ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 209 – 217

ASYMPTOTIC EQUIVALENCE FOR POSITIVE DECAYING SOLUTIONS OF THE GENERALIZED EMDEN-FOWLER EQUATIONS AND ITS APPLICATION TO ELLIPTIC PROBLEMS

KEN-ICHI KAMO

Abstract. This paper is concerned with the problem of asymptotic equivalence for positive rapidly decaying solutions of a class of second order quasilinear ordinary differential equations. Its application to exterior Dirichlet problems is also given.

1. Introduction

In this paper we consider the two differential equations of the same form

$$(p(t)|x'|^{\alpha-1}x')' = a(t)|x|^{\lambda-1}x \tag{1}$$

and

$$(p(t)|y'|^{\alpha-1}y')' = b(t)|y|^{\lambda-1}y\,. \tag{2}$$

Throughout this paper we assume the following:

(C1) $0 < \alpha < \lambda$ (super-homogeneity condition);

(C2) $p, a, b \in C([t_0, \infty); (0, \infty))$;

(C3) $\displaystyle\int_{t_0}^{\infty} p(t)^{-1/\alpha}\, dt < \infty$.

For the sake of convenience, we put

$$\pi(t) = \int_t^{\infty} p(s)^{-1/\alpha}\, ds\,, \qquad t \geq t_0\,.$$

By a solution x of (1) we mean a function x such that x and $p|x'|^{\alpha-1}x'$ are of class C^1, and x satisfies (1) near $+\infty$. Throughout this paper we shall confine ourselves to the study of those solutions which do not vanish identically near $+\infty$.

2000 *Mathematics Subject Classification*: 34D05, 35D05.
Key words and phrases: quasilinear differential equations, rapidly decaying solutions, asymptotic equivalence.
Received June 17, 2002.

It was already known in [8] that every positive solution x of (1) has exactly one of the four asymptotic behavior listed bellow:

$$\begin{array}{lll} \text{(i)} & (\textit{rapidly decaying} \text{ solution}) & \lim_{t\to\infty} \dfrac{x(t)}{\pi(t)} = 0; \\ \text{(ii)} & (\textit{slowly decaying} \text{ solution}) & \lim_{t\to\infty} \dfrac{x(t)}{\pi(t)} \in (0,\infty); \\ \text{(iii)} & (\textit{asymptotic constant} \text{ solution}) & \lim_{t\to\infty} x(t) \in (0,\infty); \\ \text{(iv)} & (\textit{unbounded} \text{ solution}) & \lim_{t\to\infty} x(t) = \infty. \end{array}$$

In this paper we discuss only rapidly decaying solutions. Necessary conditions and sufficient conditions for the existence of rapidly decaying solutions were also obtained in [8]:

Theorem A. *If* (1) *has a positive rapidly decaying solution, then*

$$\int^{\infty} a(t)\,dt = \infty\,.$$

Theorem B. *The equation* (1) *has a positive rapidly decaying solution if*

$$\int^{\infty} a(t)\pi(t)^{\lambda}\,dt = \infty\,.$$

Two topics are treated in the present paper. Firstly we show that if a and b have the same asymptotic behavior, then so do positive rapidly decaying solutions of equation (1) and of equation (2). This is the main objective of this paper. Secondly, as an application of this result, we prove the existence of positive solutions of a class of quasilinear elliptic problems with prescribed asymptotic behavior near ∞. The proof given there is based on the supersolution-subsolution method due to Kura [5].

We remark that such problems have been discussed essentially in [1, 3, 4, 7] if (C3) is violated: $\int^{\infty} p(t)^{-1/\alpha}\,dt = \infty$. On the other hand, if $a(t) < 0$, then asymptotic theory for solutions was discussed in detail, for example, [2, 6].

2. Main Result

The following comparison lemma for rapidly decaying solutions is needed to prove our main theorem:

Lemma 1. *Suppose that $a(t) \le b(t)$ for $t \in [t_0,\infty)$. Let $x(t)$ and $y(t)$ be positive rapidly decaying solutions of equations* (1) *and* (2)*, respectively, and $x(t_0) \ge y(t_0)$. Then $x(t) \ge y(t)$ for $t \in [t_0,\infty)$.*

Proof. Suppose the contrary that there exists $t_1 > t_0$ such that $x(t_1) < y(t_1)$. Let $t_2 = \max\{t < t_1 : x(t) = y(t)\}$. Then $x(t_2) = y(t_2)$ and $x(t) < y(t)$ for

$t \in (t_2, t_1)$. We can easily see that there exists $t_3 \in (t_2, t_1]$ such that $x(t_3) < y(t_3)$ and $x'(t_3) < y'(t_3)$. By Lemma 1.1 in [8], we obtain $x(t) < y(t)$ for $t \geq t_3$. Since

$$(p(t)|y'|^{\alpha-1}y')' - (p(t)|x'|^{\alpha-1}x')' = b(t)y^{\lambda} - a(t)x^{\lambda} > 0 \qquad \text{for} \quad t \geq t_3\,,$$

we see

$$\begin{aligned} p(t)|y'|^{\alpha-1}y' - p(t)|x'|^{\alpha-1}x' &= p(t)((-x')^{\alpha} - (-y')^{\alpha}) \\ &\geq p(t_3)[(-x'(t_3))^{\alpha} - (-y'(t_3))^{\alpha}] \\ &\equiv \delta_1 > 0 \end{aligned}$$

for $t \geq t_3$. This implies that

$$(-x'(t))^{\alpha} \geq (-x'(t))^{\alpha} - (-y'(t))^{\alpha} > \delta_1 p(t)^{-1}\,,$$

that is

$$-x'(t) > \delta_2 p(t)^{-1/\alpha}\,,$$

where δ_2 is a positive constant. Integrating this inequality on $[t, \infty)$, we obtain

$$x(t) > \delta_2 \int_t^{\infty} p(s)^{-1/\alpha}\, ds \qquad \text{for} \quad t \geq t_3\,,$$

which is a contradiction. Hence $y(t) \leq x(t)$ for $t \geq t_0$. This completes the proof. □

The following theorem is the main result of this paper:

Theorem 1. *Suppose that*

$$\lim_{t\to\infty} \frac{a(t)}{b(t)} = 1\,.$$

Let x and y be positive rapidly decaying solutions of (1) *and* (2)*, respectively. Then $x(t) \sim y(t)$ as $t \to \infty$.*

Proof. Put $z(t) = x(t)/y(t)$, and make the change of variable $s = \int^t y(r)^{-2}\, dr$. Then

$$\begin{aligned} &\ddot{z}(s) + \frac{b(t)y(t)^{\lambda+3} + p'(t)(-y'(t))^{\alpha}y(t)^3}{\alpha p(t)(-y'(t))^{\alpha-1}} z(s) \\ &= \frac{y(t)^2[-y'(t)y(t)z(s) - \dot{z}(s)][p'(t) + a(t)y(t)^{\lambda}z(s)^{\lambda}\{-y'(t)z(s) - \dot{z}(s)/y(t)\}^{-\alpha}]}{\alpha p(t)}\,, \end{aligned}$$

where $\cdot = d/ds$. To establish this theorem it suffices to show that $\lim_{s\to\infty} z(s) = 1$.

First we show that

$$0 < \liminf_{s\to\infty} z(s) \leq \limsup_{s\to\infty} z(s) < \infty\,, \tag{3}$$

i.e., there exist positive constants c_1 and c_2 satisfying

$$c_1 y(t) \leq x(t) \leq c_2 y(t) \quad \text{near} \quad +\infty\,.$$

Let δ be a positive constant. Then for sufficiently large t_1 we have

$$b(t) \leq (1+\delta)a(t)\,, \qquad t \geq t_1\,. \tag{4}$$

We note that the function $w(t) \equiv (1+\delta)^{1/(\alpha-\lambda)}x(t)$ solves the equation

$$(p(t)|w'|^{\alpha-1}w')' = (1+\delta)a(t)|w|^{\lambda-1}w\,. \tag{5}$$

There are two possibilities for $y(t_1)$ and $w(t_1)$:

(A) $w(t_1) \le y(t_1)$; and

(B) $w(t_1) > y(t_1)$.

Let case (A) occur. Since y and w are rapidly decaying solutions of equations (2) and (5), respectively, we see by (4) and Lemma 1 that $w(t) \le y(t)$ for $t \ge t_1$. Obviously, this implies that $x(t) = O(y(t))$.

Next let (B) occur. We can find sufficiently small constant $0 < m < 1$ such that $mw(t_1) < y(t_1)$. Notice that the function $\bar{w}(t) \equiv mw(t)$ is a rapidly decaying solution of the equation

$$(p(t)|\bar{w}'|^{\alpha-1}\bar{w}')' = \frac{1+\delta}{m^{\lambda-\alpha}}a(t)|\bar{w}|^{\lambda-1}\bar{w}\,.$$

By Lemma 1 again, we see that $\bar{w}(t) \equiv mw(t) \le y(t)$; and so we have $x(t) = O(y(t))$.

In a similar way we find that $y(t) = O(x(t))$. Hence (3) holds.

Next we show that $z \to 1$ as $s \to \infty$. Define an auxiliary function f by

$$f(s) = \left(\frac{\tilde{b}(s)}{\tilde{a}(s)}\right)^{1/(\lambda-\alpha)}\,,$$

where $\tilde{a}(s) = a(t(s))$ and $\tilde{b}(s) = b(t(s))$. Clearly we see that $f(s) \to 1$ as $s \to \infty$. Notice that if z attains an extremum at some point s_0, then we have

$$\ddot{z}(s_0) = \frac{\tilde{a}(s_0)y(t(s_0))^{\lambda+3}(-y'(t(s_0)))^{1-\alpha}}{\alpha p(t(s_0))}z(s_0)\left[z(s_0)^{\lambda-\alpha} - \frac{\tilde{b}(s_0)}{\tilde{a}(s_0)}\right]. \tag{6}$$

Hence if $\dot{z} = 0$ and $z > f(s)$, then $\ddot{z}(s) > 0$ there, by (6). This means that only minimum can occur in the region $z > f(s)$. Similarly only maximum can occur in the regeon $0 < z < f(s)$.

We may assume that $f(s)$ oscillates around 1 as $s \to \infty$ since the other case is even simpler.

For sufficiently small $\delta > 0$, we can find two strictly increasing sequences $\{s_n\}$ and $\{\tilde{s}_n\}$ satisfying

$$\begin{cases} \lim_{n\to\infty} s_n = \lim_{n\to\infty} \tilde{s}_n = \infty\,, \\ f(s_n) = 1 + \dfrac{\delta}{n}\,,\ f(s) < 1 + \dfrac{\delta}{n}\,,\ s > s_n\,, \\ f(\tilde{s}_n) = 1 - \dfrac{\delta}{n}\,,\ f(s) > 1 - \dfrac{\delta}{n}\,,\ s > \tilde{s}_n\,,\ n \in N\,. \end{cases}$$

Obviously, if $z(s)$ attains extremum at points (s, v) satisfying

$$s_n \le s \le s_{n+1},\ z(s) > 1 + \frac{\delta}{n}\,,$$

then only minimum can occur. Similarly, if $z(s)$ attains extremum at points (s,z) satisfying

$$\tilde{s}_n \le s \le \tilde{s}_{n+1}\,,\ 0 < z(s) < 1 - \frac{\delta}{n}\,,$$

then only maximum can occur. We define the sets Ω_n and $\tilde{\Omega}_n,\ n \in N$, by

$$\Omega_n = \{(s,z) \mid s_n \le s \le s_{n+1}, 1 \le z \le z(s_n)\}\,,$$

$$\tilde{\Omega}_n = \{(s,z) \mid \tilde{s}_n \le s \le \tilde{s}_{n+1},\ z(\tilde{s}_n) \le z \le 1\}\,,$$

and put

$$\Omega = \left(\bigcup_{n=1}^{\infty} \Omega_n\right) \bigcup \left(\bigcup_{n=1}^{\infty} \tilde{\Omega}_n\right)\,.$$

Suppose that $\big(s_*, z(s_*)\big) \in \Omega$. (The other case can be treated similarly.) If z remains in IntΩ (the interior of Ω) for all $s \ge s_*$, then clearly $\lim_{s\to\infty} z(s) = 1$. Suppose the contrary that z intersects $\partial\Omega$. Let $\bar{s} > s_*$, be the least value of s's where z intersects $\partial\Omega$. We may suppose that $z(\bar{s}) > 1$ (and we automatically have $z(\bar{s}) > f(\bar{s})$) since the other case is proved as in this case. If $\dot{z}(\bar{s}) > 0$, then z never goes back to IntΩ again, and hence one can easily show that v is eventually monotone. If $\dot{z}(\bar{s}) < 0$, then either z goes back to IntΩ again, or z never hits the set IntΩ for $s \ge \bar{s}$. In the latter case we find that z is eventually monotone, while if z goes out of IntΩ and goes into IntΩ infinitely many times in a neighborhood of $+\infty$, then we find that $z \to 1$ as $s \to \infty$. The case that $\dot{z}(\bar{s}) = 0$ can be treated similarly.

Continuing the tracing of the solution curve in this manner, we find that z is eventually monotone or that $z(s) \to 1$ as $s \to \infty$. Hence z has a finite limit l. By (3) we find that $l \in (0,\infty)$.

Since rapidly decaying solution y has the property $\lim_{t\to\infty} p(t)[-y'(t)]^\alpha = 0$, employing l'Hospital's rule we obtain

$$\begin{aligned} l &= \lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to\infty} \frac{-x'(t)}{-y'(t)} = \lim_{t\to\infty} \left(\frac{p(t)(-x'(t))^\alpha}{p(t)(-y'(t))^\alpha}\right)^{1/\alpha} \\ &= \lim_{t\to\infty} \left(\frac{-[p(t)(-x'(t))^\alpha]'}{-[p(t)(-y'(t))^\alpha]'}\right)^{1/\alpha} = \lim_{t\to\infty} \left(\frac{a(t)x(t)^\lambda}{b(t)y(t)^\lambda}\right)^{1/\alpha} = l^{\lambda/\alpha}\,. \end{aligned}$$

This implies that $l = 1$, that is, $x(t) \sim y(t)$ as $t \to \infty$. This completes the proof. □

Let $c > 0$, $0 < \alpha < \beta$ and $\beta\lambda - \alpha\sigma - \alpha\lambda - \alpha < 0$. Then, the typical equation

$$(t^\beta |v'|^{\alpha-1} v')' = ct^\sigma |v|^{\lambda-1} v$$

has a positive rapidly decaying solution u_0 explicitly given by

$$u_0(t) = \hat{c}t^{-k}\,, \tag{7}$$

where

$$k = \frac{\sigma + \alpha + 1 - \beta}{\lambda - \alpha} > 0\,, \quad \text{and} \quad \hat{c}^{\lambda-\alpha} = \frac{k^\alpha(\alpha(k+1) - \beta)}{c}\,.$$

This simple observation and Theorem 2 give the following corollary used in Section 3:

Corollary 1. *Consider the equation*

$$(t^{\beta}|u'|^{\alpha-1}u')' = q(t)|u|^{\lambda-1}u\,, \tag{8}$$

where α, β and λ are constants such that $0 < \alpha < \lambda$ and $\alpha < \beta$, and q is a positive continuous function satisfying $q(t) \sim ct^{\sigma}$ for some $c > 0$, $\sigma \in R$. If $\beta\lambda - \alpha\sigma - \alpha\lambda - \alpha < 0$, then every positive rapidly decaying solution of (8) *has the asymptotic form*

$$u(t) \sim u_0(t)\,,$$

where u_0 is given by (7)*.*

3. Application

In this section we show that Corollary 3 can be applied to show the existence of some solutions of the exterior Dirichlet problem for the elliptic equation:

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) = q(x)|u|^{\lambda-1}u\,, \quad \text{in} \quad \Omega\,, \tag{9}$$

$$u = g(x)\,, \quad \text{on} \quad \partial\Omega\,, \tag{10}$$

where Ω is an exterior domain in R^N, $N \geq 2$, with boundary $\partial\Omega$ of class C^1, $0 < m-1 < \lambda$, $g \in C^1\big(\partial\Omega; (0,\infty)\big)$ and $q \in C\big(\bar{\Omega}; (0,\infty)\big)$. We assume throughout the section that

$$q(x) \sim c_1|x|^{\sigma_1} \quad \text{as} \quad |x| \to \infty \quad \text{for some constants} \quad c_1 > 0 \quad \text{and} \quad \sigma_1 \in R\,.$$

A function u is said to be a solution (subsolution, supersolution) of equation (9) in Ω if $u \in W^{1,m}_{\text{loc}}(\Omega)$ and

$$\int_{\Omega} \{|\nabla u|^{m-2}\nabla u \cdot \nabla\phi + q(x)|u|^{\lambda-1}u\phi\}\,dx = 0 \quad (\leq 0,\ \geq 0)\,,$$

for all $\phi \in C_0^{\infty}(\Omega)$ with $\phi \geq 0$ in Ω. See [5] for details.

To find positive solutions of problem (9)–(10) we use the supersolution-subsolution method which can be formulated, in our context, as follows:

Proposition 1 (Cf. Theorem 4.4 in [5])**.** *Let v and w be a subsolution and a supersolution of* (9) *in Ω, respectively, such that $v \leq w$ a.e. in Ω and $v \leq g \leq w$ a.e. on $\partial\Omega$. Then problem* (9)–(10) *has a solution u such that $v \leq u \leq w$ a.e. in Ω.*

Theorem 2. *If $m < N$ and $N\lambda - mN - m\sigma_1 + N + \sigma_1 - m\lambda < 0$, then problem* (9)–(10) *has a positive solution u satisfying*

$$u(x) \sim a|x|^{-\nu}\,, \quad \text{as} \quad |x| \to \infty \tag{11}$$

where

$$\begin{aligned} &\nu = \frac{m+\sigma_1}{\lambda - m + 1} > 0 \quad \text{and} \\ &a^{\lambda-m+1} = \frac{(Nm + m\lambda + m\sigma_1 - N\lambda - N - \sigma_1)(m+\sigma_1)^{m-1}}{c_1(\lambda - m + 1)^m}\,. \end{aligned} \tag{12}$$

We introduce notations used here. We may assume without loss of generality that $0 \notin \bar{\Omega}$ and q is defined in R^N. Let

$$\begin{aligned} g_* &= \min_{\partial\Omega} g(x)\,, & g^* &= \max_{\partial\Omega} g(x)\,;\\ r_* &= \operatorname{dist}(0,\partial\Omega)\,, & r^* &= \max\{|x|; x \in \partial\Omega\}\,;\\ q_*(r) &= \min_{|x|=r} q(x) \quad \text{and} & q^*(r) &= \max_{|x|=r} q(x)\,. \end{aligned}$$

Proof of Theorem 5. A function $\bar{u}$ satisfying

$$\operatorname{div}(|\nabla \bar{u}|^{m-2}\nabla \bar{u}) \le q_*(|x|)\bar{u}^{\lambda}\,, \quad |x| \ge r_* \tag{13}$$

is a supersolution of equation (9). Similarly a function $\underline{u}$ satisfying

$$\operatorname{div}(|\nabla \underline{u}|^{m-2}\nabla \underline{u}) \ge q^*(|x|)\underline{u}^{\lambda},\ |x| \ge r_* \tag{14}$$

is a subsolution of equation (9). Since these inequalities have radial symmetry, it is natural to construct such $\bar{u}$ and $\underline{u}$ as radially symmetric functions. By putting $\bar{u}(x) = \bar{v}(r)$ and $\underline{u}(x) = \underline{v}(r)$, $r = |x|$, (13) and (14) reduce to

$$(r^{N-1}|\bar{v}'|^{m-2}\bar{v}')' \le r^{N-1}q_*(r)\bar{v}^{\lambda}\,, \quad r \ge r_* \tag{15}$$

and

$$(r^{N-1}|\underline{v}'|^{m-2}\underline{v}')' \ge r^{N-1}q^*(r)\underline{v}^{\lambda}\,, \quad r \ge r_*\,, \tag{16}$$

respectively, where $' = d/dr$.

First we construct a subsolution $\underline{u}$. Consider the problem for the ordinary differential equation

$$\begin{cases} (r^{N-1}|\underline{w}'|^{m-2}\underline{w}')' = r^{N-1}q^*(r)\underline{w}^{\lambda}\,, & r \ge r_*\,,\\ \underline{w}(r_*) = g_*\,. \end{cases}$$

From Theorem 4.2 in [8], this problem has at least one positive rapidly decaying solution $\underline{w}$. Since $m < N$ and $N\lambda - mN - m\sigma_1 + N + \sigma_1 - m\lambda < 0$, we find from Corollary 3 that

$$\underline{w}(r) \sim ar^{-\nu} \quad \text{as} \quad r \to \infty\,, \tag{17}$$

where a and ν are given by (12). Hence, the function given by $\underline{v}(r) \equiv \underline{w}(r)$ satisfies (16) (with $\ge$ replaced by $=$). This means that the function $\underline{u}(x) \equiv \underline{v}(|x|)$ is a subsolution of (9) satisfying $\underline{u}(x) \le g(x)$ on $\partial\Omega$.

Next we must construct a supersolution $\bar{u}$ so that $\underline{u} \le \bar{u}$ in Ω and $\bar{u} \ge g$ on $\partial\Omega$. Let $\bar{w}$ be a positive rapidly decaying solution of the problem

$$\begin{cases} (r^{N-1}|\bar{w}'|^{m-2}\bar{w}')' = r^{N-1}q_*(r)\bar{w}^{\lambda}\,, & r \ge r^*\,,\\ \bar{w}(r^*) = g^*\,. \end{cases}$$

Corollary 3 implies, as before, that

$$\bar{w}(r) \sim ar^{-\nu} \quad \text{as} \quad r \to \infty\,, \tag{18}$$

where a and ν are given in (12). Put

$$\bar{z}(r) = \begin{cases} \bar{w}(r)\,, \quad r \geq r^*\,, \\ g^* + \frac{(m-1)(r^*)^{(1-N)/(m-1)}(-\bar{w}'(r^*))}{m-N}\left[(r^*)^{(m-N)/(m-1)} - r^{(m-N)/(m-1)}\right]\,, \\ \hfill r_* \leq r \leq r^*\,. \end{cases}$$

Then, obviously $\bar{z}$ is of class $C^1[r_*, \infty)$ and satisfies

$$\begin{cases} (r^{N-1}|\bar{z}'|^{m-2}\bar{z}')' \leq r^{N-1}q_*(r)\bar{z}^\lambda\,, & r \geq r_*\,, \\ \bar{z}(r) \geq g^*\,, & r_* \leq r \leq r^*\,. \end{cases}$$

Moreover Lemma 1 implies that $\underline{w} \leq \bar{z}$, $r \geq r_*$. Therefore the function $\bar{u}$ given by $\bar{u}(x) = \bar{z}(|x|)$ becomes a supersolution of (9) satisfying

$$\underline{u} \leq \bar{u} \quad \text{for} \quad |x| \geq r_*\,, \quad \text{and} \quad \bar{u} \geq g \quad \text{on} \partial\Omega\,.$$

Proposition 4 guarantees that boundary value problem (9)–(10) has at least one solution u satisfying

$$\underline{u}(x) \leq u(x) \leq \bar{u}(x)\,, \quad \text{a.e.} \quad x \in \Omega\,.$$

Since the asymptotic behavior (17) and (18) yield

$$\underline{u}(x) \sim \bar{u}(x) \sim a|x|^{-\nu} \quad \text{as} \quad |x| \to \infty\,,$$

u must satisfy (11). This completes the proof. □

When $m = 2 < N$, the positive solution obtained in Theorem 5 satisfies

$$u(x) \sim c|x|^{-(\sigma_1+2)/(\lambda-1)} \text{ as } |x| \to \infty$$

for a suitable constant $c > 0$. Since $(\sigma_1+2)/(\lambda-1) > N-2$ under our assumptions, this means that semilinear elliptic equation $\Delta u = q(x)|u|^{\lambda-1}u$ can possess positive solutions decaying faster than $|x|^{2-N}$ as $|x| \to \infty$.

Acknowledgement. The author would like to express his sincere thanks to Professor Ivan Kiguradze for a number of helpful comments and suggestions.

References

[1] Cecchi, M., Došlá, Z. and Marini, M., *On the dynamics of the generalized Emden-Fowler equation*, Georgian Math. J. **7** (2000), 269–282.

[2] Elbert, Á. and Kusano, T., *Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations*, Acta Math. Hungar. **56** (1990), 325–336.

[3] Kamo, K., Usami, H., *Asymptotic forms of positive solutions of second-order quasilinear ordinary differential equations*, Adv. Math. Sci. Appl. **10** (2000), 673–688.

[4] Kamo, K., Usami, H., *Asymptotic forms of positive solutions of second-order quasilinear ordinary differential equations with sub-homogeneity*, Hiroshima Math. J. **31** (2001), 35–49.

[5] Kura, T., *The weak supersolution-subsolution method for second order quasilinear elliptic equations*, Hiroshima Math. J. **19** (1989), 1–36.

[6] Mirzov, J. D., *On some analogs of Strum's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **53** (1976), 418–425.

[7] Mizukami, M., Naito, M. and Usami, H., *Asymptotic behavior of solutions of a class of second order quasilinear ordinary differential equations*, Hiroshima Math. J. **32** (2002), 51–78.

[8] Tanigawa, T., *Existence and asymptotic behavior of positive solutions of second order quasilinear differential equations*, Adv. Math. Sci. Appl. **9** (1999), 907–938.

Statistics and Cancer Control Division
Research Center for Cancer Prevention and Screening
National Cancer Center 5-1-1 Tsukiji Chuo-ku
Tokyo 104-0045, Japan
E-mail: kkamo@gan2.res.ncc.go.jp