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CONVERGENCE OF CUBATURE-DIFFERENCES METHOD FOR MULTIDIMENSIONAL SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. Here we propose and justify the cubature-differences method for the multidimensional singular integro-differential equations with Hilbert kernel. The convergence of the method is proved and the error estimate is obtained.

Introduction

In the papers [1]–[4] the quadrature-differences methods for the various classes of the 1-dimensional periodic singular integro-differential equations with Hilbert kernels were justified. The convergence of the methods was proved and the errors estimates were obtained. Here we propose and justify the cubature-differences method for the 2-dimensional[1] linear periodic singular integro-differential equations. The convergence of the method is proved and the error estimate is obtained.

It is known (see e.g. [7], [9]) that the theory of the singular integral equations in multidimensicnal case is less developed than in 1-dimensional case. Thus, for instance, if for 1-dimensional singular integral equations simple necessary and sufficient conditions of solvability is known, then for multidimensional equations there are some, only sufficient, conditions of solvability and corresponding classes of solvable equations but the situation in general is still unclear. Here we consider the class of equations which dominant part maps the set of trigonometrical polynomials to itself.

The same situation is with the theories of approximate methods in 1- and multidimensional cases. For 1-dimensional singular integral equations polynomial collocation method is justified in [6] for all solvable equations. It means that we don't need any special conditions in addition to solvability of the equation for invertibility of the operator approximating the dominant part. In mutidimesional

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[1] 2-dimensional case is considered only for the sake of simplicity. All results could be easily generalised to the case of d ($d \geq 3$) dimensions.

case the same result doesn't exist. So we need instead to assume, as it is mentioned above, that dominant part maps the set of trigonometrical polynomials to itself.

To approximate derivatives in 1-dimentional case any converging differences (with some easy-to-check restrictions for approximating the highest order derivative) could be used. It means that for equations with smooth solutions one can achieve the highest possible rate of convergence using the differences of appropriate order. In mutidimesional case there aren't any rules of constructing appropriate differences, so we use fixed second order differences and can't obtain the rate of covergence higher than 2.

1. Statement of the problem

Let's denote by $\mathbf{N}, \mathbf{N_0}, \mathbf{Z}, \mathbf{R}$ and $\boldsymbol{\Delta}$ the Cartesian squares of the sets of N-natural, N_0-natural including zero, Z-interger and R-real numbers and the interval $\Delta = (-\pi; \pi] \subset R$ respectevely. For the elements of this sets (2-components vectors) beside the usual operations of adding, substracting and multipling the number we will define the following operations

$$\mathbf{l} \cdot \mathbf{k} = l_1 k_1 + l_2 k_2\,,\ \mathbf{l} * \mathbf{k} = (l_1 k_1, l_2 k_2)\,,\ \mathbf{l}^2 = l_1^2 + l_2^2\,,\ \mid \mathbf{l} \mid = l_1 + l_2\,,\ [\mathbf{l}] = l_1 l_2\,,$$

and relations of the partial order

$$\mathbf{l} < \mathbf{k} \equiv (l_1 < k_1) \& (l_2 < k_2)\,, \quad \mathbf{l} \le \mathbf{k} \equiv (l_1 \le k_1) \& (l_2 \le k_2)\,,$$
$$\mathbf{l} = (l_1, l_2)\,,\ \mathbf{k} = (k_1, k_2)\,.$$

For the fixed $s \in R$ let's denote by H^s Sobolev space of 2-dimensional 2π-periodical by each variable complex-valued functions with the norm

$$\|u\|_s = \|u\|_{H^s} = \Big(\sum_{\mathbf{k} \in \mathbf{Z}} (1 + \mathbf{k}^2)^s \mid \widehat{u}(\mathbf{k}) \mid^2 \Big)^{1/2}$$

and inner product

$$\langle u, v \rangle_s = \sum_{\mathbf{k} \in \mathbf{Z}} (1 + \mathbf{k}^2)^s \widehat{u}(\mathbf{k}) \bar{\widehat{v}}(\mathbf{k})\,,$$

where

$$\widehat{u}(\mathbf{k}) = (2\pi)^{-2} \int_{\boldsymbol{\Delta}} u(\boldsymbol{\tau}) \bar{e}_{\mathbf{k}}(\boldsymbol{\tau})\, d\boldsymbol{\tau}$$

are the Fourier coefficients of the function $u(\boldsymbol{\tau})$ by the system of trigonometric monomials

$$e_{\mathbf{k}}(\boldsymbol{\tau}) = \exp(i \mathbf{k} \cdot \boldsymbol{\tau})\,, \qquad \mathbf{k} \in \mathbf{Z}\,, \boldsymbol{\tau} \in \boldsymbol{\Delta}\,.$$

For the following we will asume that $s > 1$ providing (see e.g. [10]) the embedding of H^s in the space of continuous functions.

Consider the linear singular integro-differential equation

$$ABu + Tu = f\,, \tag{1}$$

where A is 2-dimensional singular integral operator

$$Au \equiv a_{00}(\mathbf{t})u(\mathbf{t}) + a_{01}(\mathbf{t})(J_{01}u)(\mathbf{t}) + a_{10}(\mathbf{t})(J_{10}u)(\mathbf{t}) + a_{11}(\mathbf{t})(J_{11}u)(\mathbf{t})\,, \qquad A : H^s \to H^s\,,$$

with singular integrals

$$(J_{01}u)(\mathbf{t}) = (2\pi)^{-1}\int_{-\pi}^{\pi} u(t_1,\tau_2)\operatorname{ctg}\frac{\tau_2 - t_2}{2}\,d\tau_2\,,$$

$$(J_{10}u)(\mathbf{t}) = (2\pi)^{-1}\int_{-\pi}^{\pi} u(\tau_1,t_2)\operatorname{ctg}\frac{\tau_1 - t_1}{2}\,d\tau_1\,,$$

$$(J_{11}u)(\mathbf{t}) = (2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} u(\tau_1,\tau_2)\operatorname{ctg}\frac{\tau_1 - t_1}{2}\operatorname{ctg}\frac{\tau_2 - t_2}{2}\,d\tau_2\,d\tau_1$$

which are to be interpreted as the Cauchy-Lebesgues principal value, B is elliptical differential operator

$$Bu \equiv (Bu)(\mathbf{t}) = \sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=m} b_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\mathbf{t})(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u)(\mathbf{t})\,,\ B : H^{s+2m} \to H^s\,,\ m \in \mathrm{N}\,,$$

with derivatives

$$D^{\boldsymbol{\alpha}}u = \frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial {t_1}^{\alpha_1}\partial {t_2}^{\alpha_2}} \quad \text{of order} \quad \boldsymbol{\alpha} = (\alpha_1,\alpha_2) \in \mathbf{N}_0\,,$$

and $T : H^{s+2m} \to H^s$ is known linear operator. Coefficients $a_{kl}(\mathbf{t}),\ k,l = 0,1$, $b_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\mathbf{t})$, $|\,\boldsymbol{\alpha}\,|=|\,\boldsymbol{\beta}\,|= m$, and the right-hand side $f(\mathbf{t})$ of the equation (1) we will consider, for the sake of simplicity, belonging to H^∞.

2. Calculation scheme

Let's fix $\mathbf{n} = (n_1, n_2) \in \mathbf{N}$, denote by

$$\mathbf{I_n} = I_{n_1} \times I_{n_2}\,, \quad I_{n_j} = \{k_j \mid k_j \in Z, |\,k_j\,| \le n_j\}\,, \quad j = 1,2,$$

index set and difine the grid

$$\boldsymbol{\Delta}_n = \{\mathbf{t_k} = (t_{k_1}, t_{k_2}) \mid \mathbf{k} = (k_1,k_2) \in \mathbf{I_n}, t_{k_j} = k_j h_j, h_j = 2\pi/(2n_j+1), j = 1,2\}\,.$$

on $\boldsymbol{\Delta}$. Approximate solution of the equation (1) we will seek as a periodic grid function (vector of values) $u_{\mathbf{n}} = u_{\mathbf{n}}(\mathbf{t})$ defined on $\boldsymbol{\Delta}_n$.

Differential operators $D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ of the equation (1) we will approximate by the operators

$$D_{\mathbf{n}}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u_{\mathbf{n}} = \frac{1}{2}(\partial^{\boldsymbol{\alpha}}\bar{\partial}^{\boldsymbol{\beta}} + \bar{\partial}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}})u_{\mathbf{n}}\,, \tag{2}$$

where

$$\partial^{\boldsymbol{\alpha}} u_{\mathbf{n}} = \partial_1^{\alpha_1}\partial_2^{\alpha_2} u_{\mathbf{n}}\,, \qquad \bar\partial^{\boldsymbol{\alpha}} u_{\mathbf{n}} = \bar\partial_1^{\alpha_1}\bar\partial_2^{\alpha_2} u_{\mathbf{n}}\,,$$
$$\partial_j u_{\mathbf{n}} = h_j^{-1}(u_{\mathbf{n}}(\mathbf{t}+h_j\boldsymbol{\delta}_j) - u_{\mathbf{n}}(\mathbf{t}))\,, \qquad \bar\partial_j u_{\mathbf{n}} = h_j^{-1}(u_{\mathbf{n}}(\mathbf{t}) - u_{\mathbf{n}}(\mathbf{t}-h_j\boldsymbol{\delta}_j))\,,$$

$\boldsymbol{\delta}_j = (\delta_{j1}, \delta_{j2})$, $j = 1, 2$, and δ_{jk} is Kronecker symbol.

Singular integrals are to be approximated by the cubatures and quadratures. To do this we will integrate interpolative Lagrange polynomial

$$(P_{\mathbf{n}} u_{\mathbf{n}})(\boldsymbol{\tau}) = \sum_{\mathbf{k}\in\mathbf{I_n}} u_{\mathbf{n}}(\mathbf{t_k})\xi_{\mathbf{n}}(\boldsymbol{\tau}, \mathbf{t_k})\,,$$
$$\xi_{\mathbf{n}}(\boldsymbol{\tau}, \mathbf{t_k}) = \prod_{j=1,2} \frac{\sin((2n_j+1)(\tau_j - t_{k_j})/2)}{(2n_j+1)\sin((\tau_j - t_{k_j})/2)}\,,$$
$$\boldsymbol{\tau} = (\tau_1, \tau_2) \in \boldsymbol{\Delta},\ \mathbf{t_k} = (t_{k_1}, t_{k_2}) \in \boldsymbol{\Delta}_n\,.$$

Then the integrals will take the form

$$\begin{aligned}
(J_{01}P_{\mathbf{n}}u_{\mathbf{n}})(\mathbf{t_k}) &= (2n_2+1)^{-1}\sum_{l_2\in I_{n_2}} \gamma^{(n_2)}_{k_2-l_2} u_{\mathbf{n}}(t_{k_1}, t_{l_2})\,,\\
(J_{10}P_{\mathbf{n}}u_{\mathbf{n}})(\mathbf{t_k}) &= (2n_1+1)^{-1}\sum_{l_1\in I_{n_1}} \gamma^{(n_1)}_{k_1-l_1} u_{\mathbf{n}}(t_{l_1}, t_{k_2})\,,\\
(J_{11}P_{\mathbf{n}}u_{\mathbf{n}})(\mathbf{t_k}) &= [2\mathbf{n}+\mathbf{1}]^{-1}\sum_{\mathbf{l}\in\mathbf{I_n}} \gamma^{(n_1)}_{k_1-l_1}\gamma^{(n_2)}_{k_2-l_2} u_{\mathbf{n}}(\mathbf{t_l})\,,
\end{aligned} \tag{3}$$

$\mathbf{t_k} \in \boldsymbol{\Delta}_n$, $\mathbf{1} = (1, 1)$, and the coefficients $\gamma_r^{(q)}$ are

$$\gamma_r^{(q)} = \Big\{ \operatorname{tg}\frac{r\pi}{2(2q+1)},\ r \text{ even},\ -\operatorname{ctg}\frac{r\pi}{2(2q+1)},\ r \text{ odd}\Big\}.$$

Operator T we will approximate by any covergent operator $T_{\mathbf{n}}$.

Substituting the numerical deffirential formulas (2), cubature and quadrature sums (3), the values of the coefficients $a_{kl}(\mathbf{t})$, $k, l = 0, 1$, $b_{\boldsymbol{\alpha\beta}}(\mathbf{t})$, $|\ \boldsymbol{\alpha}\ |=|\ \boldsymbol{\beta}\ |= m$, of the operator $(T_{\mathbf{n}}\mathbf{u_n})(\mathbf{t})$ and the right-hand side $f(\mathbf{t})$ in the nodes of the grid $\boldsymbol{\Delta}_n$ in the equation (1) we will obtain the system of linear algebraic equations

$$\begin{aligned}
&a_{00}(\mathbf{t_k}) \sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=m} b_{\boldsymbol{\alpha\beta}}(\mathbf{t_k})(D_{\mathbf{n}}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} u_{\mathbf{n}})(\mathbf{t_k})\\
&+ a_{01}(\mathbf{t_k})(2n_2+1)^{-1}\sum_{l_2\in I_{n_2}}\gamma^{(n_2)}_{k_2-l_2}\sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=m} b_{\boldsymbol{\alpha\beta}}(t_{k_1}, t_{l_2})(D_{\mathbf{n}}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} u_{\mathbf{n}})(t_{k_1}, t_{l_2})\\
&+ a_{10}(\mathbf{t_k})(2n_1+1)^{-1}\sum_{l_1\in I_{n_1}}\gamma^{(n_1)}_{k_1-l_1}\sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=m} b_{\boldsymbol{\alpha\beta}}(t_{l_1}, t_{k_2})(D_{\mathbf{n}}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} u_{\mathbf{n}})(t_{l_1}, t_{k_2})
\end{aligned} \tag{4}$$

$$+ a_{11}(\mathbf{t_k})[2\mathbf{n}+\mathbf{1}]^{-1}\sum_{\mathbf{l}\in\mathbf{I_n}}\gamma^{(n_1)}_{k_1-l_1}\gamma^{(n_2)}_{k_2-l_2}\sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=m} b_{\boldsymbol{\alpha\beta}}(t_{l_1},t_{l_2})(D_{\mathbf{n}}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}u_{\mathbf{n}})(t_{l_1},t_{l_2})$$

$$+ (T_{\mathbf{n}}u_{\mathbf{n}})(\mathbf{t_k}) = f(\mathbf{t_k})\,, \quad \mathbf{t_k}\in\mathbf{\Delta}_n\,,$$

of the cubature-differences method.

3. Preliminaries

Let's denote by $H^s_{\mathbf{n}}$ the set of grid functions (vectors of values) on $\mathbf{\Delta}_n$ with the norm

$$\|u_{\mathbf{n}}\|_{s,\mathbf{n}} = \|u_{\mathbf{n}}\|_{H^s_{\mathbf{n}}} = \Big(\sum_{\mathbf{k}\in\mathbf{I_n}}(1+\mathbf{k}^2)^s|\widehat{u}_{\mathbf{n}}(\mathbf{k})^{(\mathbf{n})}|^2\Big)^{1/2}$$

and inner product

$$\langle u_{\mathbf{n}}, v_{\mathbf{n}}\rangle_{s,\mathbf{n}} = \sum_{\mathbf{k}\in\mathbf{I_n}}(1+\mathbf{k}^2)^s\widehat{u}_{\mathbf{n}}(\mathbf{k})^{(\mathbf{n})}\bar{\widehat{v_{\mathbf{n}}}}(\mathbf{k})^{(\mathbf{n})}\,,$$

where

$$\widehat{u}_{\mathbf{n}}(\mathbf{k})^{(\mathbf{n})} = [2\mathbf{n}+\mathbf{1}]^{-1}\sum_{\mathbf{l}\in\mathbf{I_n}}u_{\mathbf{n}}(\mathbf{t_l})\bar{e}_{\mathbf{k}}(\mathbf{t_l})\,, \quad \mathbf{k}\in\mathbf{I_n}\,,$$

are Fourier-Lagrange coefficients of the function $u_{\mathbf{n}}(\mathbf{t})$ by the grid $\mathbf{\Delta}_n$.

The sets H^s and $H^s_{\mathbf{n}}$ we will bind by the operators

$$p_{\mathbf{n}}u = (u(\mathbf{t_k}))_{\mathbf{k}\in\mathbf{I_n}}\,,\ p_{\mathbf{n}}: H^s\to H^s_{\mathbf{n}}\,,$$

$$(P_{\mathbf{n}}u_{\mathbf{n}})(\boldsymbol{\tau}) = \sum_{\mathbf{k}\in\mathbf{I_n}}u_{\mathbf{n}}(\mathbf{t_k})\xi_{\mathbf{n}}(\boldsymbol{\tau},\mathbf{t_k})\,,\ P_{\mathbf{n}}: H^s_{\mathbf{n}}\to H^s\,,$$

and denote by $E_{\mathbf{n}}(u)_s$ the best approximation of the function $u\in H^s$ by the trigonometrical polynomials of order not higher than $\mathbf{n}$. It is known that in Hilbert space the polynomial of the best approximation of the function is its partial sum of the Fourier series

$$(S_{\mathbf{n}}u)(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbf{I_n}}\widehat{u}(\mathbf{k})e_{\mathbf{k}}(\mathbf{t})\,, \qquad E_{\mathbf{n}}(u)_s = \|u - S_{\mathbf{n}}u\|_s\,.$$

Lemma 1. *For any $u\in H^s$, $s\in \mathrm{R}$, $s>1$ and $\mathbf{n}\in\mathbf{N}$ the following estimations are valid*

$$\|P_{\mathbf{n}}\|_{H^s_{\mathbf{n}}\to H^s} = 1\,, \qquad \|p_{\mathbf{n}}\|_{H^s\to H^s_{\mathbf{n}}} \le 2M(\mathbf{n},s)\sqrt{\zeta(2s-1)}\,,$$

$$\|P_{\mathbf{n}}p_{\mathbf{n}}u - u\|_s \le (1+2M(\mathbf{n},s)\sqrt{\zeta(2s-1)})E_{\mathbf{n}}(u)_s\,,$$

where $M(\mathbf{n},s) = (\frac{\sqrt{n_1^2+n_2^2}}{\min(n_1,n_2)})^s$, $\mathbf{n}=(n_1,n_2)$, and $\zeta(t)$ is the Riemann's ζ-function bounded and decreasing for $t>1$.

Proof. The equation $\|P_{\mathbf{n}}u_{\mathbf{n}}\|_s = \|u_{\mathbf{n}}\|_{s,\mathbf{n}}$ for any $u_{\mathbf{n}} \in H^s_{\mathbf{n}}$ follows directly from the definitions of the norms in the spaces H^s and $H^s_{\mathbf{n}}$. So $\|P_{\mathbf{n}}\|_{H^s_{\mathbf{n}} \to H^s} = 1$ is valid trivially.

To obtain the norm of the operator $p_{\mathbf{n}}$ let's take the arbitrary function $u \in H^s$ and write according to the difinition of the norm in $H^s_{\mathbf{n}}$

$$\|p_{\mathbf{n}}u\|^2_{s,\mathbf{n}} = \sum_{\mathbf{k}\in\mathbf{I_n}} (1+\mathbf{k}^2)^s |\widehat{u}(\mathbf{k})^{(\mathbf{n})}|^2 ,$$

where

$$\widehat{u}(\mathbf{k})^{(\mathbf{n})} = [2\mathbf{n}+\mathbf{1}]^{-1} \sum_{\mathbf{l}\in\mathbf{I_n}} u(\mathbf{t_l})\bar{e}_{\mathbf{k}}(\mathbf{t_l}) , \quad \mathbf{k} \in \mathbf{I_n}$$

are Fourier-Lagrange coefficients of the function $u(\mathbf{t})$ with respect to the grid $\mathbf{\Delta}_n$. Substituting the values of the function $u(\mathbf{t})$ in the nodes of the grid $\mathbf{\Delta}_n$ by the values of its Fourier series we will obtain

$$\begin{aligned}\widehat{u}(\mathbf{k})^{(\mathbf{n})} &= [2\mathbf{n}+\mathbf{1}]^{-1} \sum_{\mathbf{l}\in\mathbf{I_n}} \Big(\sum_{\mathbf{m}\in\mathbf{Z}} \widehat{u}(\mathbf{m}) e_{\mathbf{m}}(\mathbf{t_l}) \Big) \bar{e}_{\mathbf{k}}(\mathbf{t_l}) \\ &= [2\mathbf{n}+\mathbf{1}]^{-1} \sum_{\mathbf{m}\in\mathbf{Z}} \sum_{\mathbf{l}\in\mathbf{I_n}} \widehat{u}(\mathbf{m}) e_{\mathbf{m}}(\mathbf{t_l}) \bar{e}_{\mathbf{k}}(\mathbf{t_l}) = \sum_{\mathbf{m}\in\mathbf{Z}} \widehat{u}\big(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1})\big) .\end{aligned}$$

Then, following [5], we will write

$$\begin{aligned}\|p_{\mathbf{n}}u\|^2_{s,\mathbf{n}} = &\sum_{\mathbf{k}\in\mathbf{I_n}} (1+\mathbf{k}^2)^s \Big| \sum_{\mathbf{m}\in\mathbf{Z}} \widehat{u}\big(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1})\big) \Big|^2 \\ = &\sum_{\mathbf{k}\in\mathbf{I_n}} \Big| \sum_{\mathbf{m}\in\mathbf{Z}} (1+\mathbf{k}^2)^{\frac{s}{2}} \big(1+(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2\big)^{-\frac{s}{2}} \\ &\times \widehat{u}\big(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1})\big) \big(1+(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2\big)^{\frac{s}{2}} \Big|^2 \\ \le &\sum_{\mathbf{k}\in\mathbf{I_n}} \Big(\sum_{\mathbf{m}\in\mathbf{Z}} |\widehat{u}\big(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1})\big)|^2 \big(1+(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2\big)^s \\ &\times \sum_{\mathbf{m}\in\mathbf{Z}} \big((1+\mathbf{k}^2)/(1+(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2)\big)^s \Big) \\ \le &\max_{\mathbf{k}\in\mathbf{I_n}} \Big(\sum_{\mathbf{m}\in\mathbf{Z}} \big((1+\mathbf{k}^2)/\big(1+(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2\big)\big)^s \Big) \|u\|^2_s .\end{aligned}$$

The chains of the inequalities

$$\begin{aligned}&\max_{\mathbf{k}\in\mathbf{I_n}} \Big(\sum_{\mathbf{m}\in\mathbf{Z}} \big((1+\mathbf{k}^2)/\big(1+(\mathbf{k}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2\big)\big)^s \Big) \\ &\quad \le \sum_{\mathbf{m}\in\mathbf{Z}} \big((1+\mathbf{n}^2)/\big(1+(\mathbf{n}+\mathbf{m}*(2\mathbf{n}+\mathbf{1}))^2\big)\big)^s \\ &\quad \le 2^{s+2} \big(n_1^2+n_2^2\big)^s \sum_{\mathbf{m}\in\mathbf{N}} \big(n_1^2(2m_1-1)^2+n_2^2(2m_2-1)^2\big)^{-s}\end{aligned}$$

$$\begin{aligned}&\le 4M^2(\mathbf{n},s)\sum_{\mathbf{m}\in\mathbf{N}}(m_1+m_2-1)^{-2s} = 4M^2(\mathbf{n},s)\sum_{m\in N} m^{1-2s}\\ &= 4M^2(\mathbf{n},s)\zeta(2s-1)\,,\end{aligned}$$

$$\begin{aligned}\|P_{\mathbf{n}}p_{\mathbf{n}}u-u\|_s &\le \|P_{\mathbf{n}}p_{\mathbf{n}}u - P_{\mathbf{n}}p_{\mathbf{n}}S_{\mathbf{n}}u\|_s + \|S_{\mathbf{n}}u-u\|_s\\ &\le \bigl(1+2M(\mathbf{n},s)\sqrt{\zeta(2s-1)}\,\bigr)E_{\mathbf{n}}(u)_s\end{aligned}$$

finish the proof of the Lemma 1. □

To prove the convergence of the method we need the function $M(\mathbf{n},s)$ to be bounded. So we'll restrict the set of indices to one where $M(\mathbf{n},s)$ is bounded. Let's for some $c,s\in R$ define the set

$$\mathbf{N}(c,s)=\{\mathbf{n}\mid \mathbf{n}\in\mathbf{N},\ M(\mathbf{n},s)\le c\}\,.$$

Obviously, $\mathbf{N}(c,s)=\emptyset$ for $c<2^{s/2}$ and $\mathbf{N}(c,s)=\{\mathbf{n}\mid \mathbf{n}=(j,j,\dots,j),\ j\in\mathbf{N}\}$ for $c=2^{s/2}$. For the following we'll mean that all indices $\mathbf{n}$, $\mathbf{n}_0$, $\mathbf{n}_1$ mentioned below belong to $\mathbf{N}(c,s)$, and $\mathbf{n}\to\infty$ means that $\mathbf{n}$ gets the values of some sequence

$$(\mathbf{n}_j)_{j\in N},\quad \mathbf{n}_j\in\mathbf{N}(c,s),\quad \mathbf{n}_j<\mathbf{n}_{j+1},\quad j=1,2,\dots$$

Lemma 2. *For any $s\le p$ and $u\in H^p$*

$$E_{\mathbf{n}}(u)_s\le(1+\mathbf{n}^2)^{(s-p)/2}E_{\mathbf{n}}(u)_p.$$

Proof.

$$\begin{aligned}E_{\mathbf{n}}(u)_s &= \|u-S_{\mathbf{n}}u\|_s = \Bigl(\sum_{\mathbf{k}\notin\mathbf{I_n}}(1+\mathbf{k}^2)^s|\widehat{u}(\mathbf{k})|^2\Bigr)^{1/2}\\ &= \Bigl(\sum_{\mathbf{k}\notin\mathbf{I_n}}(1+\mathbf{k}^2)^p(1+\mathbf{k}^2)^{s-p}|\widehat{u}(\mathbf{k})|^2\Bigr)^{1/2}\le(1+\mathbf{n}^2)^{(s-p)/2}E_{\mathbf{n}}(u)_p\,.\end{aligned}$$ □

4. Justification

Theorem. *Let for some $c,s\in R,\ s>1,\ c\ge 2^{s/2}$ the equation* (1) *and calculation scheme* (2) – (4) *of the method satisfy the following conditions:*

1) *for any $\mathbf{n}$ operator A maps the set of all trigonometric polynomials of order not higher than $\mathbf{n}$ to itself,*

2) *B is elliptic operator i.e. for any point $\mathbf{t}\in\boldsymbol{\Delta}$ and real numbers $\tau_{\boldsymbol{\alpha}}$, $\tau_{\boldsymbol{\beta}}$ the following inequality is valid* [2]

$$\sum_{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=m} b_{\boldsymbol{\alpha\beta}}(\mathbf{t})\tau_{\boldsymbol{\alpha}}\tau_{\boldsymbol{\beta}}\ge C\sum_{|\boldsymbol{\alpha}|=m}\tau_{\boldsymbol{\alpha}}^2\,,$$

3) *operator $T: H^{s+2m}\to H^{s+\varepsilon}$ is bounded for some $\varepsilon\in R,\ \varepsilon>0$,*

[2] Here and further C denotes generic real positive constants, independent from $\mathbf{n}$.

4) *operator* $T_{\mathbf{n}}$ *approximates operator* T *with respect to* $p_{\mathbf{n}}$*, i.e. for any function* $u \in H^s$

$$\|T_{\mathbf{n}} p_{\mathbf{n}} u - p_{\mathbf{n}} T u\|_{s,\mathbf{n}} = \eta_{\mathbf{n}} \to 0 \quad \textit{for} \quad \mathbf{n} \to \infty\,,$$

5) *the equation* (1) *has a unique solution* $u^* \in H^{s+2m}$ *for any right-hand side* $f \in H^s$.

Then for all $\mathbf{n}$*, beginning from some* $\mathbf{n}_0$*, the system of equations* (4) *is uniquely solvable and approximate solutions* $u^*_{\mathbf{n}}$ *converge to the exact solution* u^* *of the equation* (1)

$$\|u^*_{\mathbf{n}} - p_{\mathbf{n}} u^*\|_{s+2m} \to 0, \ \ \mathbf{n} \to \infty\,.$$

If, in addition, $u^* \in H^{s+2m+2}$*, then the error estimate*

$$\|u^*_{\mathbf{n}} - p_{\mathbf{n}} u^*\|_{s+2m} \le C(\mathbf{h}^2 + \eta_{\mathbf{n}})\,, \qquad \mathbf{h} = (h_1, h_2)\,, \ \ h_j = 2\pi/(2n_j + 1)\,, \ \ j = 1, 2,$$

is valid.

Proof. Let us take an arbitrary constant $r \in R$ which is not an eigenvalue of the problem $Bu + ru = 0, \ u \in H^{s+2m}$ and make in the equation (1) a substitution

$$v = Bu + ru\,, \qquad v \in H^s\,. \tag{5}$$

The existence of such constant follows from the properties of the spectrum of the elliptical operators (see e.g. [8]). Then

$$u = Gv\,, \quad Bu = v - rGv\,, \tag{6}$$

where G is the inverse to $Bu + ru$ and the equation (1) will take the form

$$Kv \equiv Av - rAGv + TGv = f\,, \qquad K : H^s \to H^s\,, \tag{7}$$

being still equivalent to the original one. The equivalence here means, that solvability of one of the equation yields solvability of another and their solutions are joined by the relationships (5), (6). Now let us rewrite the system of equations (4) as an operator equation

$$A_{\mathbf{n}} B_{\mathbf{n}} u_{\mathbf{n}} + T_{\mathbf{n}} u_{\mathbf{n}} = f_{\mathbf{n}}\,, \tag{8}$$

$$A_{\mathbf{n}} = p_{\mathbf{n}} A P_{\mathbf{n}}\,, \quad f_{\mathbf{n}} = p_{\mathbf{n}} f\,,$$

$$(B_{\mathbf{n}} u_{\mathbf{n}})(\mathbf{t_k}) = \sum_{|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = m} b_{\boldsymbol{\alpha}\boldsymbol{\beta}}(\mathbf{t_k})(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}_{\mathbf{n}} u_{\mathbf{n}})(\mathbf{t_k})\,, \qquad \mathbf{t_k} \in \boldsymbol{\Delta}_n\,,$$

and make a substitution

$$v_{\mathbf{n}} = B_{\mathbf{n}} u_{\mathbf{n}} + r u_{\mathbf{n}}\,, \qquad v_{\mathbf{n}} \in H^s_{\mathbf{n}}\,. \tag{9}$$

As it is shown in [12] equation (9) is uniquely solvable for all $\mathbf{n}$, beginning from some $\mathbf{n_1}$, and for $v_{\mathbf{n}} = p_{\mathbf{n}} v$ solutions $u_{\mathbf{n}} = G_{\mathbf{n}} v_{\mathbf{n}} = G_{\mathbf{n}} p_{\mathbf{n}} v$ converge to the solution $u = Gv$ of the equation (5). Here $G_{\mathbf{n}}$ is inverse to the operator $B_{\mathbf{n}} u_{\mathbf{n}} + r u_{\mathbf{n}}$ and

$$u_{\mathbf{n}} = G_{\mathbf{n}} v_{\mathbf{n}}\,, \qquad B_{\mathbf{n}} u_{\mathbf{n}} = v_{\mathbf{n}} - r G_{\mathbf{n}} v_{\mathbf{n}}\,. \tag{10}$$

By the substitution (9) we will get an equation

$$K_{\mathbf{n}} v_{\mathbf{n}} \equiv A_{\mathbf{n}} v_{\mathbf{n}} - r A_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} + T_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} = f_{\mathbf{n}}\,, \qquad K_{\mathbf{n}} : H^s_{\mathbf{n}} \to H^s_{\mathbf{n}}\,, \tag{11}$$

which is equivalent to the equation (8). As above the equivalence here means, that solvability of one of the equations yields solvability of another and their solutions are joined by the relationships (9), (10).

The invertibility of the operators $K_{\mathbf{n}} : H^s_{\mathbf{n}} \to H^s_{\mathbf{n}}$ we'll prove following [11]. To do this we have to establish the following:

a) $\|P_{\mathbf{n}} f_{\mathbf{n}} - f\|_s \to 0$ for $\mathbf{n} \to \infty$;

b) the sequence of operators $(K_{\mathbf{n}})$ approximates operator K compactly;

c) K is invertible.

The validity of a) follows immediately from the definition of $f_{\mathbf{n}}$ and the Lemma 1.

$$\|P_{\mathbf{n}} f_{\mathbf{n}} - f\|_s = \|P_{\mathbf{n}} p_{\mathbf{n}} f - f\|_s \le C E_{\mathbf{n}}(f)_s .$$

To check b) we will show first that the sequence $(K_{\mathbf{n}})$ approximates the operator K with respect to $P_{\mathbf{n}}$. For arbitrary $v_{\mathbf{n}} \in H^s_{\mathbf{n}}$ we will write

$$\begin{aligned} \|P_{\mathbf{n}} K_{\mathbf{n}} v_{\mathbf{n}} - K P_{\mathbf{n}} v_{\mathbf{n}}\|_s &\le \|P_{\mathbf{n}} A_{\mathbf{n}} v_{\mathbf{n}} - A P_{\mathbf{n}} v_{\mathbf{n}}\|_s \\ &\quad + |r|\, \|P_{\mathbf{n}} A_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} - A G P_{\mathbf{n}} v_{\mathbf{n}}\|_s + \|P_{\mathbf{n}} T_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} - T G P_{\mathbf{n}} v_{\mathbf{n}}\|_s \end{aligned} \tag{12}$$

and estimate each summand of the right-hand side independently. From the definition of the operator $A_{\mathbf{n}}$ and condition 1) of the Theorem it follows that the first summand is equal to zero. For the second summand, using once more the definition of the operator $A_{\mathbf{n}}$, condition 1) of the Theorem and boundness of the operators A and $P_{\mathbf{n}}$, we will have

$$\begin{aligned} |r|\, \|P_{\mathbf{n}} A_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} - A G P_{\mathbf{n}} v_{\mathbf{n}}\|_s &\le C \|P_{\mathbf{n}} p_{\mathbf{n}} A P_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} - A G P_{\mathbf{n}} v_{\mathbf{n}}\|_s \\ &\le C \|P_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} - G P_{\mathbf{n}} v_{\mathbf{n}}\|_s \\ &\le C (\|G_{\mathbf{n}} v_{\mathbf{n}} - p_{\mathbf{n}} G P_{\mathbf{n}} v_{\mathbf{n}}\|_{s,\mathbf{n}} + E_{\mathbf{n}}(G P_{\mathbf{n}} v_{\mathbf{n}})_s) . \end{aligned}$$

For the third summand, using Lemma 1 and boundness of the operators $T_{\mathbf{n}}$, we will obtain

$$\begin{aligned} \|P_{\mathbf{n}} T_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} - T G P_{\mathbf{n}} v_{\mathbf{n}}\|_s &\le C \big(\|G_{\mathbf{n}} v_{\mathbf{n}} - p_{\mathbf{n}} G P_{\mathbf{n}} v_{\mathbf{n}}\|_{s,\mathbf{n}} \\ &\quad + \|T_{\mathbf{n}} p_{\mathbf{n}} G P_{\mathbf{n}} v_{\mathbf{n}} - p_{\mathbf{n}} T G P_{\mathbf{n}} v_{\mathbf{n}}\|_{s,\mathbf{n}} + E_{\mathbf{n}}(T G P_{\mathbf{n}} v_{\mathbf{n}})_s \big) . \end{aligned}$$

Finally, the estimation (12) will take the form

$$\begin{aligned} \|P_{\mathbf{n}} K_{\mathbf{n}} v_{\mathbf{n}} - K P_{\mathbf{n}} v_{\mathbf{n}}\|_s &\le C \big(\|G_{\mathbf{n}} v_{\mathbf{n}} - p_{\mathbf{n}} G P_{\mathbf{n}} v_{\mathbf{n}}\|_{s,\mathbf{n}} \\ &\quad + \|T_{\mathbf{n}} p_{\mathbf{n}} G P_{\mathbf{n}} v_{\mathbf{n}} - p_{\mathbf{n}} T G P_{\mathbf{n}} v_{\mathbf{n}}\|_{s,\mathbf{n}} \\ &\quad + E_{\mathbf{n}}(G P_{\mathbf{n}} v_{\mathbf{n}})_s + E_{\mathbf{n}}(T G P_{\mathbf{n}} v_{\mathbf{n}})_s \big) , \end{aligned}$$

which, taking into account the condition 4) of the Theorem, convergence of the operators $(G_{\mathbf{n}})$ and convergence to zero of the best approximations of the functions $G P_{\mathbf{n}} v_{\mathbf{n}}$ and $T G P_{\mathbf{n}} v_{\mathbf{n}}$, means the approximation of the operator K by the sequence of the operators $(K_{\mathbf{n}})$ with respect to $P_{\mathbf{n}}$.

Let us assume now, that the sequence $(v_{\mathbf{n}})$, $v_{\mathbf{n}} \in H^s_{\mathbf{n}}$ is bounded $\|v_{\mathbf{n}}\|_{s,\mathbf{n}} \le 1$, and prove that the sequence $(P_{\mathbf{n}} K_{\mathbf{n}} v_{\mathbf{n}} - K P_{\mathbf{n}} v_{\mathbf{n}})$ is compact in $H^s_{\mathbf{n}}$. We will write

$$P_{\mathbf{n}} K_{\mathbf{n}} v_{\mathbf{n}} - K P_{\mathbf{n}} v_{\mathbf{n}} = r A G P_{\mathbf{n}} v_{\mathbf{n}} - T G P_{\mathbf{n}} v_{\mathbf{n}} - r A P_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} + P_{\mathbf{n}} T_{\mathbf{n}} G_{\mathbf{n}} v_{\mathbf{n}} ,$$

and prove the compactness of each summand of the right-hand side. The operators $G : H^s \to H^{s+2m}$, $T : H^{s+2m} \to H^{s+\varepsilon}$ $A : H^{s+2m} \to H^{s+2m}$ are bounded, so the sequences $(rAGP_{\mathbf{n}}v_{\mathbf{n}})$ and $(TGP_{\mathbf{n}}v_{\mathbf{n}})$ are bounded in $H^{s+\gamma}, \gamma = \min(2m, \varepsilon)$ and thus compact in H^s. The operators $G_{\mathbf{n}} : H^s_{\mathbf{n}} \to H^{s+2m}_{\mathbf{n}}$ and $T_{\mathbf{n}}G_{\mathbf{n}} : H^s_{\mathbf{n}} \to H^{s+\varepsilon}_{\mathbf{n}}$ are also bounded so the polynomials $P_{\mathbf{n}}G_{\mathbf{n}}v_{\mathbf{n}}$ and $P_{\mathbf{n}}T_{\mathbf{n}}G_{\mathbf{n}}v_{\mathbf{n}}$ are bounded in $H^{s+\gamma}$ and thus sequences $(rAP_{\mathbf{n}}G_{\mathbf{n}}v_{\mathbf{n}})$ and $(P_{\mathbf{n}}T_{\mathbf{n}}G_{\mathbf{n}}v_{\mathbf{n}})$ are also compact in H^s, which gives the compactness of the sequence $(P_{\mathbf{n}}K_{\mathbf{n}}v_{\mathbf{n}} - KP_{\mathbf{n}}v_{\mathbf{n}})$.

The validity of c) follows from the condition 5) of the Theorem and equivalence of the equations (1) and (7).

Therefore, according to the Theorem 6.1 [11], for all $\mathbf{n}$, beginning from some $\mathbf{n}_0$, $\mathbf{n}_0 \geq \mathbf{n}_1$, the equations (11), (8), and thus the system of the equations (4) are uniquely solvable and the approximate solutions $(u^*_{\mathbf{n}})$ of the system of equations (4) converge to the exact solution u^* of the equation (1) with a rate

$$\begin{aligned}\|u^*_{\mathbf{n}} - p_{\mathbf{n}}u^*\|_{s+2m,\mathbf{n}} &\leq C\|p_{\mathbf{n}}(ABu^* + Tu^*) - (A_{\mathbf{n}}B_{\mathbf{n}}p_{\mathbf{n}}u^* + T_{\mathbf{n}}p_{\mathbf{n}}u^*)\|_{s,\mathbf{n}} \\ &\leq \ C\big(E_{\mathbf{n}}(Bu^*)_s + \|p_{\mathbf{n}}Bu^* - B_{\mathbf{n}}p_{\mathbf{n}}u^*\|_{s,\mathbf{n}} + \|p_{\mathbf{n}}Tu^* - T_{\mathbf{n}}p_{\mathbf{n}}u^*\|_{s,\mathbf{n}}\big)\,.\end{aligned}$$

If, moreover, $u^* \in H^{s+2m+2}$, then $Bu^* \in H^{s+2}$ and as it is shown in [11],

$$\|p_{\mathbf{n}}Bu^* - B_{\mathbf{n}}p_{\mathbf{n}}u^*\|_{s,\mathbf{n}} \leq C\mathbf{h}^2\,.$$

On the other hand, according to the Lemma 2, and using obvious inequality $(1 + \mathbf{n}^2)^{-q} \leq C(\mathbf{h}^2)^q$, $q \in R$, $q > 0$, we will have

$$E_{\mathbf{n}}(Bu^*)_s \leq (1 + \mathbf{n}^2)^{-1}E_{\mathbf{n}}(Bu^*)_{s+2} \leq C(\mathbf{h}^2)\,,$$

which, together with the condition 4) of the Theorem gives the requested estimation

$$\|u^*_{\mathbf{n}} - p_{\mathbf{n}}u^*\|_{s+2m,\mathbf{n}} \leq C(\mathbf{h}^2 + \eta_{\mathbf{n}})\,.$$

The Theorem is proved. □

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