsilon\,.$$

0.2. Some notions of distributions and Blaschke distribution.

For a function f, $f\colon \Omega \to C^n$, $\Omega \subseteq \mathcal{R}^n$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \in \mathcal{N} \cup \{0\}$, $x \in \Omega$, $D_x^\alpha f$ denotes the differential operator

$$D_x^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}\,, \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n\,.$$

$C^\infty(\mathcal{R}^n)$ denotes the space of all complex valued infinitely differentiable functions on $\mathcal{R}^n$ and $C_0^\infty(\mathcal{R}^n)$ denotes the subspace of $C^\infty(\mathcal{R}^n)$ that consists of those functions of $C^\infty(\mathcal{R}^n)$ which have compact support. Support of a function f, denoted by $\operatorname{supp}(f)$, is the closure of $\{x \mid f(x) \ne 0\}$ in $\mathcal{R}^n$.

$D = D(\mathcal{R}^n)$ denotes the space of $C_0^\infty(\mathcal{R}^n)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in D as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset \mathcal{R}^n$ such that $\operatorname{supp}(\varphi_\lambda) \subseteq K$ for each λ, $\operatorname{supp}(\varphi) \subseteq K$ and for every n-tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on K as $\lambda \to \lambda_0$.

$D' = D'(\mathcal{R}^n)$ is the space of all continuous linear functionals on D, where continuity means that $\varphi_\alpha \to \varphi$ in D as $\lambda \to \lambda_0$ implies $\langle T, \varphi_\lambda \rangle \to \langle T, \varphi \rangle$ as $\lambda \to \lambda_0$, $T \in D'$.

Note. $\langle T, \varphi \rangle$ denotes the value of the functional T, when it acts on the function φ.

D' is called the space of distributions.

$S = S(\mathcal{R}^n)$ denotes the space of all infinitely differentiable complex valued function φ on $\mathcal{R}^n$ satisfying

$$\sup_{t \in \mathcal{R}^n} |t^\beta D^\alpha \varphi(t)| < \infty$$

for all n-tuple α and β of nonnegative integers. Convergence in S is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in S$ converges to $\varphi \in S$ in S as $\lambda \to \lambda_0$ if and only if

$$\lim_{\lambda \to \lambda_0} \sup_{t \in \mathcal{R}^n} |t^\beta D_t^\alpha [\varphi_\lambda(t) - \varphi(t)]| = 0$$

for all n-tuple α and β of nonnegative integers.

Again, S' is the space of all continuous, linear functionals on S, called the space of tempered distributions.

Let φ be an element of one of the above function spaces D or S, and f be a function for which

$$\langle T_f, \varphi \rangle = \int_{\mathcal{R}^n} f(t)\varphi(t)\, dt\,, \quad \varphi \in D \ \ (\varphi \in S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f.

Now, let $B(z)$ be the Blaschke product, $z = x + iy \in \Pi^+$, with zeros z_n that belong to the upper half plane. In [7] it is proven that $\langle B^+, \varphi \rangle$, where

$$\langle B^+, \varphi \rangle = \lim_{y \to 0^+} \int_{-\infty}^{\infty} B(z)\varphi(x)\, dx\,, \quad z = x + iy \in \Pi^+, \quad \varphi \in D(\mathcal{R})\,, \tag{0.2.1}$$

is distribution on D, named upper Blaschke distribution on D.

Note. This is a new notion in the theory of distributions and has useful application in the problems of approximation. The introduced Blaschke distributions in [7] were used for representing some distributions in D' as a limit of sequence of Blaschke distributions.

The following theorem gives another application of the Blaschke distribution.

1. Main result

Theorem 1.1. *Let $f(z) \in H^\infty(\Pi^+)$ and $\|f\|_{H^\infty} \leq 1$. Let T_{f^*} be the distribution in $S'(\mathcal{R})$ generated with the boundary value f^* of the function $f(z)$. Then* *for every* $\varepsilon > 0$, *and for every* $\varphi \in S(\mathcal{R})$ *there are upper Blaschke distributions* $B_1^+, B_2^+, \dots, B_n^+$ *and positive numbers* $\lambda_1, \lambda_2, \dots, \lambda_n$, $\sum_{k=1}^{n} \lambda_k = 1$ *such that*

$$\left| \langle T_{f^*}, \varphi \rangle - \sum_{k=1}^{n} \lambda_k \langle B_k^+, \varphi \rangle \right| < \varepsilon \,. \tag{1.1}$$

Proof. Let $f(z) \in H^\infty(\Pi^+)$, $\|f\|_{H^\infty} \le 1$. Let $\varepsilon > 0$ and $\varphi \in S(\mathcal{R})$ be arbitrary chosen. Because $S(\mathcal{R}) \subset L^1(\mathcal{R})$, it follows that $\varphi \in L^1(\mathcal{R})$.

Let $\varepsilon_1 = \dfrac{\varepsilon}{\|\varphi\|_{L^1}} > 0$. Then because of the Marshal theorem, there are Blaschke products $B_1(z), B_2(z), \dots, B_n(z)$ with zeros in the upper half plane Π^+, and positive numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, $\sum_{k=1}^{n} \lambda_k = 1$ such that

$$\left\| f(z) - \sum_{k=1}^{n} \lambda_k B_k(z) \right\|_{H^\infty} < \varepsilon_1 \,. \tag{1.2}$$

From (1.2), we have that for $B_k(z)$, λ_k, $k \in \{1, 2, \dots, n\}$ hold

$$\left| f(z) - \sum_{k=1}^{n} \lambda_k B_k(z) \right| < \varepsilon_1 \,, \quad \forall z \in \Pi^+ \,. \tag{1.3}$$

Because the Blaschke products $B_1(z), B_2(z), \dots, B_n(z)$ have zeros in Π^+, they define upper Blaschke distributions $B_1^+, B_2^+, \dots, B_n^+$ respectively, as in [7]. Now, let

$$\langle B_k^+, \varphi \rangle = \lim_{y \to 0^+} \int_{-\infty}^{\infty} B_k(z) \varphi(x) \, dx, \quad z = x + iy \in \Pi,^+ \quad \varphi \in S(\mathcal{R}) \,. \tag{1.4}$$

We will prove that $B_k^+ \in S'(\mathcal{R})$, for $k \in \{1, 2, \dots, n\}$. Because of the theorem of characterization of tempered distributions given in [8], it is enough to prove that $B_k^+ * \alpha$ are continuous and bounded functions on $\mathcal{R}$, for every $\alpha \in D(\mathcal{R})$. So, let $\alpha \in D(\mathcal{R})$, $\operatorname{supp}(\alpha) = K$, $t \in \mathcal{R}$ and $K_1 = t - K$. Then

$$\begin{aligned}
(B_k^+ * \alpha)(t) &= \langle B_{kx}^+, \ \alpha(t - x) \rangle \\
&= \lim_{y \to 0^+} \int_{-\infty}^{\infty} B_k(x + iy) \alpha(t - x) \, dx \\
&= \lim_{y \to 0^+} \int_{K_1} B_k(x + iy) \alpha(t - x) \, dx \,.
\end{aligned}$$

First, we will show that $B_k^+ * \alpha$ is bounded function on $\mathcal{R}$:

$$
\begin{aligned}
|(B_k^+ * \alpha)(t)| &= \Big| \lim_{y\to 0^+} \int\limits_{K_1} B_k(x+iy)\alpha(t-x)\,dx \Big| \overset{|B_k(x+iy)|\le 1}{\le} \int\limits_{K_1} |\alpha(t-x)|\,dx \\
&\le M \cdot m(K) < \infty\,,
\end{aligned}
$$

where $m(K)$ is the Lebesque measure of K.

Now, we will prove the continuity of $B_k^+ * \alpha$ on $\mathcal{R}$. Let $\varepsilon > 0$, $t_0 \in \mathcal{R}$ and let $K_0 = t_0 - K$. Since α is continuous, there exists $\delta > 0$, so that $|t - t_0| < \delta$ implies $|\alpha(t) - \alpha(t_0)| < \varepsilon$ i.e. if $x \in \mathcal{R}$ is any real number, the last is equivalent with: there exists $\delta > 0$, so that $|(t-x)-(t_0-x)| < \delta$ implies $|\alpha(t-x) - \alpha(t_0-x)| < \varepsilon$. Now

$$
\begin{aligned}
&|(B_k^+ * \alpha)(t) - (B_k^+ * \alpha)(t_0)| \\
&\quad = \Big| \lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} B_k(x+iy)\alpha(t-x)\,dx - \lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} B_k(x+iy)\alpha(t_0-x)\,dx \Big| \\
&\quad \le \Big| \lim_{y\to 0^+} \int\limits_{K_2} B_k(x+iy)[\alpha(t-x) - \alpha(t_0-x)]\,dx \Big| \overset{|B_k(x+iy)|\le 1}{\le} \\
&\quad \le \int\limits_{K_2} |\alpha(t-x) - \alpha(t_0-x)|\,dx \\
&\quad \le \varepsilon(2m(K) + \delta) = \varepsilon_1 \quad \text{when} \quad |t - t_0| < \delta
\end{aligned}
$$

(K_2 is a compact set that contains K_0 i K_1.)

On the other hand, using the properties of the space H^∞, it is clear that the boundary function f^* of the function $f(z)$ exists, $f^* \in L^\infty$ and $f(x+iy) \to f^*(x)$, in L^∞, as $y \to 0^+$, $x + iy \in \Pi^+$.

Even more, theorem 5.3 in [3] claims that $f(x + iy) \to f^*(x)$ in $S'(\mathcal{R})$, as $y \to 0^+, x + iy \in \Pi^+$ i.e.

$$
\lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} f(x+iy)\varphi(x)\,dx = \langle T_{f^*}, \varphi\rangle, x+iy \in \Pi^+\,, \quad \varphi \in S(\mathcal{R})\,. \tag{1.5}
$$

Now, we get that

$$
\begin{aligned}
&\Big| \langle T_{f^*}, \varphi\rangle - \sum_{k=1}^{n} \lambda_k \langle B_k^+, \varphi\rangle \Big| \\
&\quad \overset{(1.4)}{\underset{(1.5)}{=}} \Big| \lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} f(x+iy)\varphi(x)\,dx - \sum_{k=1}^{n} \lambda_k \lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} B_k(x+iy)\varphi(x)\,dx \Big|
\end{aligned}
$$

$$
\begin{aligned}
&= \Big| \lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)\,dx - \lim_{y\to 0^+} \sum_{k=1}^{n} \lambda_k \int_{-\infty}^{\infty} B_k(x+iy)\varphi(x)\,dx \Big| \\
&= \Big| \lim_{y\to 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x)\,dx - \lim_{y\to 0^+} \int_{-\infty}^{\infty} \Big[\sum_{k=1}^{n} \lambda_k B_k(x+iy) \Big] \varphi(x)\,dx \Big| \\
&= \Big| \lim_{y\to 0^+} \int_{-\infty}^{\infty} \Big[f(x+iy) - \sum_{k=1}^{n} \lambda_k B_k(x+iy) \Big] \varphi(x)\,dx \Big| \\
&\le \lim_{y\to 0^+} \int_{-\infty}^{\infty} \Big| f(x+iy) - \sum_{k=1}^{n} \lambda_k B_k(x+iy) \Big| \, |\varphi(x)|\,dx \\
&\overset{(1.3)}{\le} \lim_{y\to 0^+} \int_{-\infty}^{\infty} \varepsilon_1 |\varphi(x)|\,dx = \varepsilon_1 \int_{-\infty}^{\infty} |\varphi(x)|\,dx \\
&= \varepsilon_1 \|\varphi\|_{L^1} = \frac{\varepsilon}{\|\varphi\|_{L^1}} \|\varphi\|_{L^1} = \varepsilon\,.
\end{aligned}
$$

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