ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 33 – 40

SOME SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

E. SAVAŞ AND R. SAVAŞ

ABSTRACT. In this paper we introduce a new concept of λ-strong convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces. It is also shown that if a sequence is λ-strongly convergent with respect to an Orlicz function then it is λ-statistically convergent.

1. Introduction

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. Let X be a linear space. A function $p : X \to \mathbb{R}$ is called *paranorm*, if

(P.1) $p(0) \geq 0$

(P.2) $p(x) \geq 0$ for all $x \in X$

(P.3) $p(-x) = p(x)$ for all $x \in X$

(P.4) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ (triangle inequality)

(P.5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda (n \to \infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (continuity of multiplication by scalars).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total*. It is well known that the metric of any linear metric space is given by some total paranorm (cf. [14, Theorem 10.4.2, p.183]).

Let $\Lambda = (\lambda_n)$ be a non decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.

The generalized de la Vallée-Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k \,,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number ℓ (see [2]) if $t_n(x) \to \ell$ as $n \to \infty$.

2000 *Mathematics Subject Classification*: 40D05, 40A05.

Key words and phrases: sequence spaces, Orlicz function, de la Vallée-Poussin means.
Received January 8, 2002.

We write

$$[V,\lambda]_0 = \left\{ x = x_k : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\}$$

$$[V,\lambda] = \left\{ x = x_k : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - \ell e| = 0\,, \quad \text{for some} \quad \ell \in C \right\}$$

and

$$[V,\lambda]_\infty = \left\{ x = x_k : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty \right\}.$$

For the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method. In the special case where $\lambda_n = n$ for $n = 1, 2, 3, \ldots$, the sets $[V,\lambda]_0$, $[V,\lambda]$ and $[V,\lambda]_\infty$ reduce to the sets ω_0, ω and ω_∞ introduced and studied by Maddox [5].

Following Lindenstrauss and Tzafriri [4], we recall that an Orlicz function M is a continuous, convex, non-decreasing function defined for $x \geq 0$ such that $M(0) = 0$ and $M(x) \geq 0$ for $x > 0$.

If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called a modulus function, defined and discussed by Nakano [8], Ruckle [10], Maddox [6] and others.

Lindenstrauss and Tzafriri used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some} \quad \rho > 0 \right\}.$$

The space l_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(x) = x^p$, $1 \leq p < \infty$, the space l_M coincide with the classical sequence space l_p.

Recently Parashar and Choudhary [9] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[C,1,p]$, $[C,1,p]_0$ and $[C,1,p]_\infty$. It may be noted that the spaces of strongly summable sequences were discussed by Maddox [5].

Quite recently E. Savaş [11] has also used an Orlicz function to construct some sequence spaces.

In the present paper we introduce a new concept of λ-strong convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces. Furthermore it is shown that if a sequence is λ-strongly convergent with respect to an Orlicz function then it is λ-statistically convergent.

We now introduce the generalizations of the spaces of λ-strongly.

We have

Definition 1. Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers.

We define the following sequence spaces:

$$[V, M, p] = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k - \ell|}{\rho} \right) \right]^{p_k} = 0 \text{ for some } l \text{ and } \rho > 0 \right\}$$

$$[V, M, p]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} = 0 \text{ for some } \rho > 0 \right\}$$

$$[V, M, p]_\infty = \left\{ x = (x_k) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} < \infty \text{ for some } \rho > 0 \right\}.$$

We denote $[V, M, p]$, $[V, M, p]_0$ and $[V, M, p]_\infty$ as $[V, M]$, $[V, M]_0$ and $[V, M]_\infty$ when $p_k = 1$ for all k. If $x \in [V, M]$ we say that x is of λ-strongly convergent with respect to the Orlicz function M. If $M(x) = x$, $p_k = 1$ for all k, then $[V, M, p] = [V, \lambda]$, $[V, M, p]_0 = [V, \lambda]_0$ and $[V, M, p]_\infty = [V, \lambda]_\infty$. If $\lambda_n = n$ then, $[V, M, p]$, $[V, M, p]_0$ and $[V, M, p]_\infty$ reduce the $[C, M, p]$, $[C, M, p]_0$ and $[C, M, p]_\infty$ which were studied Parashar and Choudhary [9].

2. Main Results

In this section we examine some topological properties of $[V, M, p]$, $[V, M, p]_0$ and $[V, M, p]_\infty$ spaces.

Theorem 1. *For any Orlicz function M and any sequence $p = (p_k)$ of strictly positive real numbers, $[V, M, p]$, $[V, M, p]_0$ and $[V, M, p]_\infty$ are linear spaces over the set of complex numbers.*

Proof. We shall prove only for $[V, M, p]_0$. The others can be treated similarly. Let $x, y \in [V, M, p]_0$ and $\alpha, \beta \in C$. In order to prove the result we need to find some $\rho_3 > 0$ such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\alpha x_k + \beta y_k|}{\rho_3} \right) \right]^{p_k} = 0\,.$$

Since $x, y \in [V, M, p]_0$, there exist a positive some ρ_1 and ρ_2 such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho_1} \right) \right]^{p_k} = 0 \quad \text{and} \quad \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|y_k|}{\rho_2} \right) \right]^{p_k} = 0\,.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex,

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\alpha x_k+\beta y_k|}{\rho_3}\right)\right]^{p_k} &\le \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\alpha x_k|}{\rho_3}+\frac{|\beta y_k|}{\rho_3}\right)\right]^{p_k}\\ &\le \frac{1}{\lambda_n}\sum_{k\in I_n}\frac{1}{2^{p_k}}\left[M\left(\frac{|x_k|}{\rho_1}\right)+M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k}\\ &\le \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\rho_1}\right)+M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k}\\ \le K\cdot\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\rho_1}\right)\right]^{p_k} &+ K\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|y_k|}{\rho_2}\right)\right]^{p_k}\to 0\end{aligned}$$

as $n\to\infty$, where $K=\max\left(1,2^{H-1}\right)$, $H=\sup p_k$, so that $\alpha x+\beta y\in[V,M,p]_0$. This completes the proof. □

Theorem 2. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[V,M,p]_0$ is a total paranormed spaces with*

$$g(x)=\inf\left\{\rho^{p_n/H}:\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1\,,\quad n=1,2,3,\dots\right\}.$$

where $H=\max(1,\sup p_k)$.

Proof. Clearly $g(x)=g(-x)$. By using Theorem 1, for a $\alpha=\beta=1$, we get $g(x+y)\le g(x)+g(y)$. Since $M(0)=0$, we get $\inf\{\rho^{p_n/H}\}=0$ for $x=0$.

Conversely, suppose $g(x)=0$, then

$$\inf\left\{\rho^{p_n/H}:\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1\right\}=0\,.$$

This implies that for a given $\varepsilon>0$, there exists some ρ_ε $(0<\rho_\varepsilon<\varepsilon)$ such that

$$\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\rho_\varepsilon}\right)\right]^{p_k}\right)^{1/H}\le 1\,.$$

Thus,

$$\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\varepsilon}\right)\right]^{p_k}\right)^{1/H}\le\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_k|}{\rho_\varepsilon}\right)\right]^{p_k}\right)^{1/H}\le 1\,,$$

for each n.

Suppose that $x_{n_m}\ne 0$ for some $m\in I_n$. Let $\varepsilon\to 0$, then $\left(\frac{|x_{n_m}|}{\varepsilon}\right)\to\infty$. It follows that

$$\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|x_{n_m}|}{\varepsilon}\right)\right]^{p_k}\right)^{1/H}\to\infty$$

which is a contradiction. Therefore $x_{n_m} = 0$ for each m. Finally, we prove that scalar multiplication is continuous. Let μ be any complex number. By definition

$$g(\mu x) = \inf \left\{ \rho^{p_n/H} : \left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\mu x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1\,, \quad n = 1, 2, 3, \dots \right\}.$$

Then

$$g(\mu x) = \inf \left\{ (|\mu| s)^{p_n/H} : \left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1\,, \quad n = 1, 2, 3, \dots \right\}$$

where $s = \rho/|\mu|$. Since $|\mu|^{p_n} \leq \max(1, |\mu|^{\sup p_n})$, we have

$$\begin{aligned} g(\mu x) &\leq (\max(1, |\mu|^{\sup p_n}))^{1/H} \\ &\quad \times \inf \left\{ s^{p_n/H} : \left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1\,, \quad n = 1, 2, 3, \dots \right\} \end{aligned}$$

which converges to zero as x converges to zero in $[V, M, p]_0$.

Now suppose $\mu_m \to 0$ and x is fixed in $[V, M, p]_0$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} < (\varepsilon/2)^H \quad \text{for some} \quad \rho > 0 \quad \text{and all} \quad n > N\,.$$

This implies that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} < \varepsilon/2 \quad \text{for some} \quad \rho > 0 \quad \text{and all} \quad n > N\,.$$

Let $0 < |\mu| < 1$, using convexity of M, for $n > N$, we get

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\mu x_k|}{\rho} \right) \right]^{p_k} < \frac{1}{\lambda_n} \sum_{k \in I_n} \left[|\mu| M\left(\frac{|x_k|}{\rho} \right) \right]^{p_k} < (\varepsilon/2)^H\,.$$

Since M is continuous everywhere in $[0, \infty)$, then for $n \leq N$

$$f(t) = \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t x_k|}{\rho} \right) \right]^{p_k}$$

is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < (\varepsilon/2)^H$ for $0 < t < \delta$. Let K be such that $|\mu_m| < \delta$ for $m > K$ then for $m > K$ and $n \leq N$

$$\left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\mu_m x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon/2\,.$$

Thus

$$\left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\mu_m x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon$$

for $m > K$ and all n, so that $g(\mu x) \to 0$ $(\mu \to 0)$. □

Definition 2 ([1])**.** An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$, $u \ge 0$.

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \le K(l)M(u)$, for all values of u and for $l > 1$.

Theorem 3. *For any Orlicz function M which satisfies Δ_2-condition, we have $[V, \lambda] \subseteq [V, M]$.*

Proof. Let $x \in [V, \lambda]$ so that

$$T_n = \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - \ell| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for some} \quad \ell\,.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t \le \delta$. Write $y_k = |x_k - \ell|$ and consider

$$\frac{1}{\lambda_n} \sum_{k \in I_n} M\left(|y_k|\right) = \sum\nolimits_1 + \sum\nolimits_2$$

where the first summation is over $y_k \le \delta$ and the second summation over $y_k > \delta$. Since, M is continuous

$$\sum\nolimits_1 < \lambda_n \varepsilon$$

and for $y_k > \delta$ we use the fact that $y_k < y_k/\delta < 1 + y_k/\delta$. Since M is non decreasing and convex, it follows that

$$M(y_k) < M\left(1 + \delta^{-1} y_k\right) < \frac{1}{2} M(2) + \frac{1}{2} M\left(2\delta^{-1} y_k\right)$$

Since M satisfies Δ_2-condition there is a constant $K > 2$ such that $M\left(2\delta^{-1} y_k\right) \le \frac{1}{2} K \delta^{-1} y_k M(2)$, therefore

$$\begin{aligned} M(y_k) &< \frac{1}{2} K \delta^{-1} y_k M(2) + \frac{1}{2} K \delta^{-1} y_k M(2) \\ &= K \delta^{-1} y_k M(2)\,. \end{aligned}$$

Hence

$$\sum\nolimits_2 M(y_k) \le K \delta^{-1} M(2) \lambda_n T_n$$

which together with $\sum_1 \le \varepsilon \lambda_n$ yields $[V, \lambda] \subseteq [V, M]$. This completes proof. □

The method of the proof of Theorem 3 shows that for any Orlicz function M which satisfies Δ_2-condition; we have $[V, \lambda]_0 \subset [V, M]_0$ and $[V, \lambda]_\infty \subset [V, M]_\infty$.

Theorem 4. *Let $0 \le p_k \le q_k$ and $\left(q_k / p_k\right)$ be bounded. Then $[V, M, q] \subset [V, M, p]$.*

The proof of Theorem 4 used the ideas similar to those used in proving Theorem 7 of Parashar and Choudhary [9].

We now introduce a natural relationship between strong convergence with respect to an Orlicz function and λ-statistical convergence. Recently, Mursaleen [7] introduced the concept of statistical convergence as follows:

Definition 3. A sequence $x = (x_k)$ is said to be λ-statistically convergent or s_λ-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0\,,$$

where the vertical bars indicate the number of elements in the enclosed set.

In this case we write $s_\lambda - \lim x = L$ or $x_k \to L(s_\lambda)$ and $s_\lambda = \{x : \exists L \in R : s_\lambda - \lim x = L\}$.

Later on, λ-statistical convergence was generalized by Savaş [12].

We now establish an inclusion relation between $[V, M]$ and s_λ.

Theorem 5. *For any Orlicz function M, $[V, M] \subset s_\lambda$.*

Proof. Let $x \in [V, M]$ and $\varepsilon > 0$. Then

$$\begin{aligned}
\frac{1}{\lambda_n} \sum_{k \in I_n} M\left(\frac{|x_k - \ell|}{\rho}\right) &\geq \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - l| \geq \varepsilon} M\left(\frac{|x_k - \ell|}{\rho}\right) \\
&\geq \frac{1}{\lambda_n} M\left(\varepsilon/\rho\right) \cdot |\{k \in I_n : |x_k - \ell| \geq \varepsilon\}|
\end{aligned}$$

from which it follows that $x \in s_\lambda$.

To show that s_λ strictly contains $[V, M]$, we proceed as in [7]. We define $x = (x_k)$ by $x_k = k$ if $n - \left[\sqrt{\lambda_n}\right] + 1 \leq k \leq n$ and $x_k = 0$ otherwise. Then $x \notin \ell_\infty$ and for every ε $(0 < \varepsilon \leq 1)$

$$\frac{1}{\lambda_n} |\{k \in I_n : |x_k - 0| \geq \varepsilon\}| = \frac{\left[\sqrt{\lambda_n}\right]}{\lambda_n} \to 0 \quad \text{as} \quad n \to \infty$$

i.e. $x_k \to 0\,(s_\lambda)$, where $[\ ]$ denotes the greatest integer function. On the other hand,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} M\left(\frac{|x_k - 0|}{\rho}\right) \to \infty \qquad (n \to \infty)$$

i.e. $x_k \not\to 0\,[V, M]$. This completes the proof. □

References

[1] Krasnoselski, M. A. and Rutitsky, Y. B., *Convex functions and Orlicz Spaces*, Groningen, the Netherlands, 1961.

[2] Leindler, L., *Über de la Vallee Pousinsche Summierbarkeit allgemeiner Orthogonalreihen*, Acta Math. Hung. **16** (1965), 375–378.

[3] Lindenstrauss, J., *Some aspects of the theory of Banach spaces*, Adv. Math. **5** (1970), 159–180.

[4] Lindenstrauss, J. and Tzafriri, L., *On Orlicz sequence spaces*, Israel J. Math. **10** (3) (1971), 379–390.

[5] Maddox, I. J., *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser. (2), **18** (1967), 345–355.

[6] Maddox, I. J., *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 161–166.

[7] Mursaleen, λ-*statistical convergence*, Math. Slovaca **50**, No.1 (2000), 111–115.

[8] Nakano, H., *Concave modulus*, J. Math. Soc. Japan **5** (1953), 29–49.

[9] Parashar, S. D. and Choudhary, B., *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math. **25** (14) (1994), 419–428.

[10] Ruckle, W. H., *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973), 973–978.

[11] Savaş, E., *Strongly almost convergent sequences defined by Orlicz functions*, Comm. Appl. Anal. **4** (2000), 453–458.

[12] Savaş, E., *Strong almost convergence and almost* λ-*statistical convergence*, Hokkaido Math. J. **24**(3), (2000), 531–536.

[13] Savaş, E. and Rhoades, B. E., *On some new sequence spaces of invariant means defined by Orlicz functions*, Math. Inequal. Appl. **5**(2) (2002), 271–281.

[14] Wilansky, A., *Summability through Functional Analysis*, North-Holland Math. Stud. **85** (1984).

Department of Mathematics, Yüzüncü Yil Üniversity
Van 65080, Turkey
E-mail: ekremsavas@yahoo.com