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SOLUTION OF A QUADRATIC STABILITY ULAM TYPE PROBLEM

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Abstract. In 1940 S. M. Ulam (Intersci. Publ., Inc., New York 1960) imposed at the University of Wisconsin the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". According to P. M. Gruber (Trans. Amer. Math. Soc. 245 (1978), 263–277) the afore-mentioned problem of S. M. Ulam belongs to the following general problem or Ulam type problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this objects by objects, satisfying the property exactly?" In 1941 D. H. Hyers (Proc. Nat. Acad. Sci. 27 (1941), 411–416) established the stability Ulam problem with Cauchy inequality involving a non-negative constant. Then in 1989 we (J. Approx. Theory, 57 (1989), 268–273) solved Ulam problem with Cauchy functional inequality, involving a product of powers of norms. Finally we (Discuss. Math. 12 (1992), 95–103) established the general version of this stability problem. In this paper we solve a stability Ulam type problem for a general quadratic functional inequality. Moreover, we introduce an approximate eveness on approximately quadratic mappings of this problem. These problems, according to P. M. Gruber (1978), are of particular interest in probability theory and in the case of functional equations of different types. Today there are applications in actuarial and financial mathematics, sociology and psychology, as well as in algebra and geometry.

Definition 1. Let X be a linear space and also let Y be a real linear space. Then a mapping $Q_2 : X \to Y$ is called *quadratic*, if the functional equation

$$(*) \qquad Q_2\left(\frac{x_1+x_2}{2}\right) + Q_2\left(\frac{-x_1+x_2}{2}\right) = \frac{1}{2}[Q_2(x_1) + Q_2(x_2)]$$

holds for all vectors $(x_1, x_2) \in X^2$.

The term *quadratic* is introduced in this paper, because the algebraic identity

$$\left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{-x_1+x_2}{2}\right)^2 = \frac{1}{2}(x_1^2 + x_2^2)$$

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holds for all $x \in X$. An additional reason for this new term is because

$$(**) \qquad Q_2(2^n x) = (2^n)^2 Q_2(x)$$

holds for all $x \in X$ and all $n \in N$. In fact, substitution $x_1 = x_2 = 0$ in functional equation $(*)$ yields

$$(1) \qquad Q_2(0) = 0\,.$$

Moreover, substitution $x_1 = 0, x_2 = 2x$ in $(*)$ with (1) yield

$$(2) \qquad Q_2(x) = 2^{-2} Q_2(2x)\,.$$

Replacing x with $2x$ in (2) one concludes that

$$Q_2(2x) = 2^{-2} Q_2(2^2 x)\,, \quad \text{or}$$

$$(2a) \qquad 2^{-2} Q_2(2x) = 2^{-4} Q_2(2^2 x)\,.$$

Identities (2)–(2a) yield that

$$(2b) \qquad Q_2(x) = 2^{-4} Q_2(2^2 x)\,.$$

By induction on $n \in N$ with $x \to 2^{n-1} x$ in (2) one gets

$$(3) \qquad Q_2(x) = 2^{-2n} Q_2(2^n x)$$

or equivalently $(**)$, for all $x \in X$, and $n \in N$.

Theorem 1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is an approximately quadratic mapping; that is, a mapping f for which there exists a constant c (independent of x_1, x_2) ≥ 0 such that the quadratic functional inequality*

$$(4) \qquad \left\| f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{-x_1 + x_2}{2}\right) - \frac{1}{2}[f(x_1) + f(x_2)] \right\| \leq c\,,$$

holds for all vectors $(x_1, x_2) in X^2$.

Then the limit

$$(4') \qquad Q_2(x) = \lim_{n \to \infty} 2^{-2n} f(2^n x)$$

exists for all $x \in X$ and $Q_2 : X \to Y$ is the unique quadratic mapping satisfying equation $(*)$, such that Q_2 is near f; that is,

$$(6') \qquad \|f(x) - Q_2(x)\| \leq c\,,$$

holds for all $x \in X$ with constant c (independent of x) ≥ 0. Moreover, identity

$$(6a') \qquad Q_2(x) = 2^{-2n} Q_2(2^n x)\,,$$

holds for all $x \in X$ and all $n \in N$.

Note that substitution $x_1 = x_2 = 0$ in (4) yields

$$(4a) \qquad \|f(0)\| \leq c\,.$$

Moreover, from (4a) and (4′) one gets that

$$\|Q_2(0)\| = \lim_{n \to \infty} 2^{-2n} \|f(0)\| \leq (\lim_{n \to \infty} 2^{-2n}) c = 0\,, \quad \text{or}$$
$$\|Q_2(0)\| = 0\,, \quad \text{or} \quad Q_2(0) = 0\,, \quad \text{or} \quad (1)\,.$$

Proof of existence

Substitution $x_1 = 0$, $x_2 = 2x$ into (4) yields

$$\|2f(x) - \frac{1}{2}[f(0) + f(2x)]\| \le c\,, \quad \text{or}$$

from triangle inequality one obtains

$$\|f(x) - 2^{-2}f(2x)\| \le \frac{c}{2} + \frac{1}{4}\|f(0)\|\,. \tag{4b}$$

Then from (4a)–(4b) one concludes that

$$\|f(x) - 2^{-2}f(2x)\| \le \frac{3}{4}c = c(1 - 2^{-2})\,, \tag{5}$$

holds for all $x \in X$.

Replacing x with $2x$ in (5) one gets that

$$\|f(2x) - 2^{-2}f(2^2x)\| \le c(1 - 2^{-2})\,, \quad \text{or}$$

$$\|2^{-2}f(2x) - 2^{-4}f(2^2x)\| \le c2^{-2}(1 - 2^{-2})\,, \tag{5a}$$

holds for all $x \in X$.

Inequalities (5)–(5a) and triangle inequality yield

$$\|f(x) - 2^{-4}f(2^2x)\| \le \|f(x) - 2^{-2}f(2x)\| + \|2^{-2}f(2x) - 2^{-4}f(2^2x)\|\,, \quad \text{or}$$

$$\|f(x) - 2^{-4}f(2^2x)\| \le c[(1 - 2^{-2}) + 2^{-2}(1 - 2^{-2})]\,, \quad \text{or}$$

$$\|f(x) - 2^{-4}f(2^2x)\| \le c(1 - 2^{-4})\,, \tag{5b}$$

for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to 2^{n-1}x$ in (5) *claim* that *general inequality*

$$\|f(x) - 2^{-2n}f(2^nx)\| \le c(1 - 2^{-2n})\,, \tag{6}$$

holds for all $x \in X$, and all $n \in N$.

In fact, (5) with $x \to 2^{n-1}x$ imply

$$\|f(2^{n-1}x) - 2^{-2}f(2^nx)\| \le c(1 - 2^{-2})\,, \quad \text{or}$$

$$\|2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^nx)\| \le c2^{-2(n-1)}(1 - 2^{-2})\,, \tag{6a}$$

for all $x \in X$.

By induction hypothesis with $n \to n - 1$ in (6) inequality

$$\|f(x) - 2^{-2(n-1)}f(2^{n-1}x)\| \le c(1 - 2^{-2(n-1)})\,, \tag{6b}$$

holds for all $x \in X$.

Thus functional inequalities (6a)–(6b) and triangle inequality yield

$$\|f(x) - 2^{-2n} f(2^n x)\| \leq \|f(x) - 2^{-2(n-1)} f(2^{n-1} x)\| + \|2^{-2(n-1)} f(2^{n-1}x) - 2^{-2n} f(2^n x)\|, \quad \text{or}$$

$$\|f(x) - 2^{-2n} f(2^n x)\| \leq c[(1 - 2^{-2(n-1)}) + 2^{-2(n-1)}(1 - 2^{-2})] = c(1 - 2^{-2n}),$$

completing the proof of (6).

Claim now that the sequence

$$\{2^{-2n} f(2^n x)\}$$

converges.

Note that from general inequality (6) and the *completeness* of Y, one proves that the above mentioned sequence is a *Cauchy sequence*.

In fact, if $i > j > 0$, then

$$\|2^{2i} f(2^i x) - 2^{-2j} f(2^j x)\| = 2^{-2j} \|2^{-2(i-j)} f(2^i x) - f(2^j x)\|, \tag{7}$$

for all $x \in X$, and all $i, j \in N$.

Setting $h = 2^j x$ in (7) and employing the general inequality (6) one concludes that

$$\begin{aligned}
&\|2^{-2i} f(2^i x) - 2^{-2j} f(2^j x)\| = 2^{-2j} \|2^{-2(i-j)} f(2^{i-j} h) - f(h)\|, \quad \text{or} \\
&\|2^{-2i} f(2^i x) - 2^{-2j} f(2^j x)\| \leq 2^{-2j} c(1 - 2^{-2(i-j)}), \quad \text{or} \\
&\|2^{-2i} f(2^i x) - 2^{-2j} f(2^j x)\| \leq c(2^{-2j} - 2^{-2i}) < c 2^{-2j}, \quad \text{or} \\
&\lim_{j\to\infty} \|2^{-2i} f(2^i x) - 2^{-2j} f(2^j x)\| = 0,
\end{aligned} \tag{7a}$$

completing the proof that the sequence $\{2^{-2n} f(2^n x)\}$ converges.

Hence $Q_2 = Q_2(x)$ is a *well-defined mapping* via the formula (4′). This means that the limit (4′) *exists* for all $x \in X$.

In addition *claim* that Q_2 satisfies the functional equation (∗) for all vectors $(x_1, x_2) \in X^2$. In fact, it is clear from functional inequality (4) and the limit (4′) that inequality

$$2^{-2n} \left\| f\left(\frac{2^n x_1 + 2^n x_2}{2}\right) + f\left(\frac{-2^n x_1 + 2^n x_2}{2}\right) - \frac{1}{2}[f(2^n x_1) + f(2^n x_2)] \right\| \leq 2^{-2n} c,$$

holds for all $x_1, x_2 \in X$, and all $n \in N$.

Therefore

$$\left\| \lim_{n\to\infty} 2^{-2n} f\left(2^n \frac{x_1 + x_2}{2}\right) + \lim_{n\to\infty} 2^{-2n} f\left(2^n \frac{-x_1 + x_2}{2}\right) - \frac{1}{2}\left[\lim_{n\to\infty} 2^{-2n} f(2^n x_1) + \lim_{n\to\infty} 2^{-2n} f(2^n x_2)\right] \right\| \leq (\lim_{n\to\infty} 2^{-2n}) c = 0, \quad \text{or}$$

$$\left\| Q_2\left(\frac{x_1 + x_2}{2}\right) + Q_2\left(\frac{-x_1 + x_2}{2}\right) - \frac{1}{2}[Q_2(x_1) + Q_2(x_2)] \right\| = 0, \quad \text{or}$$

Q_2 satisfies the functional equation $(*)$ for all $(x_1, x_2) \in X^2$.

Thus Q_2 is a *quadratic mapping*.

It is clear now from general inequality (6), $n \to \infty$, and formula (4′) that inequality (6′) holds in X, completing *the existence proof* of this Theorem 1.

Proof of uniqueness

Let $Q_2' : X \to Y$ be another quadratic mapping satisfying functional equation $(*)$, such that inequality

$$\|f(x) - Q_2'(x)\| \le c\,, \tag{6''}$$

holds for all $x \in X$ with constant c (independent of x) ≥ 0.

If there exists a quadratic mapping $Q_2 : X \to Y$ satisfying equation $(*)$, then

$$Q_2(c) = Q_2'(x)\,, \tag{8}$$

holds for all $x \in X$.

To prove the afore-mentioned *uniqueness* employ (3) or (6a′) for Q_2 and Q_2', as well, so that

$$Q_2'(x) = 2^{-2n} Q_2'(2^n x) \tag{3'}$$

holds for all $x \in X$ and all $n \in N$.

Moreover triangle inequality and functional inequalities (6′) - (6″) imply that

$$\|Q_2(2^n x) - Q_2'(2^n x)\| \le \|Q_2(2^n x) - f(2^n x)\| + \|f(2^n x) - Q_2'(2^n x)\|\,, \quad \text{or}$$

$$\|Q_2(2^n x) - Q_2'(2^n x)\| \le c + c = 2c\,, \tag{9}$$

for all $x \in X$ and all $n \in N$.

Then from (3), (3′), and (9) one proves that

$$\|Q_2(x) - Q_2'(x)\| = \|2^{-2n} Q_2(2^n x) - 2^{-2n} Q_2'(2^n x)\|\,, \quad \text{or}$$

$$\|Q_2(x) - Q_2'(x)\| \le 2^{1-2n} c\,, \tag{9a}$$

holds for all $x \in X$ and all $n \in N$.

Therefore from (9a), and $n \to \infty$, one gets that

$$\lim_{n\to\infty} \|Q_2(x) - Q_2'(x)\| \le (\lim_{n\to\infty} 2^{1-2n}) c = 0\,, \quad \text{or} \quad \|Q_2(x) - Q_2'(x)\| = 0\,, \quad \text{or}$$

$$Q_2(x) = Q_2'(x)$$

holds for all $x \in X$, completing the proof of *uniqueness* and thus the *stability* of Theorem 1.

Note that the best approximation constant is c. In fact, take

$$f(x) = c$$

in inequality (6′), $n \to \infty$, and limit (4′) (: $Q_2(x) = 0$).

Definition 2. Let X be a linear space and also let Y be a real linear space. Then a mapping $Q_2 : X \to Y$ is called *general quadratic*, if the functional equation

$$(10) \qquad Q_2(a_1x_1 + a_2x_2) + Q_2(-a_1x_1 + a_2x_2) = 2[a_1^2 Q_2(x_1) + a_2^2 Q_2(x_2)]$$

holds for all vectors $(x_1, x_2) \in X^2$, and all fixed positive reals $a_1, a_2 : 0 < a_2 < l = \sqrt{a_1^2 + a_2^2} < 1$ or $l > a_2 > 1$ or $l = \sqrt{2} > a_2 = 1 = a_1$.

Note that

$$(11) \qquad Q_2(x) = a_2^{2n} Q_2(a_2^{-n}x)\,,$$

holds for all $x \in X$, all $n \in N$ and $0 < a_2 < l < 1$.

Claim that identity (11) holds. In fact, substitution $x_1 = x_2 = 0$ in equation (10) yields

$$(11a) \qquad Q_2(0) = 0\,.$$

Moreover, substitution $x_1 = 0$, $x_2 = a_2^{-1}x(: 0 < a_2 < l < 1)$ in (10) with (1) yield

$$(11b) \qquad Q_2(x) = a_2^2 Q_2(a_2^{-1}x)\,.$$

Replacing x with $a_2^{-1}x$ in (11b) and then employing (11b) one gets

$$(11c) \qquad Q_2(x) = a_2^2 Q_2(a_2^{-1}x) = a_2^4 Q_2(a_2^{-2}x)\,.$$

By induction on $n \in N$ with $x \to a_2^{-(n-1)}x$ in (11b) one gets the required identity (11).

Similarly by substitution $x_1 = 0$, $x_2 = x$ in (10) and then $x \to a_2x$ one concludes that

$$(11') \qquad Q_2(x) = a_2^{-2n} Q_2(a_2^n x)\,,$$

holds for all $x \in X$, all $n \in N$ and $l > a_2 > 1$. Also by substitution $x_1 = x_2 = x$ in (10) with $a_1 = a_2 = 1$ one concludes that

$$(11'') \qquad Q_2(x) = 2^{-2n} Q_2(2^n x)\,,$$

holds for all $x \in X$ and all $n \in N$.

Formulas (11)–(12) are important to prove *uniqueness* of mapping Q_2 in the following general Theorem 2.

General Theorem 2. *Let X be a normed linear space, Y be a real complete normed linear space, and $f : X \to Y$. Assume in addition that $\sqrt{a_1^2 + a_2^2} = l > 0$ for all fixed reals a_i $(i = 1, 2) : 0 < a_2 < l < 1$ or $l > a_2 > 1$ or $l = \sqrt{2} > a_2 = 1 = a_1$. Moreover the general quadratic functional inequality*

$$(12) \qquad \|f(a_1x_1 + a_2x_2) + f(-a_1x_1 + a_2x_2) - 2[a_1^2 f(x_1) + a_2^2 f(x_2)]\| \leq c\,,$$

holds for all vectors $(x_1, x_2) \in X^2$ *with constant* c (*independent of* $x_1, x_2) \geq 0$ *and initial condition*

$$\|f(0)\| \leq \begin{cases} \frac{c}{2(1-l^2)}\,, & \textit{if} \quad 0 < a_2 < l < 1; \\ \frac{c}{2}\,, & \textit{if} \quad l = \sqrt{2} > a_2 = 1 = a_1; \\ \frac{c}{2(l^2-1)}\,, & \textit{if} \quad l > a_2 > 1. \end{cases} \tag{12a}$$

Then the limit

$$Q_2(x) = \lim_{n\to\infty} \begin{cases} a_2^{2n} f(a_2^{-n}x)\,, & \textit{if} \quad 0 < a_2 < l < 1; \\ 2^{-2n} f(2^n x)\,, & \textit{if} \quad l = \sqrt{2} > a_2 = 1 = a_1; \\ a_2^{-2n} f(a_2^n x)\,, & \textit{if} \quad l > a_2 > 1 \end{cases} \tag{12b}$$

exists *for all* $x \in X$ *and* $Q_2 : X \to Y$ *is the* unique *general quadratic mapping satisfying the functional equation (10) for all vectors* $(x_1, x_2) \in X^2$*, such that inequality*

$$\|f(x) - Q_2(x)\| \leq \begin{cases} \frac{c}{2(1-l^2)}\,, & \textit{if} \quad 0 < a_2 < l < 1; \\ \frac{c}{2}\,, & \textit{if} \quad l = \sqrt{2} > a_2 = 1 = a_1; \\ \frac{[2l^2-(a_2^2+1)]c}{2(l^2-1)(a_2^2-1)}\,, & \textit{if} \quad l > a_2 > 1 \end{cases} \tag{12c}$$

and identity

$$Q_2(x) = \begin{cases} a_2^{2n} Q_2(a_2^{-n}x)\,, & \textit{if} \quad 0 < a_2 < l < 1; \\ 2^{-2n} Q_2(2^n x)\,, & \textit{if} \quad l = \sqrt{2} > a_2 = 1 = a_1; \\ a_2^{-2n} Q_2(a_2^n x)\,, & \textit{if} \quad l > a_2 > 1 \end{cases} \tag{12d}$$

hold for all $x \in X$ *and all* $n \in N$ *with* c (*independent of* $x) \geq 0$.

Proof of Theorem 2.

Case I $(0 < a_2 < l < 1)$

Substitution $x_1 = x_2 = 0$ in (12) yields that

$$\|f(0)\| \leq \frac{c}{2(1-l^2)}\,, \qquad 0 < l < 1\,. \tag{13}$$

Employing $x_1 = 0$, $x_2 = a_2^{-1}x$ in (12) one finds

$$\|f(x) - [a_1^2 f(0) + a_2^2 f(a_2^{-1}x)]\| \leq \frac{c}{2}\,. \tag{14}$$

Triangle inequality and (13) – (14) imply

$$\|f(x) - a_2^2 f(a_2^{-1}x)\| \leq \frac{c}{2} + a_1^2\|f(0)\|\,, \quad \text{or}$$

$$\|f(x) - a_2^2 f(a_2^{-1}x)\| \leq \frac{c + 2a_1^2\|f(0)\|}{2(1-a_2^2)}(1-a_2^2)\,, \quad \text{or}$$

$$\|f(x) - a_2^2 f(a_2^{-1}x)\| \leq \frac{c + 2(l^2-a_2^2)\frac{c}{2(1-l^2)}}{2(1-a_2^2)}(1-a_2^2)\,, \quad \text{or}$$

$$\|f(x) - a_2^2 f(a_2^{-1}x)\| \leq \frac{c}{2(1-l^2)}(1-a_2^2)\,, \tag{15}$$

for all $x \in X$ and $0 < a_2 < l < 1$.

Then the induction on $n \in N$ with $x \to a_2^{-(n-1)}x$ in (15) implies *the general inequality*

$$\|f(x) - a_2^{2n} f(a_2^{-n}x)\| \leq \frac{c}{2(1-l^2)}(1 - a_2^{2n}), \tag{15a}$$

for all $x \in X$, all $n \in N$ and $0 < a_2 < l < 1$.

Inequality (15a), with $n \to \infty$, and limit formula

$$Q_2(x) = \lim_{n\to\infty} a_2^{2n} f(a_2^{-n}x), \quad 0 < a_2 < l < 1, \tag{15b}$$

yield that inequality

$$\|f(x) - Q_2(x)\| \leq \frac{c}{2(1-l^2)}, \tag{15c}$$

holds for all $x \in X$ and $0 < l < 1$.

Case II $(l > a_2 > 1)$

Similar case to the afore-mentioned case I.

In fact, substitution $x_1 = x_2 = 0$ in (12) implies

$$\|f(0)\| \leq \frac{c}{2(l^2-1)}, \quad l > 1. \tag{13a}$$

Employing $x_1 = 0$, $x_2 = x$ in (12) and triangle inequality one gets

$$\|f(a_2x) - [a_1^2 f(0) + a_2^2 f(x)]\| \leq \frac{c}{2}, \quad \text{or}$$

$$\left\|[f(x) - a_2^{-2} f(a_2x)] - \left[-\frac{a_1^2}{a_2^2} f(0)\right]\right\| \leq \frac{c}{2a_2^2}, \quad \text{or}$$

$$\|f(x) - a_2^{-2} f(a_2x)\| \leq \frac{c + 2a_1^2\|f(0)\|}{2(a_2^2-1)}(1 - a_2^{-2}), \quad \text{or}$$

$$\|f(x) - a_2^{-2} f(a_2x)\| \leq \frac{c + 2(l^2 - a_2^2)\frac{c}{2(l^2-l)}}{2(a_2^2-1)}(1 - a_2^{-2}), \quad \text{or}$$

$$\|f(x) - a_2^{-2} f(a_2x)\| \leq \frac{[2l^2 - (a_2^2+1)]c}{2(l^2-1)(a_2^2-1)}(1 - a_2^{-2}), \tag{16}$$

for all $x \in X$ and $l > a_2 > 1$.

Then induction on $n \in N$ with $x \to a_2^{n-1}x$ in (16) implies *the general inequality*

$$\|f(x) - a_2^{-2n} f(a_2^n x)\| \leq \frac{2l^2 - (a_2^2+1)}{2(l^2-1)(a_2^2-1)} c(1 - a_2^{-2n}), \tag{16a}$$

for all $x \in X$, all $n \in N$ and $l > a_2 > 1$.

Inequality (16a), with $n \to \infty$, and formula

$$Q_2(x) = \lim_{n\to\infty} a_2^{-2n} f(a_2^n x)\,, \quad l > a_2 > 1\,, \tag{16b}$$

yield that inequality

$$\|f(x) - Q_2(x)\| \le \frac{2l^2 - (a_2^2+1)}{2(l^2-1)(a_2^2-1)}c\,, \tag{16c}$$

holds for all $x \in X$ and $l > a_2 > 1$.

Case III $(l = \sqrt{2} > a_2 = 1 = a_1)$.
Employing $a_2 = 1 = a_1$ one gets that $l = \sqrt{a_1^2 + a_2^2} = \sqrt{2} > 1$, and from (13a)

$$\|f(0)\| \le \frac{c}{2}\,. \tag{13b}$$

Substitution $x_1 = x_2 = x$ in (12) with $a_1 = a_2 = 1$, triangle inequality and (13b) yield

$$\|f(2x) + f(0) - 4f(x)\| \le c\,, \quad \text{or}$$

$$\|f(x) - 2^{-2}f(2x)\| \le \frac{3}{8}c = \frac{c}{2}(1-2^{-2})\,, \tag{17}$$

for all $x \in X$.

Then induction on $n \in N$ with $x \to 2^{n-1}x$ in (17) implies *the general inequality*

$$\|f(x) - 2^{-2n}f(2^n x)\| \le \frac{c}{2}(1-2^{-2n})\,, \tag{17a}$$

for all $x \in X$, all $n \in N$ and $l = \sqrt{2} > a_2 = 1 = a_1$. □

The rest of the proof is omitted as similar to the proof of Theorem 1.

General Theorem 3. *Let X be a normed linear space, Y be a real complete normed linear space, and $f : X \to Y$. Assume in addition that all fixed reals a_1 are positive. Moreover the general quadratic functional inequality*

$$\|f(a_1x_1 + x_2) + f(-a_1x_1 + x_2) - 2[a_1^2 f(x_1) + f(x_2)]\| \le c\,, \tag{18}$$

holds for all vectors $(x_1, x_2) \in X^2$ with constant c (independent of x_1, x_2) ≥ 0 and initial condition

$$\|f(0) \le \frac{c}{2a_1^2}\,. \tag{18a}$$

Then the limit

$$Q_2(x) = \lim_{n\to\infty} \begin{cases} a_1^{2n} f(a_1^{-n}x)\,, & \textit{if} \quad 0 < a_1 < 1 \\ 2^{-2n} f(2^n x)\,, & \textit{if} \quad a_1 = 1 \\ a_1^{-2n} f(a_1^n x)\,, & \textit{if} \quad a_1 > 1 \end{cases} \tag{18b}$$

exists *for all* $x \in X$ *and* $Q_2 : X \to Y$ *is the* unique *general quadratic mapping satisfying the functional equation*

$$Q_2(a_1x_1 + x_2) + Q_2(-a_1x_1 + x_2) = 2[a_1^2 Q_2(x_1) + Q_2(x_2)] \tag{10a}$$

for all vectors $(x_1, x_2) \in X^2$, *such that inequality*

$$\|f(x) - Q_2(x)\| \le \begin{cases} \frac{(1+a_1^2)^2 c}{2a_1^4(1-a_1^2)}\,, & \text{if} \quad 0 < a_1 < 1 \\ \frac{c}{2}\,, & \text{if} \quad a_1 = 1 \\ \frac{(a_1^2+1)^2 c}{2a_1^4(a_1^2-1)}\,, & \text{if} \quad a_1 > 1 \end{cases} \tag{18c}$$

and identity

$$Q_2(x) = \begin{cases} a_1^{2n} Q_2(a_1^{-n}x)\,, & \text{if} \quad 0 < a_1 < 1 \\ 2^{-2n} Q_2(2^n x)\,, & \text{if} \quad a_1 = 1 \\ a_1^{-2n} Q_2(a_1^n x)\,, & \text{if} \quad a_1 > 1 \end{cases} \tag{18d}$$

hold for all $x \in X$ *and all* $n \in N$ *with constant* c (*independent of* x) ≥ 0.

Lemma 1. *If* $f : X \to Y$ *satisfies the assumptions of above general Theorem 3, then* f *is* approximately even*; that is, functional inequality*

$$\|f(x) - f(-x)\| \le \frac{a_1^2 + 1}{a_1^4} c \tag{19}$$

holds for all $x \in X$ *with constant* c (*independent of* x) ≥ 0.

Proof of Lemma 1.

(i) *First* assume $a_1 : 0 < a_1 < 1$.

In fact, substitution $x_1 = x_2 = 0$ in (18) yields (18a).

Then replacing $x_1 = a_1^{-1}x$, $x_2 = 0$ in (18) and employing triangle inequality one gets

$$\|f(x) + f(-x) - 2a_1^2 f(a_1^{-1}x)\| \le c + 2\|f(0)\|\,, \quad \text{or (from (18a))}$$

functional inequality

$$\|f(x) + f(-x) - 2a_1^2 f(a_1^{-1}x)\| \le \frac{a_1^2 + 1}{a_1^2} c\,, \tag{20}$$

for all $x \in X$ with $c \ge 0$.

But the functional identity

$$\begin{aligned} 2a_1^2[f(a_1^{-1}x) - f(-a_1^{-1}x)] = {} & \{2a_1^2 f(a_1^{-1}x) - [f(x) + f(-x)]\} \\ & + [f(-x) + f(x) - 2a_1^2 f(-a_1^{-1}x)]\,, \end{aligned} \tag{21}$$

holds for all $x \in X$.

Substituting x with $-x$ in (20) one finds functional inequality

$$\|f(-x) + f(x) - 2a_1^2 f(-a_1^{-1}x)\| \leq \frac{a_1^2+1}{a_1^2}c \tag{20a}$$

for all $x \in X$ with $c \geq 0$.

Therefore triangle inequality, identity (21) and functional inequalities (20)–(20a) yield

$$\begin{aligned}\|2a_1^2[f(a_1^{-1}x) - f(-a_1^{-1}x)]\| \leq\ & \|2a_1^2 f(a_1^{-1}x) - [f(x) + f(-x)]\| \\ & + \|f(-x) + f(x) - 2a_1^2 f(-a_1^{-1}x)\| \leq 2\frac{a_1^2+1}{a_1^2}c, \quad \text{or}\end{aligned}$$

$$\|f(a_1^{-1}x) - f(-a_1^{-1}x)\| \leq \frac{a_1^2+1}{a_1^4}c, \tag{19a}$$

for all $x \in X$ with $c \geq 0$.

Hence replacing x with $a_1 x$ in (19a) one concludes that functional inequality (19) holds for all $x \in X$ and all $a_1 : 0 < a_1 < 1$.

(ii) *Second* assume $a_1 : a_1 = 1$.

In fact, replacing $x_1 = x$, $x_2 = 0$ in (18) with $a_1 = 1$ and considering (18a) one concludes that

$$\begin{aligned}\|f(x) + f(-x) - 2[f(x) + f(0)]\| \leq c, \quad & \text{or} \\ \|f(x) - f(-x)\| \leq c + 2\|f(0)\|, \quad & \text{or} \\ \|f(x) - f(-x)\| \leq 2c, \quad & c \geq 0 \qquad (19\text{b})\end{aligned}$$

holds for all $x \in X$. Thus inequality (19b) is a special case of inequality (19) for $a_1 = 1$.

(iii) *Finally* assume $a_1 : a_1 > 1$.

In fact, replacing $x_1 = x$, $x_2 = 0$ in (18) and employing triangle inequality one finds that functional inequality

$$\|f(a_1x) + f(-a_1x) - 2a_1^2 f(x)\| \leq c + 2\|f(0)\|, \quad \text{or} \quad (\text{from (18a)})$$

$$\|f(a_1x) + f(-a_1x) - 2a_1^2 f(x)\| \leq \frac{a_1^2+1}{a_1^2}c, \tag{22}$$

holds for all $x \in X$ with $c \geq 0$.

Also the functional identity

$$\begin{aligned}2a_1^2[f(x) - f(-x)] =& \{2a_1^2 f(x) - [f(a_1x) + f(-a_1x)]\} \\ & + [f(-a_1x) + f(a_1x) - 2a_1^2 f(-x)], \qquad (23)\end{aligned}$$

holds for all $x \in X$.

Substituting x with $-x$ in (22) one concludes that functional inequality

$$\|f(-a_1x) + f(a_1x) - 2a_1^2 f(-x)\| \leq \frac{a_1^2+1}{a_1^2}c \tag{22a}$$

holds for all $x \in X$ with $c \geq 0$.

Thus triangle inequality, identity (23) and functional inequalities (22)–(22a) imply

$$\|2a_1^2[f(x) - f(-x)]\| \leq \|2a_1^2 f(x) - [f(a_1x) + f(-a_1x)]\| + \|f(-a_1x) + f(a_1x) - 2a_1^2 f(-x)\| \leq 2\frac{a_1^2+1}{a_1^2}c\,, \quad \text{or}$$

functional inequality

$$\|f(x) - f(-x)\| \leq \frac{a_1^2+1}{a_1^4}c\,, \quad c \geq 0\,, \tag{19c}$$

completing the proof of inequality (19) for $a_1 > 1$, $c \geq 0$ and all $x \in X$.

Hence the proof of Lemma 1 is complete. □

Lemma 2. *If $Q_2 : X \to Y$ satisfies the assumptions of above general Theorem 3, then Q_2 is* even*; that is, functional equation*

$$Q_2(-x) = Q_2(x) \tag{24}$$

holds for all $x \in X$.

Proof of Lemma 2.

(i) Assume *first* $a_1 : 0 < a_1 < 1$.

In fact, substitution $x_1 = x_2 = 0$ in equation (10a) yields

$$Q_2(0) = 0\,. \tag{25}$$

Then replacing $x_1 = a_1^{-1}x$, $x_2 = 0$ in (10a) and employing (25) one finds the functional equation

$$Q_2(x) + Q_2(-x) = 2a_1^2 Q_2(a_1^{-1}x)\,, \quad 0 < a_1 < 1\,, \tag{26}$$

for all $x \in X$.

Substitution x with $-x$ in (26) one obtains equation

$$Q_2(-x) + Q_2(x) = 2a_1^2 Q_2(a_1^{-1}x)\,, \quad 0 < a_1 < 1\,, \tag{26a}$$

for all $x \in X$.

Equations (26)–(26a) yield

$$Q_2(-a_1^{-1}x) = Q_2(a_1^{-1}x)\,, \quad 0 < a_1 < 1\,, \tag{24a}$$

for all $x \in X$.

Replacing x with $a_1 x$ in (24a) one gets equation (24) for $0 < a_1 < 1$ and all $x \in X$.

(ii) Assume *second* $a_1 : a_1 = 1$.

Therefore from (10a) one finds equation

$$Q_2(x_1 + x_2) + Q_2(-x_1 + x_2) = 2[Q_2(x_1) + Q_2(x_2)]\,, \tag{10b}$$

for all vectors $(x_1, x_2) \in X^2$.

Replacing $x_1 = x$, $x_2 = 0$ in equation (10b) and considering (25) one concludes equation (24) for $a_1 = 1$ and all $x \in X$.

(iii) Assume *finally* $a_1 : a_1 > 1$.

In fact, replacing $x_1 = x$, $x_2 = 0$ in equation (10a) and employing (25) one gets equation

$$Q_2(a_1 x) + Q_2(-a_1 x) = 2a_1^2 Q_2(x)\,, \quad a_1 > 1\,, \tag{27}$$

for all $x \in X$.

Substituting x with $-x$ in (27) one obtains equation

$$Q_2(-a_1 x) + Q_2(a_1 x) = 2a_1^2 Q_2(-x)\,, \quad a_1 > 1\,, \tag{27a}$$

for all $x \in X$.

Equations (27)–(27a) imply the required equation (24) for $a_1 > 1$ and all $x \in X$.

Thus the proof of Lemma 2 is complete. $\square$

Proof of Theorem 3. To prove the *existence* part of Theorem 3 one employs above Lemma 1 and establishes the following *two functional inequalities*

$$(\mathrm{I}_1)\ \ \|f(x) - a_1^{2n} f(a_1^{-n}x)\| \le \frac{(1 + a_1^2)^2 c}{2a_1^4(1 - a_1^2)}(1 - a_1^{2n})\,,$$

$$(\mathrm{I}_2)\ \ \|f(x) - a_1^{-2n} f(a_1^{n}x)\| \le \frac{(a_1^2 + 1)^2 c}{2a_1^4(a_1^2 - 1)}(1 - a_1^{-2n})$$

for all $x \in X$ and all fixed positive reals $a_1 : a_1 > 1$.

Note case $a_1 = 1$ has been established at Theorem 2 (inequality (17a)).

Claim that inequality (I_1) holds.

In fact, equation

$$2f(x) - 2a_1^2 f(a_1^{-1}x) = [f(x) + f(-x) - 2a_1^2 f(a_1^{-1}x)] + [f(x) - f(-x)]\,, \tag{28}$$

holds for all $x \in X$, and $0 < a_1 < 1$.

Thus employing equation (28), triangle inequality, inequality (20) and Lemma 1 (inequality (19)) one concludes that

$$2\|f(x) - a_1^2 f(a_1^{-1}x)\| \le \|f(x) + f(-x) - 2a_1^2 f(a_1^{-1}x)\| + \|f(x) - f(-x)\|\,, \quad \text{or}$$

$$2\|f(x) - a_1^2 f(a_1^{-1}x)\| \le \frac{a_1^2+1}{a_1^2}c + \frac{a_1^2+1}{a_1^4}c\,, \quad \text{or}$$

$$\|f(x) - a_1^2 f(a_1^{-1}x)\| \le \frac{(1+a_1^2)^2 c}{2a_1^4(1-a_1^2)}(1-a_1^2)\,, \quad 0 < a_1 < 1\,, \tag{29}$$

holds for all $x \in X$.

Therefore by induction on $n \in N$ and replacing x with $a_1^{-(n-1)}x$ in (29) one completes the proof of inequality (I_1).

Note that employing inequality (I_1) with $n \to \infty$ and limit (18b) for $0 < a_1 < 1$ one gets inequality (18c) for $0 < a_1 < 1$:

$$\|f(x) - Q_2(x)\| \le \frac{(1+a_1^2)^2 c}{2a_1^4(1-a_1^2)}\,, \quad 0 < a_1 < 1\,, \tag{30}$$

holds for all $x \in X$.

Claim now that inequality (I_2) holds.

In fact, equation

$$\begin{aligned} 2f(a_1x) - 2a_1^2 f(x) = {} & [f(a_1x) + f(-a_1x) - 2a_1^2 f(x)] \\ & + [f(a_1x) - f(-a_1x)]\,, \end{aligned} \tag{28a}$$

holds for all $x \in X$, and $a_1 > 1$.

Thus employing equation (28a), triangle inequality, inequality (22) and Lemma 1 (inequality (19) with $x \to a_1x$) one finds that

$$\begin{aligned} 2\|f(a_1x) - a_1^2 f(x)\| \le {} & \|f(a_1x) + f(-a_1x) - 2a_1^2 f(x)\| \\ & + \|f(a_1x) - f(-a_1x)\| \le \frac{a_1^2+1}{a_1^2}c + \frac{a_1^2+1}{a_1^4}c\,, \quad \text{or} \end{aligned}$$

$$\|f(a_1x) - a_1^2 f(x)\| \le \frac{(a_1^2+1)^2 c}{2a_1^4}\,, \quad \text{or}$$

$$\|f(x) - a_1^{-2} f(a_1x)\| \le \frac{(a_1^2+1)^2 c}{2a_1^4(a_1^2-1)}(1-a_1^{-2})\,, \quad a_1 > 1 \tag{29a}$$

for all $x \in X$.

Therefore by employing induction on $n \in N$ and substituting x with $a_1^{n-1}x$ in (29a) one completes the proof of inequality (I_2).

Note that employing inequality (I_2) with $n \to \infty$, and limit (18b) for $a_1 > 1$ one gets inequality (18c) for $a_1 > 1$:

$$\|f(x) - Q_2(x)\| \leq \frac{(a_1^2+1)^2 c}{2a_1^4(a_1^2-1)}\,, \quad a_1 > 1\,, \tag{30a}$$

for all $x \in X$.

To prove the *Uniqueness* part of Theorem 3 one employs above Lemma 2 and establishes the following *two functional equations*

(F_1) $\qquad Q_2(x) = a_1^{2n} Q_2(a_1^{-n}x)$

(F_2) $\qquad Q_2(x) = a_1^{-2n} Q_2(a_1^{n}x)$

for all $x \in X$ and all fixed positive reals $a_1 : a_1 > 1$.

Claim that equation (F_1) holds.

In fact, substitution $x_1 = a_1^{-1}x$, $x_2 = 0$ in equation (10a) and using Lemma 2 (formulas (24)–(25)) one gets that

$$Q_2(x) = a_1^2 Q_2(a_1^{-1}x)\,, \quad \text{if} \quad 0 < a_1 < 1\,, \tag{31}$$

for all $x \in X$.

Induction on $n \in N$ with $x \to a_1^{-(n-1)}x$ completes the proof of equation (F_1).

Claim now that equation (F_2) holds.

In fact, substitution $x_1 = x$, $x_2 = 0$ in (10a) and using Lemma 2 (formula (24)) with $x \to a_1 x$ and formula (25) one finds

$$Q_2(x) = a_1^{-2} Q_2(a_1 x)\,, \quad \text{if} \quad a_1 > 1\,, \tag{31a}$$

for all $x \in X$.

Thus applying induction on $n \in N$ with $x \to a_1^{n-1}x$ one completes the proof of equation (F_2). $\square$

The rest of the proof of Theorem 3 is omited as similar to the proof of Theorem 1.

Examples.

(1) Let $f : R \to R$ be a real function, such that $f(x) = x^2 + k$ with k a real constant and inequality

$$\|[(a_1x_1 + a_2x_2)^2 + k] + [(-a_1x_1 + a_2x_2)^2 + k] - 2[a_1^2(x_1^2 + k) + a_2^2(x_2^2 + k)]\| \leq c$$

holds with condition $\qquad |k| \leq \begin{cases} \dfrac{c}{2(1-l^2)}\,, & \text{if} \quad 0 < a_2 < l < 1 \\ \dfrac{c}{2}\,, & \text{if} \quad l > a_2 = 1 = a_1 \\ \dfrac{c}{2(l^2-1)}\,, & \text{if} \quad l > a_2 > 1 \end{cases}$

Then the general quadratic mapping $Q_2 : R \to R$, such that $Q_2(x) = x^2$, for all $x \in X$, is *unique* and satisfies (10) and (12b)–(12c)–(12d).

(2) Let $f : R \to R$ be a real function, such that $f(x) = x^2 + k$ with k a real constant and inequality

$$\|[(a_1x_1 + x_2)^2 + k] + [(-a_1x_1 + x_2)^2 + k] - 2[a_1^2(x_1^2 + k) + (x_2^2 + k)]\| \leq c$$

holds with condition

$$|k| \leq \frac{c}{2a_1^2}, \quad a_1 > 0.$$

Then the general quadratic mapping $Q_2 : R \to R$, such that $Q_2(x) = x^2$, for all $x \in X$, is *unique* and satisfies (10a) and (18b)–(18c)–(18d).

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