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IDEAL-THEORETIC CHARACTERIZATIONS OF VALUATION AND PRÜFER MONOIDS

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Abstract. It is well known that an integral domain is a valuation domain if and only if it possesses only one finitary ideal system (Lorenzen r-system of finite character). We prove an analogous result for root-closed (cancellative) monoids and apply it to give several new characterizations of Prüfer (multiplication) monoids and integral domains.

1. Introduction and Preliminaries

It is well known that a great part of classical valuation theory and the theory of valuation rings can be formulated in a purely multiplicative context. For this point of view, the reader is refered to [3], Chap. 15 ff and to the survey article [6]. The central notion in this purely multiplicative theory ist the that of a valuation monoid, and the theory of ideal systems (Lorenzen r-systems) has to take the place of ordinary ideal theory. The theory of ideal systems on a valuation monoid is very simple. There the system of ordinary semigroup ideals is the only finitary ideal system. It was proved by K. E. Aubert [1] that valuation rings can be characterized by this property. In this note, we show that this is no longer the case for valuation monoids, and we also show which additional condition is necessary.

This paper is organized as follows. In this first section we recall the necessary facts from the theory of ideal systems. In section 2, we recall the results of K. E. Aubert (Theorem 1) and give the promised characterization of valuation monoids by means of their ideal systems (Theorem 2). In section 3 we globalize this characterization by means of spectral ideal systems in order to obtain new ideal-theoretic characterizations of Prüfer monoids (Theorem 3) and Prüfer domains (Theorem 4).

By a *monoid* D we always mean a commutative multiplicative semigroup possessing a unit element $1 \in D$ (such that $1a = a$ for all $a \in D$), a zero element $0 \in D$ (such that $0a = 0$ for all $a \in D$), and satisfying the cancellation law (if

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$a, b, c \in D$ and $ab = ac$, then either $a = 0$ or $b = c$). We set $D^\bullet = D \setminus \{0\}$ and denote by $D^\times$ the group of invertible elements of D. By an *quotient groupoid* of D we mean an overmonoid $K \supset D$ such that $K^\bullet$ is a quotient group of $D^\bullet$, that means, $K = \{a^{-1}b \mid a \in D^\bullet,\ b \in D\}$. For any subsets $X, Y \subset K$, we set

$$(X : Y) = \{z \in K \mid zY \subset X\} \quad \text{and} \quad X^{-1} = (D : X)\,.$$

If D is an integral domain with quotient field K, then (disregarding the additive structure) D is a monoid with quotient groupoid K.

We shall consider ideal systems (Lorenzen r-systems) on a monoid D as defined in [3], and we shall throughout use the terminology and notations introduced there. In particular, for an ideal system r on D, we denote by $\mathcal{I}_r(D)$ the set of all r-ideals and by $\mathcal{I}_{r,\mathrm{fin}}(D)$ the set of all r-finitely generated r-ideals of D. $\mathcal{I}_r(D)$ and $\mathcal{I}_{r,\mathrm{fin}}(D)$ are commutative semigroups with respect to the r-multiplication, defined by $I \cdot_r J = (IJ)_r$ (where $IJ = \{xy \mid x \in I, y \in J\}$). The ideal system r is called finitely cancellative if $\mathcal{I}_{r,\mathrm{fin}}(D)$ satisfies the cancellation law for the r-multiplication. If q and r are ideal systems on D, we write $q \le r$ and call q finer than r, if $\mathcal{I}_r(D) \subset \mathcal{I}_q(D)$ (equivalently: $X_q \subset X_r$ for all $X \subset D$). An ideal system r on D is called finitary if

$$X_r = \bigcup_{E \in \mathbb{P}_{\mathrm{fin}}(X)} E_r \quad \text{for all} \quad X \subset D\,,$$

where $\mathbb{P}_{\mathrm{fin}}(X)$ denotes the set of all finite subsets of X.

On a monoid D, we consider the the ideal system $v(D)$ of divisorial ideals and the finitary ideal system $t(D)$ and $s(D)$, defined by $X_{v(D)} = (X^{-1})^{-1}$, $X_{s(D)} = XD$ and $X_{t(D)} = \bigcup\{E_{v(D)} \mid E \in \mathbb{P}_{\mathrm{fin}}(X)\}$. Recall that, for any ideal system r on D we have $s(D) \le r \le v(D)$, and if r is finitary, then even $r \le t(D)$. On an integral domain D, we shall also consider the finitary ideal system $d(D)$ of ordinary ring ideals.

Let D be a monoid, r a finitary ideal system on D and K a quotient groupoid of D. D is called r-closed if $(J : J) = D$ for all non-zero r-finitely generated r-ideals J of D, and D is called root-closed if, for all $x \in K$ and $n \ge 1$, $x^n \in D$ implies $x \in D$. A monoid D is $s(D)$-closed if and only if it is root-closed, and an integral domain D is $d(D)$-closed if and only if it is integrally closed. On an r-closed monoid D, the finitary ideal system r_{a} is defined by

$$X_{r_{\mathrm{a}}} = \bigcup_{\substack{B \in \mathbb{P}_{\mathrm{fin}}(D)\\ B \cap D^\bullet \ne \emptyset}} ((XB)_r : B) \quad \text{for all} \quad X \subset D\,.$$

The ideal system r_{a} is finitely cancellative, $r \le r_{\mathrm{a}}$, and the importance of the ideal system r_{a} is given by the following result.

Proposition 1. *Let D be a monoid and r a finitary ideal system on D. Then r is finitely cancellative if and only if D is r-closed and $r = r_a$.*

Proof. If r is finitely cancellative, then [3], Theorem 13.3 shows that $(J : J) = D$ for all $J \in \mathcal{I}_{r,\mathrm{fin}}(D)$ and hence D is r-closed. If D is r-closed, then D is r-cancellative if and only if $r = r_{\mathrm{a}}$ by [3], Theorem 19.1. □

We shall now consider the ideal system s_{a} on a root-closed monoid more closely and use it to characterize valuation monoids. For a subset X of a monoid and $n \geq 1$, we set

$$X^n = \{x_1 \cdot \ldots \cdot x_n \mid x_\nu \in X\} \quad \text{and} \quad X^{[n]} = \{x^n \mid x \in X\}\,.$$

Proposition 2. *Let D be a root-closed monoid and $X \subset D$.*

1. $X_{s_{\mathrm{a}}} = \{x \in D \mid x^n \in X^n \quad \textit{for some} \quad n \geq 1\}\,.$
2. $(X^n)_{s_{\mathrm{a}}} = (X^{[n]})_{s_{\mathrm{a}}}\,.$

Proof. 1. By [3], Proposition 19.3.

2. Since $X^{[n]} \subset X^n$, it is sufficient to show that $X^n \subset (X^{[n]})_{s_{\mathrm{a}}}$. If $x = x_1 \cdot \ldots \cdot x_n \in X^n$ (where $x_\nu \in X$), then

$$x^n = x_1^n \cdot \ldots \cdot x_n^n \in (X^{[n]})^n$$

and therefore $x \in (X^{[n]})_{s_{\mathrm{a}}}$ by definition. □

2. Valuation monoids

A monoid D is called a valuation monoid if, for all $a, b \in D$, either $a \in bD$ or $b \in aD$. An integral domain is a valuation domain if its multiplicative monoid is a valuation monoid. The following theorem gathers the known facts concerning the ideal theory of valuation monoids.

Theorem 1. *If D is a valuation monoid, then $s(D) = t(D)$ (and consequently this is the only finitary ideal system on D).*

If D is an integral domain and $s(D) = d(D)$, then D is a valuation domain.

Proof. The first assertion is proved in [3], Theorem 15.3, and the second one follows by [1], Theorem 1 or [3], Ex. 15.2. □

Note that assetion 2. of Theorem 2 generalizes to rings with zero divisors, see [1], Lemma 3 or [4], Lemma 5.3. The following example however shows the existence of a monoid possessing but one ideal system and yet not being a valuation monoid.

Example. A monoid D satisfying $s(D) = v(D)$ (and thus admitting only one ideal system at all) which is yet not a valuation monoid.

We consider the multiplicative monoid

$$D = \{2^n, -2^n \mid n \geq 1\} \cup \{0, 1\} \subset (\mathbb{Z}, \cdot)\,.$$

Since $2 \notin (-2)D$ and $-2 \notin 2D$, D is not a valuation monoid. Let $M = D \setminus \{1\}$ be the maximal s-ideal of D. Then the non-principal s-ideals of D are the ideals

$2^n M$ for $n \geq 0$. Indeed, if J is a non-principal s-ideal of D and $n \geq 1$ is minimal such that $2^n \in J$ or $-2^n \in J$, then (as J is not principal) $\{2^n, -2^n\} \subset J$ and therefore $J = 2^{n-1}M$. Hence it suffices to prove that M is a v-ideal. It is easily checked that $K = \{2^n, -2^n \mid n \in \mathbb{Z}\} \cup \{0\}$ is a quotient groupoid of D, $M^{-1} = \{2^n, -2^n \mid n \geq 0\} \cup \{0\}$ and $M_v = (M^{-1})^{-1} = M$.

Theorem 2. *For a monoid D, the following assertions are equivalent:*

a) *D is a valuation monoid.*
b) *D is root-closed, and $s(D) = t(D)$.*
c) *D is root-closed, and $s(D) = s(D)_{\mathrm{a}}$.*

Proof. a) $\Longrightarrow$ b) By Theorem 1, since every valuation monoid is root-closed.

b) $\Longrightarrow$ c) Obvious, since $s(D) \leq s(D)_{\mathrm{a}} \leq t(D)$.

c) $\Longrightarrow$ a) If $a, b \in D^\bullet$, then Proposition 2 implies

$$ab \in (\{a,b\}^2)_{s(D)} = (\{a,b\}^{[2]})_{s(D)} = (\{a^2, b^2\})_{s(D)} = a^2 D \cup b^2 D$$

and therefore $ab \in a^2 D$ or $ab \in b^2 D$, whence $b \in aD$ or $a \in bD$. □

3. r-Prüfer monoids and domains

We recall the notion of spectral ideal systems from [4]. Let D be a monoid, and let r and q be finitary ideal systems on D such that $q \leq r$. Then $r[q] : \mathbb{P}(D) \to \mathbb{P}(D)$ is defined by

$$X_{r[q]} = \bigcup_{\substack{E \in \mathbb{P}_f(D) \\ E_r = D}} (X_q : E) = \bigcap_{P \in r\text{-}\max(D)} (X_q)_P \quad \text{for} \quad X \subset D\,,$$

where r-max$\,(D)$ denotes the set of all r-maximal r-ideals of D, and $(\cdot)_P = (D \setminus P)^{-1}(\cdot)$ denotes the localization with respect to D. For the convenience of the reader we recall the main properties of $r[q]$, for details see [4], section 3.

Proposition 3. *Let D be a monoid, and let r and q be finitary ideal systems on D such that $q \leq r$.*

1. *$r[q]$ is a finitary ideal system on D satisfying $q \leq r[q] \leq r$.*
2. *If $P \in r$-max$\,(D)$, then $r[q]_P = q_P$.*
3. *$r[q] = r$ holds if and only if $r_P = q_P$ for all $P \in r$-max$\,(D)$. In particular, $q[q] = q$.*
4. *r-max$\,(D) = r[q]$-max$\,(D)$.*

We recall the definition of an r-Prüfer monoid. A monoid D with a finitary ideal system r is called an r-Prüfer monoid if every non-zero r-finitely generated r-ideal is r-invertible (equivalently, $\mathcal{I}_{r,\mathrm{fin}}(D)$ is a groupoid with respect to the r-multiplication). In [3], Chap. 17 several ideal and valuation theoretic characterizations of r-Prüfer monoids are given. We only note that D is an r-Prüfer monoid if and only if, for every $P \in r$-max$\,(D)$, D_P is a valuation monoid. By definition, D is a valuation monoid if and only if D is an r-local r-Prüfer monoid. In particular, every s-Prüfer monoid is a valuation monoid.

The following theorem characterizes r-Prüfer monoids by the equality of several spectral ideal systems.

Theorem 3. *Let D be a monoid, r a finitary ideal system on D and $s = s(D)$. Then the following assertions are equivalent:*

a) *D is an r-Prüfer monoid.*
b) *D is an $r[s]$-Prüfer monoid.*
c) *D is r-closed, and $r[q] = r_{\mathrm{a}}$ for every finitary ideal system q on D such that $q \le r$.*
d) *D is r-closed, and $r[s] = r_{\mathrm{a}}$.*

Proof. a) $\Longleftrightarrow$ b) Obvious, since r-max$\,(D) = r[s]$-max$\,(D)$ (by Proposition 3).

a) $\Longrightarrow$ c) If D is r-Prüfer, then r is finitely cancellative and hence $r = r_{\mathrm{a}}$ by Proposition 1. By [3], Theorem 17.1, D is r-closed. Let now q be a finitary ideal system on D such that $q \le r$. For all $P \in r$-max$\,(D)$, D_P is a valuation monoid and therefore $r_P = q_P$ by Theorem 1, and Proposition 3 implies $r[s] = r = r_{\mathrm{a}}$.

c) $\Longrightarrow$ d) Obvious.

d) $\Longrightarrow$ a) If $P \in r$-max$\,(D)$, then $r[s]_P = s_P = (r_{\mathrm{a}})_P = (r_P)_{\mathrm{a}}$ by [3], Ex. 19.2. Since $(r_P)_{\mathrm{a}} \ge (s_P)_{\mathrm{a}}$, we obtain $s_P = (s_P)_{\mathrm{a}}$, and since $s_P = s(D_P)$, Theorem 3 implies that D_P is a valuation monoid. □

Now we turn to integral domains. Let D be an integral domain and r a finitary ideal system on D satisfying $r \ge d = d(D)$. D is called an *r-Prüfer domain* (or a *Prüfer r-multiplication domain*) if D is an r-Prüfer monoid. D is called a *Prüfer domain*, if D is a d-Prüfer domain, and D is called a PVMD (*Prüfer v-multiplication domain*) if D is a $t(D)$-Prüfer domain.

Theorem 4. *Let D be an integral domain, $d = d(D)$, $s = s(D)$, and let r be a finitary ideal system on D such that $d \le r$. Then the following assertions are equivalent:*

a) *D is an r-Prüfer domain.*
b) *$r[s] = r$.*
c) *D is r-closed, and $r[d] = r_{\mathrm{a}}$.*

Corollary. *An integral domain D is a Prüfer domain if and only if $d(D)$ is finitely cancellative.*

Proof. The Corollary follows from the Theorem with $r = d$, observing $d[d] = d$ and Proposition 1. However, the Corollary is well known (see [3], Theorem 17.3), and we shall use it as a tool in the proof of Theorem 4.

a) $\Longrightarrow$ b) See [4], Proposition 5.4 for a more general result.

a) $\Longrightarrow$ c) By Theorem 3.

c) $\Longrightarrow$ a) If $P \in r\text{-max}\,(D)$, then (using [3], Ex. 19.2)

$$d_P = r[d]_P = (r_{\mathrm{a}})_P = (r_P)_{\mathrm{a}} \geq (d_P)_{\mathrm{a}} \geq d_P \,,$$

and therefore $d_P = d(D_P)$ is finitely cancellative. By the Corollary, D_P is a (local) Prüfer domain and hence a valuation domain. $\square$

r-Prüfer monoids and domains can also be characterized by properties of their overmonoids and overrings, see [5] and [2].

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