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A NOTE ON BIDIFFERENTIAL CALCULI AND BIHAMILTONIAN SYSTEMS

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Abstract. In this note we discuss the geometrical relationship between bi-Hamiltonian systems and bi-differential calculi, introduced by Dimakis and Möller–Hoissen.

1. Introduction

It is known that practically all the classical integrable systems may be described in terms of a pair of compatible Poisson structures on the phase space. Such a pair is called a bihamiltonian structure. Several interesting features of integrable systems can be described in terms of bihamiltonian structure.

In this note we will establish a link between the bi-differential calculi and bi-Hamiltonian systems. The proximity between these subjects has long been legendary, yet little has been written about this. Here I hope to shed some light on this issue.

In a series of paper Dimakis and Müller–Hoissen [2,3] and the references therein, have shown how to generate conservation laws in completely integrable systems by using a bi-differential calculus. Their papers are quite interesting. But the mathematical foundation of these papers are not clear, for example, they never considered the geometry behind their bi-differential formalism. Some attempts have been made by Crampin et. al [1]. They clarified the geometry behind the formalism of Dimakis and Müller–Hoissen.

In this article, I further investigate the geometrical structure of the bidifferential calculi and bicomplex formalism.

The paper is organized as follows. In next section we discuss about background material. In section 3 we discuss about the bidifferential calculi and its connection to bi-Hamiltonian systems [4].

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2. Background

Let M be a smooth manifold. The cotangent bundle of a manifold M is a vector bundle $T^*M := (TM)^*$, the (real) dual of the tangent bundle TM.

A differential form or an exterior form of degree k is a section of the vector bundle $\wedge^k T^*M$, the space of all k-forms, will be denoted by $\Omega^k(M)$. We put $\Omega^0(M) = C^\infty(M, \mathbf{R})$, then the space

$$\Omega(M) := \oplus_{k=0}^{n} \Omega^k(M)$$

is a graded commutative algebra. Let $\mathrm{Der}_k\,\Omega(M)$ the space of all (graded) derivation of degree k, so that $D \in \mathrm{Der}_k\,\Omega(M)$ satisfies $D : \Omega(M) \longrightarrow \Omega(M)$ with $D(\Omega^l(M)) \subset \Omega^{k+l}(M)$. For $k = 1$ we obtain the ordinary exterior derivative d.

We consider the space $\Omega(M, TM) = \oplus_{k=0}^{m} \Omega^k(M, TM)$ of all tangent bundle valued differential form on M. Also $\Omega(M, TM)$ is a graded Lie algebra with the Frölicher-Nijenhuis bracket

$$[\cdot,\cdot] \ : \ \Omega^k(M, TM) \times \Omega^l(M, TM) \longrightarrow \Omega^{k+l}(M, TM)\,. \tag{1}$$

The Frölicher-Nijenhuis operator δ is given by

$$\delta \ : \ \Omega^k(M, TM) \longrightarrow \Omega^{k+1}(M, TM)\,. \tag{2}$$

If $d \ : \ \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ be the exterior derivative the operator $\delta(K)$ for $K \in \Omega^k(M, TM)$ can be expressed as

$$\delta(K) := (-1)^{k-1} dc(K) \wedge A$$

where c is the contraction map

$$c \ : \ \Omega^k(M, TM) \longrightarrow \Omega^{k-1}(M)\,, \tag{3}$$

such that $c(\phi \otimes X) = i_X \phi$, and $A \in \Omega^1(M, TM)$.

3. Bidifferential calculi and bihamiltonian structure

In this section we will address our recipe. We will build an inductive scheme with the help of the exterior derivative d and another degree 1 derivation operator d_A, this is given below:

Construction of d_A. : Let us consider an action of $\wedge A$:

$$\wedge A \ : \ C^\infty(\wedge^k T^*M) \longrightarrow C^\infty(\wedge^{k+1} T^*M \otimes TM). \tag{4}$$

Combining (3) and (4) we define a new degree 0 operator

$$A(c) := c \circ \wedge A\,, \tag{5}$$

so that $A(c) : C^\infty(\wedge^k T^*M) \longrightarrow C^\infty(\wedge^k T^*M)$.

Hence, we think $A(c)$ as a homomorphism of the module of differential forms. Also, from the definition $A(c)$ can be identified with a tensor field of rank $(1, 1)$.

Definition 3.1.

$$d_A \ := \ A(c)d\,. \tag{6}$$

It is clear that d_A is a degree 1 operator.

The basic step in the construction of Dimakis and Müller–Hoissen is to define inductively a sequence of $(l-1)$-th forms

$$\{\mu^k\} \qquad k = 0, 1, 2, \dots$$

for which closed l-forms are exact by the rule given by

Lemma 3.2.

$$d\mu^{k+1}(M) = d_A\mu^k(M) \qquad \mu^k \in C^\infty(\wedge^l T^*M)\,. \tag{7}$$

According to Frölicher-Nijenhuis theory, an operator d_A associated to some $(1,1)$ tensor A, anticommutes with d. The necessary and sufficient condition for d_A to satisfy $d_A^2 = 0$ is that the Nijenhuis tensor must be zero.

Claim 3.3.

$$d^2 = {d_A}^2 = 0\,.$$

$$dd_A + d_Ad = 0\,.$$

It is easy to see that

$$dd_A\mu^k = -d_Ad\mu^k = -d_Ad_A\mu^{k+1} = -{d_A}^2\mu^{k+1} = 0\,. \tag{8}$$

This scheme is consistent provided $dd_A\mu^0 = -d_Ad\mu^0 = 0$.

Thus all the μ^ks are defined on the space $\Omega(M)/B(M)$ of differential forms modulo exact forms. These defined a generalized Poisson structure, the graded Poisson bracket. In the case of one form, entire picture coincides with the Poisson geometry.

3.1 Connection to the Poisson-Nijenhuis manifold and bi-Hamiltonian systems.

In this section we will state the correspondence with the bi-Hamiltonian systems. Let us consider a manifold M with symplectic structures ω_0. Then ω_0 induces a nondegenerate Poisson structure from the following canonical identification:

$$\omega_0(X_f, X_g) = {\Lambda_0}^{-1}(df, dg)\,.$$

Our basic structure $(\omega_0, A(c))$ induces a second Poisson structure on M. This is given by

$$\Lambda_1(df, dg) = \Lambda_0(A(c)df, dg), \tag{9}$$

where $A(c) \ : \ T^*M \longrightarrow T^*M$.

Given two vector bundle morphisms

$$J_{\Lambda_0}, J_{\Lambda_1} \ : \ T^*M \longrightarrow TM\,,$$

we can determine the mixed $(1,1)$ tensor (recursion operator)

$$A \ = \ J_{\Lambda_0} J_{\Lambda_1}^{-1}\,.$$

By abusing notation, let us denote the adjoint of $A(c)$ by A, it acts on the vector fields.

Definition 3.4. Let A be a tensor field of type $(1,1)$ on a manifold M. The Nijenhuis torsion of A is a tensor field $N(A)$ of type $(1,2)$ given, for any pair (X,Y) of vector fields on M, by

$$N(A)(X,Y) = [AX, AY] - A([AX,Y] + [X,AY] - A[X,Y]), \tag{10}$$

$N(A) = \frac{1}{2}[A,A]$ for the Frölicher-Nijenhuis bracket.

The tensor field A would be called Nijenhuis operator if its Nijenhuis torsion $N(A)$ vanishes.

The torsion of A vanishes as a consequence of the assumption that Λ_0 and Λ_1 are a pair compatible Poisson tensors.

Thus we obtain two Poisson bivectors $\Lambda_0(df,dg)$ and $\Lambda_1(df,dg)$, satisfying $[\Lambda_i,\Lambda_j]=0$, where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket. In this way we construct a Poisson-Nijenhuis manifold. A Poisson-Nijenhuis manifold is a bihamiltonian manifold.

Thus we define two symplectic structures

$$\omega_0(X_f,X_g) = {\Lambda_0}^{-1}(df,dg) \quad \text{and} \quad \omega_1(X_f,X_g) = {\Lambda_1}^{-1}(df,dg) \quad \text{on } M.$$

We have the following exact sequence

$$0 \longrightarrow H^0(M,\mathbf{R}) \longrightarrow C^\infty(M,\mathbf{R}) \xrightarrow{H} \mathfrak{V}(M) \xrightarrow{\gamma} H^1(M,\mathbf{R}) \longrightarrow 0 \tag{11}$$

Here $\gamma(\eta)$ is the cohomology class of $i_\eta\omega$, and $\mathfrak{V}(M)$ consists of all vector fields ξ with $\mathcal{L}_\xi\omega = 0$.

Thus we have two Poisson structures.

$$\begin{aligned}
\{f,g\}_0 &= \Lambda_0(df,dg)\,,\\
\{f,g\}_1 &= \Lambda_1(df,dg) = \Lambda_0(A^*(df),dg)\\
&= \Lambda_0(df,A^*(dg)) = -A(X_g)f = -d_Af(X_g)\,.
\end{aligned} \tag{12}$$

Hence, we say, a bi-differential calculus endows M with a Poisson-Nijenhuis structure, and A plays the role of recursion tensor [5].

3.2 Graded Poisson Structure.

In our case all the μ^k-s are graded objects, differential forms. Now, if we replace f by μ^{k+1} in equation (11), then from the inductive definition of the function μ^k, we obtain

$$\{\cdot,\mu^{k+1}\}_1 = \{\cdot,\mu^k\}_0\,. \tag{13}$$

The graded Poisson bracket for differential forms in the context of generalized Hamiltonian systems has been studied extensively by Peter Michor [6]. He extended the Poisson exact sequence to

$$0 \to H^0(M,\mathbf{R}) \to \Omega(M)/B(M) \xrightarrow{H} \Omega_\omega(M;TM) \xrightarrow{\gamma} H^{*+1}(M,\mathbf{R}) \to 0\,. \tag{14}$$

Theorem 3.5 (Michor)**.** *Let (M, Λ) be a Poisson manifold. Then the space $\Omega(M)/B(M)$ of differential forms modulo exact forms there exists a unique graded Poisson bracket $\{\cdot,\cdot\}_{gr}$, which is given the quotient modulo $B(M)$ of*

$$\{\phi, \psi\}_{gr} = i_{H_\phi} d\psi \,,$$

or

$$\begin{aligned} &\{f_0 df_1 \wedge \cdots \wedge df_k, g_0 dg_1 \wedge \cdots \wedge dg_l\}_{gr} \\ &\quad = \sum_{i,j} (-1)^{i+j} \{f_i, g_j\} df_0 \wedge \cdots \widehat{df_i} \cdots \wedge df_k \wedge dg_0 \wedge \cdots \widehat{dg_j} \cdots \wedge dg_k \,, \end{aligned} \tag{15}$$

such that $H : \Omega(M)/B(M) \longrightarrow \Omega(M;TM)$ is a homomorphism of graded Lie algebras.

The functions μ^k form a Lenard scheme.

There is an alternative bihamiltonian approach to dynamical systems. In this approach one starts with two compatible Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$ on M. The two Poisson brackets are compatible if the bracket $\lambda_1\{.,.\}_1 + \lambda_2\{.,.\}_2$ is Poisson for λ_1 and λ_2. One can construct based on these brackets a dynamical systems which is Hamiltonian with respect to any one of these brackets. The construction of dynamical systems based on the brackets is called *Lenard Scheme*. It provides a family of function in involution (w.r.t. any linear combination of the brackets).

Proposition 3.6. *The functions μ^k which obey the Lenard scheme are in involution with respect to both Poisson brackets.*

Proof. By using repeatedly the recursion relation we obtain,

$$\begin{aligned} \{\mu^j, \mu^k\}_1 &= \{\mu^j, \mu^{k-1}\}_0 \\ &= -\{\mu^{k-1}, \mu^j\}_0 \\ &= -\{\mu^{k-1}, \mu^{j+1}\}_1 \\ &= \{\mu^{j+1}, \mu^{k-1}\}_1 = \cdots = \{\mu^{j+k+1}, \mu^{-1}\}_1 = 0 \,. \end{aligned}$$

□

Hence their property of being in involutions then follows from the general argument (explained in the third lecture in [5]).

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