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SOLVABILITY OF A PERIODIC TYPE BOUNDARY VALUE PROBLEM FOR FIRST ORDER SCALAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Nonimprovable sufficient conditions for the solvability and unique solvability of the problem

$$u'(t) = F(u)(t)\,, \qquad u(a) - \lambda u(b) = h(u)$$

are established, where $F : C([a,b];R) \to L([a,b];R)$ is a continuous operator satisfying the Carathèodory conditions, $h : C([a,b];R) \to R$ is a continuous functional, and $\lambda \in R_+$.

Introduction

The following notation is used throughout the paper.

R is the set of all real numbers, $R_+ = [0, +\infty[$.

$C([a,b];R)$ is the Banach space of continuous functions $u : [a,b] \to R$ with the norm $\|u\|_C = \max\{|u(t)| : a \le t \le b\}$.

$C([a,b];R_+) = \{u \in C([a,b];R) : u(t) \ge 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u : [a,b] \to R$.

$B^i_{\lambda c}([a,b];R)$, where $c \in R_+$ and $i \in \{1,2\}$, is the set of functions $u \in C([a,b];R)$ satisfying

$$(-1)^{i+1}\big(u(a) - \lambda u(b)\big)\,\mathrm{sgn}\,\big((2-i)u(a) + (i-1)u(b)\big) \le c\,.$$

$L([a,b];R)$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|\,ds$.

$L([a,b];R_+) = \{p \in L([a,b];R) : p(t) \ge 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$.

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$\widetilde{\mathcal{L}}_{ab}$ is the set of linear operators $\ell : C([a,b];R) \to L([a,b];R)$ for which there is a function $\eta \in L([a,b];R_+)$ such that

$$|\ell(v)(t)| \le \eta(t)\|v\|_C \qquad \text{for} \qquad t \in [a,b]\,,\ v \in C([a,b];R)\,.$$

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \widetilde{\mathcal{L}}_{ab}$ transforming the set $C([a,b];R_+)$ into the set $L([a,b];R_+)$.

$\mathcal{K}_{ab}$ is the set of continuous operators $F : C([a,b];R) \to L([a,b];R)$ satisfying the Carathèodory conditions, i.e., for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|F(v)(t)| \le q_r(t) \qquad \text{for} \qquad t \in [a,b]\,,\ \|v\|_C \le r\,.$$

$K([a,b] \times A; B)$, where $A \subseteq R^2$, $B \subseteq R$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathèodory conditions, i.e., $f(\cdot, x) : [a,b] \to B$ is a measurable function for all $x \in A$, $f(t,\cdot) : A \to B$ is a continuous function for almost all $t \in [a,b]$, and for each $r > 0$ there exists $q_r \in L([a,b];R_+)$ such that

$$|f(t,x)| \le q_r(t) \qquad \text{for} \qquad t \in [a,b]\,,\ x \in A\,,\ \|x\| \le r\,.$$

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

By a solution of the equation

$$u'(t) = F(u)(t)\,, \tag{0.1}$$

where $F \in \mathcal{K}_{ab}$, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equality (0.1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of (0.1) satisfying the boundary condition

$$u(a) - \lambda u(b) = h(u)\,, \tag{0.2}$$

where $h : C([a,b];R) \to R$ is a continuous functional such that for each $r > 0$ there exists $M_r \in R_+$ such that

$$|h(v)| \le M_r \qquad \text{for} \qquad \|v\|_C \le r\,,$$

and $\lambda \in R_+$.

For ordinary differential equations, i.e., when the operator F is so–called Nemitsky operator, the problem (0.1), (0.2) have been studied in details (see [8,19–22] and references therein). The basis of the theory of general boundary value problems for functional differential equations were laid down in monographs [1] and [29] (see also [2, 3, 9, 10, 19, 24–28]). In spite of a large number of papers devoted to boundary value problems for functional differential equations, only a few efficient sufficient solvability conditions are known at present even in the linear case

$$u'(t) = \ell(u)(t) + q_0(t)\,, \tag{0.3}$$

$$u(a) - \lambda u(b) = c_0\,, \tag{0.4}$$

where $\ell \in \widetilde{\mathcal{L}}_{ab}$, $q_0 \in L([a,b];R)$, and $c_0 \in R$ (see [5–7, 11–18, 30]). Here, we try to fill this gap in a certain way. More precisely, in Sections 1 and 2 there are established nonimprovable effective sufficient conditions for the solvability and unique solvability of the problems (0.3), (0.4) and (0.1), (0.2), respectively. Section 3 is devoted to the examples verifying the optimality of the main results.

All results are concretized for the differential equations with deviating arguments of the forms

$$u'(t) = p(t)u\big(\tau(t)\big) - g(t)u\big(\mu(t)\big) + f\big(t, u(t), u\big(\nu(t)\big)\big)\,, \tag{0.5}$$

and

$$u'(t) = p(t)u\big(\tau(t)\big) - g(t)u\big(\mu(t)\big) + q_0(t)\,, \tag{0.6}$$

where $p, g \in L([a,b]; R_+)$, $q_0 \in L([a,b]; R)$, $\tau, \mu, \nu \in \mathcal{M}_{ab}$, and $f \in K([a,b] \times R^2; R)$.

1. Linear problem

We need the following well–known result from the general theory of linear boudary value problems for functional differential equations (see, e.g., [4, 24, 29]).

Theorem 1.1. *The problem* (0.3)*,* (0.4) *is uniquely solvable iff the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{0.3_0}$$

$$u(a) - \lambda u(b) = 0 \tag{0.4_0}$$

has only the trivial solution.

Remark 1.1. It follows from the Riesz–Schauder theory that if $\ell \in \widetilde{\mathcal{L}}_{ab}$ and the problem (0.3_0), (0.4_0) has a nontrivial solution, then there exist $q_0 \in L([a,b]; R)$ and $c_0 \in R$ such that the problem (0.3), (0.4) has no solution.

Definition 1.1. We will say that an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ belongs to the set $V^+(\lambda)$ (resp. $V^-(\lambda)$), if the homogeneous problem (0.3_0), (0.4_0) has only the trivial solution, and for arbitrary $q_0 \in L([a,b]; R_+)$ and $c_0 \in R_+$, the solution of the problem (0.3), (0.4) is nonnegative (resp. nonpositive).

Remark 1.2. From Definition 1.1 it immediately follows that $\ell \in V^+(\lambda)$ (resp. $\ell \in V^-(\lambda)$), iff for the equation (0.3) the certain theorem on differential inequalities holds, i.e., whenever $u, v \in \widetilde{C}([a,b]; R)$ satisfy the inequalities

$$u'(t) \le \ell(u)(t) + q_0(t)\,, \qquad v'(t) \ge \ell(v)(t) + q_0(t) \qquad \text{for} \qquad t \in [a,b]\,,$$
$$u(a) - \lambda u(b) \le v(a) - \lambda v(b)\,,$$

then $u(t) \le v(t)$ (resp. $u(t) \ge v(t)$) for $t \in [a,b]$.

1.1. Main results

Theorem 1.2. *Let* $\lambda \in [0,1[$ *and there exist an operator*

$$\ell_0 \in V^+(\lambda) \tag{1.1}$$

such that, on the set $\{v \in C([a,b];R) : v(a) - \lambda v(b) = 0\}$*, the inequality*

$$\ell(v)(t)\operatorname{sgn} v(t) \le \ell_0(|v|)(t) \qquad \textit{for} \qquad t \in [a,b] \tag{1.2}$$

holds. Then the problem (0.3), (0.4) *has a unique solution.*

Remark 1.3. Theorem 1.2 is nonimprovable in a certain sense. More precisely, the inequality (1.2) cannot be replaced by the inequality

$$\ell(v)(t)\operatorname{sgn} v(t) \le (1+\varepsilon)\ell_0(|v|)(t) \qquad \text{for} \qquad t \in [a,b]\,, \tag{1.3}$$

and the condition (1.1) cannot be replaced by the condition

$$(1-\varepsilon)\ell_0 \in V^+(\lambda)\,, \tag{1.4}$$

no matter how small $\varepsilon > 0$ would be (see On Remarks 1.3 and 1.4, and Examples 3.1 and 3.2).

Theorem 1.3. *Let* $\lambda \in [0,1[$ *and there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that, on the set* $\{v \in C([a,b];R) : v(a) - \lambda v(b) = 0\}$*, the inequality*

$$|\ell(v)(t) + \ell_1(v)(t)| \le \ell_0(|v|)(t) \qquad \textit{for} \qquad t \in [a,b] \tag{1.5}$$

holds. If, moreover,

$$\ell_0 \in V^+(\lambda)\,, \qquad -\frac{1}{2}\ell_1 \in V^+(\lambda)\,, \tag{1.6}$$

then the problem (0.3), (0.4) *has a unique solution.*

Remark 1.4. Theorem 1.3 is nonimprovable. More precisely, the condition (1.5) cannot be replaced by the condition

$$|\ell(v)(t) + \ell_1(v)(t)| \le (1+\varepsilon)\ell_0(|v|)(t) \qquad \text{for} \qquad t \in [a,b]\,, \tag{1.7}$$

and the assumption (1.6) can be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in V^+(\lambda)\,, \qquad -\frac{1}{2}\ell_1 \in V^+(\lambda)\,, \tag{1.8}$$

nor by the assumption

$$\ell_0 \in V^+(\lambda)\,, \qquad -\frac{1}{2+\varepsilon}\ell_1 \in V^+(\lambda)\,, \tag{1.9}$$

no matter how small $\varepsilon > 0$ would be (see On Remarks 1.3 and 1.4, and Examples 3.1, 3.2 and 3.3).

Remark 1.5. Let $\lambda \in [1,+\infty[$, $\ell \in \widetilde{\mathcal{L}}_{ab}$, $q_0 \in L([a,b];R)$, and $c_0 \in R$. Introduce the operator $\psi : L([a,b];R) \to L([a,b];R)$ by setting

$$\psi(w)(t) \stackrel{\text{def}}{=} w(a+b-t)\,.$$

Let φ be the restriction of ψ to the space $C([a,b];R)$. Put $\vartheta = \frac{1}{\lambda}$, $\widehat{c}_0 = -\vartheta c_0$ and

$$\widehat{\ell}(w)(t) \stackrel{\text{def}}{=} -\psi\big(\ell(\varphi(w))\big)(t)\,, \qquad \widehat{q}_0(t) \stackrel{\text{def}}{=} -\psi(q_0)(t)\,.$$

It is clear that if u is a solution of the problem (0.3), (0.4) then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{\ell}(v)(t) + \widehat{q}_0(t)\,, \qquad v(a) - \vartheta v(b) = \widehat{c}_0\,, \tag{1.10}$$

and vice versa, if v is a solution of the problem (1.10) then the function $u \stackrel{\mathrm{def}}{=} \varphi(v)$ is a solution of the problem (0.3), (0.4).

Therefore, $\ell \in V^+(\lambda)$ (resp. $\ell \in V^-(\lambda)$) if and only if $\widehat{\ell} \in V^-(\vartheta)$ (resp. $\widehat{\ell} \in V^+(\vartheta)$).

According to Remark 1.5, Theorems 1.2 and 1.3 imply

Theorem 1.4. *Let* $\lambda \in \,]1, +\infty[$ *and there exist an operator*

$$\ell_0 \in V^-(\lambda)$$

such that, on the set $\{v \in C([a,b]; R) : v(a) - \lambda v(b) = 0\}$, *the inequality*

$$\ell(v)(t) \operatorname{sgn} v(t) \geq \ell_0(|v|)(t) \qquad \text{for } t \in [a,b]$$

holds. Then the problem (0.3), (0.4) *has a unique solution.*

Theorem 1.5. *Let* $\lambda \in \,]1, +\infty[$ *and there exist* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that, on the set* $\{v \in C([a,b]; R) : v(a) - \lambda v(b) = 0\}$, *the inequality*

$$|\ell(v)(t) - \ell_1(v)(t)| \leq \ell_0(|v|)(t) \qquad \text{for} \quad t \in [a,b]$$

holds. Let, moreover,

$$-\ell_0 \in V^-(\lambda)\,, \qquad \frac{1}{2}\ell_1 \in V^-(\lambda)\,.$$

Then the problem (0.3), (0.4) *has a unique solution.*

Remark 1.6. According to Remarks 1.3–1.5, the conditions in Theorems 1.4 and 1.5 are also nonimprovable in an appropriate sense.

Remark 1.7. In [17] and [18], effective nonimprovable sufficient conditions for an operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ to belong to the set $V^+(\lambda)$ or $V^-(\lambda)$ have been established. Therefore, according to Theorems 1.2–1.5, the following corollaries are valid.

Corollary 1.1. *Let* $\lambda \in [0, 1[$ *and the functions* p, τ *satisfy at least one of the following conditions:*

a) $\tau(t) \leq t$ *for* $t \in [a,b]$ *and*

$$\lambda \exp\Big(\int_a^b p(s)\,ds\Big) < 1\,;$$

b) *there exists* $\alpha \in \,]0, 1\,[$ *such that*

$$\frac{\lambda}{1-\lambda}\int_a^b p(s)\int_a^{\tau(s)} p(\xi)\,d\xi\,ds + \int_a^t p(s)\int_a^{\tau(s)} p(\xi)\,d\xi\,ds$$
$$\leq \Big(\alpha - \frac{\lambda}{1-\lambda}\int_a^b p(s)\,ds\Big)\Big(\frac{\lambda}{1-\lambda}\int_a^b p(s)\,ds + \int_a^t p(s)\,ds\Big) \qquad \text{for} \quad t \in [a,b]\,;$$

c)

$$\lambda \exp\Big(\int_a^b p(s)\,ds\Big) + \int_a^b p(s)\sigma(s)\Big(\int_s^{\tau(s)} p(\xi)\,d\xi\Big)\exp\Big(\int_s^b p(\eta)\,d\eta\Big)\,ds < 1\,,$$

where $\sigma(t) = \frac{1}{2}\big(1 + \operatorname{sgn}\big(\tau(t) - t\big)\big)$ *for* $t \in [a, b]$*;*

d) $\lambda \neq 0$, $p \not\equiv 0$ *and there exist* $x \in \,\big]0, \ln\frac{1}{\lambda}\big]$ *such that*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)} p(s)\,ds : t \in [a, b]\right\} < \frac{\|p\|_L}{x}\left(x + \ln\frac{(1-\lambda)x}{\|p\|_L\big(e^x - 1\big)}\right),$$

while the functions g*,* μ *satisfy at least one of the following conditions:*

e) $\lambda \neq 0$ *and*

$$\int_a^b g(s)\,ds \le 2\lambda\,;$$

f) $\mu(t) \le t$ *for* $t \in [a, b]$ *and*

$$\int_a^b g(s)\,ds \le 2\,;$$

g) $\mu(t) \le t$ *for* $t \in [a, b]$ *and*

$$\int_a^b g(s)\int_{\mu(s)}^s g(\xi)\exp\Big(\frac{1}{2}\int_{\mu(\xi)}^s g(\eta)\,d\eta\Big)\,d\xi\,ds \le 4\,;$$

h) $g \not\equiv 0$, $\mu(t) \le t$ *for* $t \in [a, b]$ *and*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)\,ds : t \in [a, b]\right\} < 2\eta^*\,,$$

where

$$\eta^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\Big(\frac{x}{2}\int_a^b g(s)\,ds\Big) - 1}\right) : x > 0\right\}.$$

Then the problem (0.6), (0.4) *has a unique solution.*

Corollary 1.2. *Let* $\lambda \in \,]1, +\infty[$ *and the functions* g*,* μ *satisfy at least one of the following conditions:*

a) $\mu(t) \ge t$ *for* $t \in [a, b]$ *and*

$$\exp\left(\int_a^b g(s)\,ds\right) < \lambda\,;$$

b) *there exists* $\alpha \in \,]0,1[$ *such that*

$$\frac{1}{\lambda-1}\int_a^b g(s)\int_{\mu(s)}^b g(\xi)\,d\xi\,ds + \int_t^b g(s)\int_{\mu(s)}^b g(\xi)\,d\xi\,ds$$
$$\leq \left(\alpha - \frac{1}{\lambda-1}\int_a^b g(s)\,ds\right)\left(\frac{1}{\lambda-1}\int_a^b g(s)\,ds + \int_t^b g(s)\,ds\right)$$
$$\textit{for} \quad t \in [a,b]\,;$$

c)

$$\frac{1}{\lambda}\exp\left(\int_a^b g(s)\,ds\right) + \int_a^b g(s)\sigma(s)\left(\int_{\mu(s)}^s g(\xi)\,d\xi\right)\exp\left(\int_a^s g(\eta)\,d\eta\right)ds < 1\,,$$

where $\sigma(t) = \frac{1}{2}\big(1 + \operatorname{sgn}\big(t - \mu(t)\big)\big)$ *for* $t \in [a,b]$*;*

d) $g \not\equiv 0$ *and there exist* $x \in \,]0, \ln\lambda]$ *such that*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)\,ds : t \in [a,b]\right\} < \frac{\|g\|_L}{x}\left(x + \ln\frac{(\lambda-1)x}{\lambda\|g\|_L\big(e^x - 1\big)}\right),$$

while the functions p, τ *satisfy at least one of the following conditions:*

e)

$$\int_a^b p(s)\,ds \leq \frac{2}{\lambda}\,;$$

f) $\tau(t) \geq t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)\,ds \leq 2\,;$$

g) $\tau(t) \geq t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)\int_s^{\tau(s)} p(\xi)\exp\left(\frac{1}{2}\int_s^{\tau(\xi)} p(\eta)\,d\eta\right)d\xi\,ds \leq 4\,;$$

h) $p \not\equiv 0$, $\tau(t) \geq t$ *for* $t \in [a,b]$ *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)} p(s)\,ds : t \in [a,b]\right\} < 2\kappa^*\,,$$

where

$$\kappa^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\left(\frac{x}{2}\int\limits_a^b p(s)\,ds\right) - 1} \right) : x > 0 \right\}.$$

Then the problem (0.6), (0.4) *has a unique solution.*

1.2. Proofs of main results

Proof of Theorem 1.2. According to Theorem 1.1, it is sufficient to show that the problem (0.3_0), (0.4_0) has only the trivial solution.

Let u be a solution of (0.3_0), (0.4_0). Then, in view of (1.2), we have

$$|u(t)|' = \ell(u)(t)\operatorname{sgn} u(t) \le \ell_0(|u|)(t) \qquad \text{for} \quad t \in [a,b],$$

$$|u(a)| - \lambda |u(b)| = 0.$$

Hence, $|u|$ is a solution of the problem (0.3), (0.4_0) with $\ell \equiv \ell_0$ and $q_0(t) = |u(t)|' - \ell(|u|)(t) \le 0$ for $t \in [a,b]$. Therefore, according to (1.1), we obtain $|u(t)| \le 0$ for $t \in [a,b]$, i.e. $u \equiv 0$. □

Proof of Theorem 1.3. According to Theorem 1.1, it is sufficient to show that the problem (0.3_0), (0.4_0) has only the trivial solution.

Let u be a solution of (0.3_0), (0.4_0). Then, in view of (0.3_0), u satisfies

$$u'(t) = -\frac{1}{2}\ell_1(u)(t) + \ell(u)(t) + \frac{1}{2}\ell_1(u)(t), \qquad u(a) - \lambda u(b) = 0. \tag{1.11}$$

By virtue of the assumption $-\frac{1}{2}\ell_1 \in V^+(\lambda)$ and Theorem 1.1, the problem

$$\alpha'(t) = -\frac{1}{2}\ell_1(\alpha)(t) + \ell_0(|u|)(t) + \frac{1}{2}\ell_1(|u|)(t), \qquad \alpha(a) - \lambda\alpha(b) = 0 \tag{1.12}$$

has a unique solution α. Moreover, since $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and $-\frac{1}{2}\ell_1 \in V^+(\lambda)$,

$$\alpha(t) \ge 0 \qquad \text{for} \;\; t \in [a,b]. \tag{1.13}$$

The equality (1.12), in view of (1.5) and the condition $\ell_1 \in \mathcal{P}_{ab}$, yields

$$\alpha'(t) \ge -\frac{1}{2}\ell_1(\alpha)(t) + \ell(u)(t) + \frac{1}{2}\ell_1(u)(t) \qquad \text{for} \;\; t \in [a,b],$$

$$\big(-\alpha(t)\big)' \le -\frac{1}{2}\ell_1(-\alpha)(t) + \ell(u)(t) + \frac{1}{2}\ell_1(u)(t) \qquad \text{for} \;\; t \in [a,b].$$

The last two inequalities and (1.11), on account of the assumption $-\frac{1}{2}\ell_1 \in V^+(\lambda)$ and Remark 1.2, yield

$$|u(t)| \le \alpha(t) \qquad \text{for} \;\; t \in [a,b]. \tag{1.14}$$

On the other hand, due to (1.14) and the conditions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, the equality (1.12) results in

$$\alpha'(t) \le \ell_0(\alpha)(t) \qquad \text{for } \ t \in [a,b]\,.$$

Since $\ell_0 \in V^+(\lambda)$, the last inequality, together with $\alpha(a) - \lambda\alpha(b) = 0$, yield $\alpha(t) \le 0$ for $t \in [a,b]$ which, in view of (1.13), implies $\alpha \equiv 0$. Consequently, it follows from (1.14) that $u \equiv 0$. $\square$

2. Nonlinear problem

Throughout this section we assume that $q \in K([a,b] \times R_+; R_+)$ is nondecreasing in the second argument and satisfies

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b q(s,x)\,ds = 0\,. \tag{2.1}$$

2.1. Main results

Theorem 2.1. *Let $\lambda \in [0,1[$, $c \in R_+$,*

$$h(v)\operatorname{sgn} v(a) \le c \qquad \textit{for } \ v \in C([a,b];R)\,, \tag{2.2}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ such that on the set $B^1_{\lambda c}([a,b];R)$ the inequality

$$\big[F(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \le \ell_0(|v|)(t) + q(t, \|v\|_C) \qquad \textit{for } \ t \in [a,b] \tag{2.3}$$

is fulfilled. If, moreover,

$$\ell_0 \in V^+(\lambda), \qquad -\ell_1 \in V^+(\lambda)\,, \tag{2.4}$$

then the problem (0.1), (0.2) *has at least one solution.*

Remark 2.1. Theorem 2.1 is nonimprovable in a certain sense. More precisely, the inequality (2.3) cannot be replaced by the inequality

$$\big[F(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \le (1+\varepsilon)\ell_0(|v|)(t) + q(t, \|v\|_C)\,, \tag{2.5}$$

no matter how small $\varepsilon > 0$ would be. Moreover, the condition (2.4) can be replaced neither by the condition

$$(1-\varepsilon)\ell_0 \in V^+(\lambda)\,, \qquad -\ell_1 \in V^+(\lambda)\,, \tag{2.6}$$

nor by the condition

$$\ell_0 \in V^+(\lambda)\,, \qquad -(1-\varepsilon)\ell_1 \in V^+(\lambda)\,, \tag{2.7}$$

no matter how small $\varepsilon > 0$ would be (see On Remark 2.1 and Example 3.4).

Theorem 2.2. *Let $\lambda \in [0,1[$,*

$$[h(v) - h(w)]\operatorname{sgn}\big(v(a) - w(a)\big) \le 0 \qquad \textit{for } \ v, w \in C([a,b];R)\,, \tag{2.8}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that, on the set* $B^1_{\lambda c}([a,b];R)$*, where* $c = |h(0)|$*, the inequality*

$$[F(v)(t) - F(w)(t) + \ell_1(v-w)(t)] \operatorname{sgn}\big(v(t) - w(t)\big) \leq \ell_0(|v-w|)(t)$$

is fulfilled. If, moreover, the condition (2.4) *is satisfied, then the problem* (0.1)*,* (0.2) *has a unique solution.*

Remark 2.2. Theorem 2.2 is nonimprovable in a certain sense (see On Remark 2.2).

Remark 2.3. Let $\lambda \in [1, +\infty[$, φ, ψ be the operators defined in Remark 1.5. Put $\vartheta = \frac{1}{\lambda}$, and

$$\widehat{F}(w)(t) \stackrel{\text{def}}{=} -\psi\big(F(\varphi(w))\big)(t)\,, \qquad \widehat{h}(w) \stackrel{\text{def}}{=} -\vartheta h\big(\varphi(w)\big)\,.$$

It is clear that if u is a solution of the problem (0.1), (0.2), then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution of the problem

$$v'(t) = \widehat{F}(v)(t)\,, \qquad v(a) - \vartheta v(b) = \widehat{h}(v)\,, \tag{2.9}$$

and vice versa, if v is a solution of the problem (2.9), then the function $u \stackrel{\text{def}}{=} \varphi(v)$ is a solution of the problem (0.1), (0.2).

In view of Remarks 1.5 and 2.3, the following results are an immediate consequence of Theorems 2.1 and 2.2.

Theorem 2.3. *Let* $\lambda \in \,]1, +\infty[$, $c \in R_+$,

$$h(v) \operatorname{sgn} v(b) \geq -c \qquad \text{for } \ v \in C([a,b];R)\,, \tag{2.10}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that, on the set* $B^2_{\lambda c}([a,b];R)$*, the inequality*

$$\big[F(v)(t) - \ell_1(v)(t)\big] \operatorname{sgn} v(t) \geq -\ell_0(|v|)(t) - q(t, \|v\|_C) \qquad \text{for } \ t \in [a,b]$$

is fulfilled. If, moreover,

$$-\ell_0 \in V^-(\lambda)\,, \qquad \ell_1 \in V^-(\lambda)\,, \tag{2.11}$$

then the problem (0.1)*,* (0.2) *has at least one solution.*

Theorem 2.4. *Let* $\lambda \in \,]1, +\infty[$,

$$[h(v) - h(w)] \operatorname{sgn}\big(v(b) - w(b)\big) \geq 0 \qquad \text{for } \ v, w \in C([a,b];R)\,, \tag{2.12}$$

and let there exist $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *such that, on the set* $B^2_{\lambda c}([a,b];R)$*, where* $c = |h(0)|$*, the inequality*

$$[F(v)(t) - F(w)(t) - \ell_1(v-w)(t)] \operatorname{sgn}\big(v(t) - w(t)\big) \geq -\ell_0(|v-w|)(t)$$

is fulfilled. If, moreover, the condition (2.11) *is satisfied, then the problem* (0.1)*,* (0.2) *has a unique solution.*

Remark 2.4. According to Remarks 1.5, 2.1, 2.2, and 2.3, Theorems 2.3 and 2.4 are nonimprovable in an appropriate sense.

For the problem (0.5), (0.2), Theorems 2.1–2.4 imply the following assertions.

Corollary 2.1. *Let* $\lambda \in [0,1[\,$, $c \in R_+$, *the condition* (2.2) *be fulfilled, and*

$$f(t,x,y)\operatorname{sgn} x \le q(t,|x|) \qquad \text{for } \; t \in [a,b]\,, \; x,y \in R\,.$$

Let, moreover, the functions p, τ *satisfy at least one of the conditions* a), b), c) *or* d) *in Corollary 1.1, while the functions* g, μ *satisfy at least one of the following conditions:*

e) $\lambda \ne 0$ *and*

$$\int_a^b g(s)\,ds \le \lambda\,;$$

f) $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\int_a^b g(s)\,ds \le 1\,;$$

g) $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\int_a^b g(s) \int_{\mu(s)}^s g(\xi) \exp\left(\int_{\mu(\xi)}^s g(\eta)\,d\eta \right) d\xi\,ds \le 1\,;$$

h) $g \not\equiv 0$, $\mu(t) \le t$ *for* $t \in [a,b]$ *and*

$$\operatorname{ess\,sup}\left\{ \int_{\mu(t)}^t g(s)\,ds : t \in [a,b] \right\} < \eta^*\,,$$

where

$$\eta^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\left(x \int_a^b g(s)\,ds \right) - 1} \right) : x > 0 \right\}.$$

Then the problem (0.5), (0.2) *has at least one solution.*

Corollary 2.2. *Let* $\lambda \in [0,1[\,$, *the condition* (2.8) *be fulfilled, and*

$$[f(t,x_1,y_1)-f(t,x_2,y_2)]\operatorname{sgn}(x_1-x_2) \le 0 \qquad \text{for } \; t \in [a,b]\,, \quad x_1,y_1,x_2,y_2 \in R\,.$$

Let, moreover, the functions p, τ *satisfy at least one of the conditions* a), b), c) *or* d) *in Corollary 1.1, while the functions* g, μ *satisfy at least one of the conditions* e), f), g) *or* h) *in Corollary 2.1. Then the problem* (0.5), (0.2) *has a unique solution.*

Corollary 2.3. *Let* $\lambda \in]1,+\infty[\,$, $c \in R_+$, *the condition* (2.10) *be fulfilled, and*

$$f(t,x,y)\operatorname{sgn} x \ge -q(t,|x|) \qquad \text{for } \; t \in [a,b]\,, \; x,y \in R\,.$$

Let, moreover, the functions g, μ *satisfy at least one of the conditions* a), b), c) *or* d) *in Corollary 1.2, while the functions* p, τ *satisfy at least one of the following conditions:*

e)

$$\int_a^b p(s)\,ds \le \frac{1}{\lambda}\,;$$

f) $\tau(t) \ge t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s)\,ds \le 1\,;$$

g) $\tau(t) \ge t$ *for* $t \in [a,b]$ *and*

$$\int_a^b p(s) \int_s^{\tau(s)} p(\xi) \exp\left(\int_s^{\tau(\xi)} p(\eta)\,d\eta \right) d\xi\,ds \le 1\,;$$

h) $p \not\equiv 0$, $\tau(t) \ge t$ *for* $t \in [a,b]$ *and*

$$\operatorname{ess\,sup}\left\{ \int_t^{\tau(t)} p(s)\,ds : t \in [a,b] \right\} < \kappa^*\,,$$

where

$$\kappa^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\left(x \int_a^b p(s)\,ds \right) - 1} \right) : x > 0 \right\}.$$

Then the problem (0.5), (0.2) *has at least one solution.*

Corollary 2.4. *Let* $\lambda \in\,]1,+\infty[$, *the condition* (2.12) *be fulfilled, and*

$$[f(t,x_1,y_1) - f(t,x_2,y_2)] \operatorname{sgn}(x_1 - x_2) \ge 0 \qquad \text{for } t \in [a,b],\ x_1,y_1,x_2,y_2 \in R\,.$$

Let, moreover, the functions g, μ *satisfy at least one of the conditions* a), b), c) *or* d) *in Corollary 1.2, while the functions* p, τ *satisfy at least one of the conditions* e), f), g) *or* h) *in Corollary 2.3. Then the problem* (0.5), (0.2) *has a unique solution.*

2.2. Auxiliary propositions and proof of main results

First we formulate a result from [26, Theorem 1] in a suitable for us form.

Lemma 2.1. *Let there exist* $\ell_1 \in \widetilde{\mathcal{L}}_{ab}$ *and a positive number* ρ *such that the problem*

$$u'(t) + \ell_1(u)(t) = 0, \qquad u(a) - \lambda u(b) = 0 \tag{2.13}$$

has only the trivial solution and for every $\delta \in \,]0,1[\,$, *an arbitrary function* $u \in \widetilde{C}([a,b];R)$ *satisfying the relations*

$$u'(t) + \ell_1(u)(t) = \delta\big[F(u)(t) + \ell_1(u)(t)\big]\,, \quad u(a) - \lambda u(b) = \delta h(u)\,, \tag{2.14}$$

admits the estimate

$$\|u\|_C \le \rho\,. \tag{2.15}$$

Then the problem (0.1), (0.2) *has at least one solution.*

Definition 2.1. We say that the pair of operators (ℓ_0, ℓ_1) belongs to the set $U(\lambda)$, if $\ell_0 \in \mathcal{P}_{ab}$, $\ell_1 \in \widetilde{\mathcal{L}}_{ab}$, and there exists a positive number r such that, for arbitrary $q^* \in L([a,b];R_+)$ and $c \in R_+$, every function $u \in \widetilde{C}([a,b];R)$ satisfying the inequalities

$$[u(a) - \lambda u(b)] \operatorname{sgn} u(a) \le c\,, \tag{2.16}$$

$$[u'(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \le \ell_0(|u|)(t) + q^*(t) \qquad \text{for} \quad t \in [a,b]\,, \tag{2.17}$$

admits the estimate

$$\|u\|_C \le r\left(c + \|q^*\|_L\right)\,. \tag{2.18}$$

Lemma 2.2. *Let* $c \in R_+$,

$$h(v) \operatorname{sgn} v(a) \le c \qquad \textit{for} \quad v \in C([a,b];R)\,, \tag{2.19}$$

and let there exist $(\ell_0, \ell_1) \in U(\lambda)$ *such that, on the set* $B^1_{\lambda c}([a,b];R)$, *the inequality*

$$\big[F(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \le \ell_0(|v|)(t) + q(t, \|v\|_C) \qquad \textit{for} \quad t \in [a,b] \tag{2.20}$$

is fulfilled. Then the problem (0.1), (0.2) *has at least one solution.*

Proof. First note, that due to the condition $(\ell_0, \ell_1) \in U(\lambda)$, the homogeneous problem (2.13) has only the trivial solution.

Let r be the number appearing in Definition 2.1. According to (2.1), there exists $\rho > 2rc$ such that

$$\frac{1}{x}\int_a^b q(s,x)\,ds < \frac{1}{2r} \qquad \text{for} \qquad x > \rho\,.$$

Now assume that a function $u \in \widetilde{C}([a,b];R)$ satisfies (2.14) for some $\delta \in \,]0,1[$. Then, according to (2.19), u satisfies the inequality (2.16), i.e., $u \in B^1_{\lambda c}([a,b];R)$. By (2.20), the inequality (2.17) is fulfilled with $q^*(t) = q(t, \|u\|_C)$ for $t \in [a,b]$. Hence, by the condition $(\ell_0, \ell_1) \in U(\lambda)$ and the definition of the number ρ, the estimate (2.15) holds.

Since ρ depends neither on u nor on δ, from Lemma 2.1 it follows that the problem (0.1), (0.2) has at least one solution. $\square$

Lemma 2.3. *Let*

$$[h(v) - h(w)] \operatorname{sgn}\big(v(a) - w(a)\big) \leq 0 \qquad \text{for } v, w \in C([a,b]; R), \tag{2.21}$$

and let there exist $(\ell_0, \ell_1) \in U(\lambda)$ *such that, on the set* $B^1_{\lambda c}([a,b]; R)$, *where* $c = |h(0)|$, *the inequality*

$$\begin{aligned} \big[F(v)(t) \ - F(w)(t) + \ell_1(v-w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \\ \leq \ell_0(|v-w|)(t) \qquad \text{for } t \in [a,b] \end{aligned} \tag{2.22}$$

is fulfilled. Then the problem (0.1), (0.2) *has a unique solution.*

Proof. It follows from (2.21) that the condition (2.19) is fulfilled with $c = |h(0)|$. By (2.22), we get that, on the set $B^1_{\lambda c}([a,b]; R)$, the inequality (2.20) holds, where $q \equiv |F(0)|$. Consequently, the assumptions of Lemma 2.2 are fulfilled and so the problem (0.1), (0.2) has at least one solution. It remains to show that the problem (0.1), (0.2) has at most one solution.

Let u_1, u_2 be solutions of the problem (0.1), (0.2). Put $u(t) = u_1(t) - u_2(t)$ for $t \in [a,b]$. By (2.21) and (2.22) it is clear that

$$[u(a) - \lambda u(b)] \operatorname{sgn} u(a) \leq 0,$$

$$[u'(t) + \ell_1(u)(t)] \operatorname{sgn} u(t) \leq \ell_0(|u|)(t) \qquad \text{for } t \in [a,b].$$

Now the condition $(\ell_0, \ell_1) \in U(\lambda)$ implies $u \equiv 0$, consequently, $u_1 \equiv u_2$. □

Lemma 2.4. *Let* $\ell_0 \in \widetilde{\mathcal{L}}_{ab}$ *and the homogeneous problem*

$$v'(t) = \ell_0(v)(t), \qquad v(a) - \lambda v(b) = 0$$

have only the trivial solution. Then there exists a positive number r_0 *such that, for arbitrary* $q^* \in L([a,b]; R)$ *and* $c \in R$, *the solution* v *of the problem*

$$v'(t) = \ell_0(v)(t) + q^*(t), \qquad v(a) - \lambda v(b) = c \tag{2.23}$$

admits the estimate

$$\|v\|_C \leq r_0\big(|c| + \|q^*\|_L\big). \tag{2.24}$$

Proof. Let

$$R \times L([a,b]; R) = \{(c, q^*) : c \in R, q^* \in L([a,b]; R)\}$$

be the Banach space with the norm

$$\|(c, q^*)\|_{R\times L} = |c| + \|q^*\|_L,$$

and let Ω be an operator mapping every $(c, q^*) \in R \times L([a,b]; R)$ to the solution v of the problem (2.23). According to Theorem 1.4 in [24], $\Omega : R \times L([a,b]; R) \to C([a,b]; R)$ is a linear bounded operator. Denote by r_0 the norm of Ω. Then, clearly, for every $(c, q^*) \in R \times L([a,b]; R)$, the inequality

$$\|\Omega(c, q^*)\|_C \leq r_0\big(|c| + \|q^*\|_L\big)$$

holds. Consequently, the solution $v = \Omega(c, q^*)$ of the problem (2.23) admits the estimate (2.24). □

Lemma 2.5. *Let* $\lambda \in [0, 1[$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell_0 \in V^+(\lambda)$, *and* $-\ell_1 \in V^+(\lambda)$. *Then* $(\ell_0, \ell_1) \in U(\lambda)$.

Proof. Let $q^* \in L([a,b];R_+)$, $c \in R_+$, and $u \in \widetilde{C}([a,b];R)$ satisfy the inequalities (2.16) and (2.17). We will show that (2.18) holds, where $r = r_0$ is the number appearing in Lemma 2.4.

It is clear that

$$u'(t) = -\ell_1(u)(t) + \widetilde{q}(t)\,, \tag{2.25}$$

where

$$\widetilde{q}(t) = u'(t) + \ell_1(u)(t) \qquad \text{for } t \in [a,b]\,.$$

Evidently, according to (2.17),

$$\widetilde{q}(t)\operatorname{sgn} u(t) \le \ell_0(|u|)(t) + q^*(t) \qquad \text{for } t \in [a,b]\,. \tag{2.26}$$

Furthermore, from (2.25), in view of the assumption $\ell_1 \in \mathcal{P}_{ab}$ and the inequality (2.26), it follows that

$$\begin{aligned} [u(t)]_+' &\le \ell_1([u]_-)(t) + \ell_0(|u|)(t) + q^*(t) \\ &= -\ell_1([u]_+)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t) \\ &\qquad \text{for } t \in [a,b]\,, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} [u(t)]_-' &\le \ell_1([u]_+)(t) + \ell_0(|u|)(t) + q^*(t) \\ &= -\ell_1([u]_-)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t) \\ &\qquad \text{for } t \in [a,b]\,. \end{aligned} \tag{2.28}$$

Since $-\ell_1 \in V^+(\lambda)$, according to Theorem 1.1, the problem

$$\alpha'(t) = -\ell_1(\alpha)(t) + \ell_1(|u|)(t) + \ell_0(|u|)(t) + q^*(t)\,, \quad \alpha(a) - \lambda\alpha(b) = c \tag{2.29}$$

has a unique solution α. Moreover, from (2.27), (2.28), and (2.29), on account of the conditions $-\ell_1 \in V^+(\lambda)$ and

$$[u(a)]_+ - \lambda[u(b)]_+ \le c\,, \qquad [u(a)]_- - \lambda[u(b)]_- \le c\,,$$

it follows that

$$[u(t)]_+ \le \alpha(t)\,, \qquad [u(t)]_- \le \alpha(t) \qquad \text{for } t \in [a,b]$$

and consequently

$$|u(t)| \le \alpha(t) \qquad \text{for } t \in [a,b]\,. \tag{2.30}$$

By (2.30) and the condition $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, (2.29) results in

$$\alpha'(t) \le \ell_0(\alpha)(t) + q^*(t) \qquad \text{for } t \in [a,b]\,.$$

Since $\ell_0 \in V^+(\lambda)$ and $\alpha(a) - \lambda\alpha(b) = c$, the latter inequality yields

$$\alpha(t) \le v(t) \qquad \text{for } t \in [a,b]\,, \tag{2.31}$$

where v is a solution of the problem (2.23). Now from (2.30) and (2.31), according to Lemma 2.4, the estimate (2.18) holds. □

Theorem 2.1 follows from Lemmas 2.2 and 2.5, whereas Theorem 2.2 is a consequence of Lemmas 2.3 and 2.5.

3. On remarks 1.3, 1.4, 2.1 and 2.2

On Remarks 1.3 and 1.4. In Examples 3.1, 3.2 and 3.3, there are constructed operators $\ell \in \widetilde{\mathcal{L}}_{ab}$ such that the homogeneous problem (0.3_0), (0.4_0) has a nontrivial solution. Then, according to Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.3), (0.4) has no solution.

Example 3.1. Let $\varepsilon > 0$ and the operators $\ell, \ell_0 \in \widetilde{\mathcal{L}}_{ab}$ be defined by

$$\ell(v)(t) \overset{\text{def}}{=} (1+\varepsilon)p(t)v(b)\,, \qquad \ell_0(v)(t) \overset{\text{def}}{=} p(t)v(b)\,, \tag{3.1}$$

where $p \in L([a,b];R_+)$ is such that

$$\int_a^b p(s)\,ds = \frac{1-\lambda}{1+\varepsilon}\,. \tag{3.2}$$

According to Corollary 1.1 b) in [17], we have $\ell_0 \in V^+(\lambda)$. Obviously, the assumptions of Theorem 1.2 are fulfilled except of the condition (1.2), instead of which the condition (1.3) is satisfied. Moreover, the assumptions of Theorem 1.3 are fulfilled with $\ell_1 \equiv 0$, except for the condition (1.5), instead of which the condition (1.7) is satisfied.

On the other hand, the problem (0.3_0), (0.4_0) has a nontrivial solution

$$u(t) = \lambda + (1+\varepsilon)\int_a^t p(s)\,ds \qquad \text{for } t \in [a,b]\,.$$

Example 3.1 shows that the inequalities (1.2) and (1.5) in Theorems 1.2 and 1.3 cannot be replaced by the inequalities (1.3) and (1.7), respectively, no matter how small $\varepsilon > 0$ would be.

Example 3.2. Let $\varepsilon \in \,]0,1[$ and the operator $\ell \in \widetilde{\mathcal{L}}_{ab}$ be defined by

$$\ell(v)(t) \overset{\text{def}}{=} p(t)v(b)\,, \tag{3.3}$$

where $p \in L([a,b];R_+)$ is such that

$$\int_a^b p(s)\,ds = 1-\lambda\,. \tag{3.4}$$

Put $\ell_0 \equiv \ell$, $\ell_1 \equiv 0$. Then the conditions (1.2) in Theorem 1.2 and (1.5) in Theorem 1.3 are fulfilled. Furthermore, according to Corollary 1.1 b) in [17], we have $(1-\varepsilon)\ell_0 \in V^+(\lambda)$.

On the other hand, the problem (0.3_0), (0.4_0) has a nontrivial solution

$$u(t) = \lambda + \int_a^t p(s)\,ds \qquad \text{for } t \in [a,b]\,.$$

Example 3.2 shows that the condition (1.1) in Theorem 1.2 and the condition (1.6) in Theorem 1.3 cannot be replaced by the condition (1.4) and (1.8), no matter how small $\varepsilon > 0$ would be.

Example 3.3. Let $\varepsilon > 0$, $\delta = \frac{\varepsilon(1-\lambda)}{1+\varepsilon}$, and $\ell \in \widetilde{\mathcal{L}}_{03}$ be an operator defined by

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v\big(\tau(t)\big)\,, \tag{3.5}$$

where

$$p(t) = \begin{cases} 1-\lambda-\delta & \text{for } t \in [0,1[\\ -\frac{2-\delta}{1-\delta} & \text{for } t \in [1,2[\\ -2 & \text{for } t \in [2,3] \end{cases}, \qquad \tau(t) = \begin{cases} 3 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,2[\\ 2 & \text{for } t \in [2,3] \end{cases}.$$

Let, moreover,

$$\ell_0(v)(t) \stackrel{\text{def}}{=} p_0(t)v\big(\tau_0(t)\big)\,, \qquad \ell_1(v)(t) \stackrel{\text{def}}{=} p_1(t)v\big(\tau_1(t)\big)\,, \tag{3.6}$$

where $p_0 \equiv [p]_+$, $p_1 \equiv [p]_-$,

$$\tau_0 \equiv 3\,, \qquad \tau_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,2[\\ 2 & \text{for } t \in [2,3] \end{cases}.$$

It is clear that $\ell_0, \ell_1 \in \mathcal{P}_{03}$ and the condition (1.5) is fulfilled. Moreover,

$$\int_0^3 \ell_0(1)(s)\,ds = \int_0^1 p_0(s)\,ds = 1-\lambda-\delta < 1-\lambda\,.$$

Consequently, according to Corollary 1.1 b) in [17], $\ell_0 \in V^+(\lambda)$. It is not difficult to verify that the homogeneous problem

$$u'(t) = -\frac{1}{2+\varepsilon}\ell_1(u)(t)\,, \qquad u(0) - \lambda u(3) = 0$$

has only the trivial solution and, for arbitrary $q_0 \in L([0,3];R_+)$ and $c_0 \in R_+$, the solution of the problem

$$u'(t) = -\frac{1}{2+\varepsilon}\ell_1(u)(t) + q_0(t)\,, \qquad u(0) - \lambda u(3) = c_0$$

is nonnegative. Therefore, by Definition 1.1, we obtain $-\frac{1}{2+\varepsilon}\ell_1 \in V^+(\lambda)$.

On the other hand, the function

$$u(t) = \begin{cases} (1-\lambda-\delta)t + \lambda & \text{for } t \in [0,1[\\ (2-\delta)(1-t) + 1 - \delta & \text{for } t \in [1,2[\\ 2t - 5 & \text{for } t \in [2,3] \end{cases}$$

is a nontrivial solution of the problem (0.3_0), (0.4_0).

Example 3.3 shows that the assumption (1.6) in Theorem 1.3 cannot be replaced by (1.9), no matter how small $\varepsilon > 0$ would be.

On Remark 2.1. Let $\varepsilon > 0$, $\ell, \ell_0 \in \widetilde{\mathcal{L}}_{ab}$ be defined by (3.1), where $p \in L([a,b];R_+)$ satisfies (3.2). According to Example 3.1, the problem (0.3_0), (0.4_0) has a nontrivial solution. By Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.1), (0.2), where

$$F(v)(t) \stackrel{\text{def}}{=} \ell(v)(t) + q_0(t) \qquad \text{for } \ t \in [a,b]\,, \qquad h(v) \equiv c_0\,, \tag{3.7}$$

has no solution, while the conditions (2.2), (2.4) and (2.5) are fulfilled with $c = |c_0|$, $q \equiv |q_0|$, $\ell_1 \equiv 0$. Thus, Example 3.1 shows that the condition (2.3) in Theorem 2.1 cannot be replaced by the condition (2.5), no matter how small $\varepsilon > 0$ would be.

Let $\varepsilon \in\,]0,1[$, $\ell \in \widetilde{\mathcal{L}}_{ab}$ be defined by (3.3), where $p \in L([a,b];R_+)$ satisfies (3.4). According to Example 3.2, the problem (0.3_0), (0.4_0) has a nontrivial solution. By Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.1), (0.2), where F and h are defined by (3.7), has no solution, while the conditions (2.2), (2.3) and (2.6) are fulfilled with $c = |c_0|$, $q \equiv |q_0|$, $\ell_0 \equiv \ell$, and $\ell_1 \equiv 0$. Therefore, Example 3.2 shows that the condition (2.4) in Theorem 2.1 cannot be replaced by (2.6), no matter how small $\varepsilon > 0$ would be.

Example 3.4. Let $\lambda \in\,]0,1[$ (for the case $\lambda = 0$ see [15]), $\varepsilon \in\,]0,1[$, $\delta = \varepsilon(1-\lambda)$, $\vartheta \in\,]0, 1-\delta[$ such that $\vartheta < \frac{\lambda\varepsilon}{1-\varepsilon}$, $\ell \in \widetilde{\mathcal{L}}_{05}$ be an operator defined by (3.5), where

$$p(t) = \begin{cases} 1-\lambda-\delta & \text{for } t \in [0,1[\\ 0 & \text{for } t \in [1,2[\,\cup\,[3,4[\\ -\frac{1+\vartheta}{1-\delta} & \text{for } t \in [2,3[\\ -(1+\vartheta) & \text{for } t \in [4,5] \end{cases}, \qquad \tau(t) = \begin{cases} 5 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ 3 & \text{for } t \in [3,5] \end{cases},$$

and ℓ_0, ℓ_1 be defined by (3.6), where $p_0 \equiv [p]_+$, $p_1 \equiv [p]_-$,

$$\tau_0 \equiv 5\,, \qquad \tau_1(t) = \begin{cases} 0 & \text{for } t \in [0,1[\\ 1 & \text{for } t \in [1,3[\\ 3 & \text{for } t \in [3,5] \end{cases}.$$

Put

$$z(t) = \begin{cases} 0 & \text{for } t \in [0,1[\,\cup\,[2,3[\,\cup\,[4,5[\\ -\frac{1-\delta-\vartheta}{(1-\delta-\vartheta)(1-t)+1-\delta} & \text{for } t \in [1,2[\\ -\frac{1-\vartheta}{1-(1-\vartheta)(t-3)} & \text{for } t \in [3,4[\end{cases}.$$

It is clear that $-z \in L([0,5];R_+)$, $\ell_0, \ell_1 \in \mathcal{P}_{05}$ and

$$\int_0^5 \ell_0(1)(s)\,ds = \int_0^1 p_0(s)\,ds = 1-\lambda-\delta < 1-\lambda\,.$$

Consequently, according to Corollary 1.1 b) in [17], $\ell_0 \in V^+(\lambda)$. It is not difficult to verify that the homogeneous problem

$$u'(t) = -(1-\varepsilon)\ell_1(u)(t)\,, \qquad u(0) - \lambda u(5) = 0$$

has only the trivial solution and, for arbitrary $q_0 \in L([0,5];R_+)$ and $c_0 \in R_+$, the solution of the problem

$$u'(t) = -(1-\varepsilon)\ell_1(u)(t) + q_0(t)\,, \qquad u(0) - \lambda u(5) = c_0$$

is nonnegative. Therefore, by Definition 1.1, we obtain $-(1-\varepsilon)\ell_1 \in V^+(\lambda)$.

On the other hand, the function

$$u(t) = \begin{cases} (1-\lambda-\delta)t+\lambda & \text{for } t \in [0,1[\\ (1-\delta-\vartheta)(1-t)+1-\delta & \text{for } t \in [1,2[\\ (1+\vartheta)(2-t)+\vartheta & \text{for } t \in [2,3[\\ (1-\vartheta)(t-3)-1 & \text{for } t \in [3,4[\\ (1+\vartheta)(t-4)-\vartheta & \text{for } t \in [4,5] \end{cases}$$

is a nontrivial solution of the problem

$$u'(t) = p(t)u\big(\tau(t)\big) + z(t)u(t)\,, \qquad u(a) - \lambda u(b) = 0\,.$$

Consequently, according to Remark 1.1, there exist $q_0 \in L([a,b];R)$ and $c_0 \in R$ such that the problem (0.1), (0.2) with $F(v)(t) \stackrel{\text{def}}{=} p(t)v\big(\tau(t)\big) + z(t)v(t) + q_0(t)$ for $t \in [a,b]$, $h(v) \equiv c_0$ has no solution, while the conditions (2.2), (2.3) and (2.7) are fulfilled with $c = |c_0|$, $q \equiv |q_0|$.

Example 3.4 shows that the assumption (2.4) in Theorem 2.1 cannot be replaced by the assumption (2.7), no matter how small $\varepsilon > 0$ would be.

On Remark 2.2. Examples 3.1, 3.2 and 3.4 also show that the assumptions on the operators ℓ_0, ℓ_1 in Theorem 2.2 cannot be weakened.

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