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ON WEAK FORMS OF PREOPEN AND PRECLOSED FUNCTIONS

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Abstract. In this paper we introduce two classes of functions called weakly preopen and weakly preclosed functions as generalization of weak openness and weak closedness due to [26] and [27] respectively. We obtain their characterizations, their basic properties and their relationshisps with other types of functions between topological spaces.

1. Introduction and preliminaries

The notion of preopen [19] set plays a significant role in general topology. Preopen sets are also called nearly open and locally dense [11] by several authors in the literature. They are not only important in the context of covering properties and decompositions of continuity but also in functional analysis in the context of open mappings theorems and closed graph theorems. One of the most important generalizations of continuity is the notion of nearly continuity [24] (=precontinuity [19] or almost continuity [13]) which involves preopen sets and is investigated by different authors under different terms (c.f. [6], [12], [13], [19], [25]). In 1982, A. S. Mashhour et al. [19] introduce and studied the class of weak precontinuous functions. In 1985, D. A. Rose [26] and D. A. Rose with D. S. Janckovic [27] have defined the notions of weakly open and weakly closed functions in topology respectively. This paper is devoted to present the class of weakly preopen functions (resp. weakly preclosed functions) as a new generalization of weakly open functions (resp. weakly closed functions). We investigate some of the fundamental properties of this class of functions.

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If S is any subset of a space X, then $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$ denote the closure and the interior of S respectively. Recall that a set S is called regular open (resp. regular closed) if $S=\mathrm{Int}(\mathrm{Cl}(S))$ (resp. $S=\mathrm{Cl}(\mathrm{Int}(S))$. A point $x\in X$ is called a θ-cluster [29] point of S if $S\cap \mathrm{Cl}(U)\neq\emptyset$ for each open set U containing x. The set of all θ-cluster points of S is called the θ-closure of S and is denoted by $\mathrm{Cl}_\theta(S)$. Hence, a subset S is called θ-closed [29] if $\mathrm{Cl}_\theta(S)=S$. The complement of a θ-closed set is called θ-open set. A subset $S\subset X$ is called preopen [19] (resp. α-open [21] and

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β-open [1] (or semi-preopen [2])),if $S \subset \mathrm{Int}(\mathrm{Cl}(S))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S))$ and $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$. The complement of a preopen set is called a preclosed [12] set. The family of all preopen (resp. preclosed) sets of a space X is denoted by $PO(X,\tau)$ (resp. $PC(X,\tau)$). The intersection of all preclosed sets containing S is called the preclosure of S [12, 2] and is denoted by $\mathrm{pCl}(S)$. The preinterior [12, 2] of S is defined by the union of all preopen sets contained in S and is denoted by $\mathrm{pInt}(S)$.

A space X is called extremally disconnected (E. D.) [30] if the closure of each open set in X is open. A space X is called preconnected [13] if X can not be expressed as the union of two nonempty disjoint preopen sets.

A function $f : (X,\tau) \to (Y,\sigma)$ is called:

(i) precontinuous [19] if for each open subset V of Y, $f^{-1}(V) \in PO(X,\tau)$.

(ii) weakly open [26,27] if $f(U) \subset \mathrm{Int}(f(\mathrm{Cl}(U)))$ for each open subset U of X.

(iii) weakly closed [27] if $\mathrm{Cl}(f(\mathrm{Int}(F))) \subset f(F)$ for each closed subset F of X.

(iv) relatively weakly open [5] if $f(U)$ is open in $f(\mathrm{Cl}(U))$ for every open subset U of X.

(v) almost continuous (in the sense of T. Husain), written as (a. c. H) [13] if for each $x \in X$ and for each neighbourhood V of $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighbourhood of x.

(vi) strongly continuous [16], if for every subset A of X, $f(\mathrm{Cl}(A)) \subset f(A)$.

(vii) almost open in the sense of Singal and Singal, written as (a. o. S) [28] if the image of each regular open set U of X is open set in Y.

(viii) preopen [19] (resp. preclosed [12], β-open [1], α-open [21]) if for each open set U (resp. closed set F, open set U, open set U) of X, $f(U)$ is preopen (resp. $f(F)$ is preclosed, $f(U)$ is β-open, $f(U)$ is α-open) set in Y.

(ix) contra-open [4] (resp. contra-closed [4], contra preclosed) if $f(U)$ is closed (resp. open, preopen) in Y for each open (resp. closed, closed) set U of X.

2. Weakly preopen functions

Since precontinuity [26] is dual to preopenness [19], we define in this section the concept of weak preopenness as a natural dual to the weak precontinuity due to A. Kar and P. Bhattacharya [14].

Definition 2.1. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be weakly preopen if $f(U) \subset \mathrm{pInt}(f(Cl(U)))$ for each open set U of X.

Clearly, every weakly open function is weakly preopen and every preopen function is also weakly preopen, but the converse is not generally true. For,

Example 2.2. A weakly preopen function need not be weakly open.
Let $X = \{a,b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is not weakly open, since $\{x\} = f(\{a\}) \not\subset \mathrm{pInt}(f(\mathrm{Cl}(\{a\}))) = \emptyset$, but f is weakly preopen.

Theorem 2.3. *Let X be a regular space. Then $f : (X,\tau) \to (Y,\sigma)$ is weakly preopen if and only if f is preopen.*

Proof. The sufficiency is clear. Necessity. Let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subset \mathrm{Cl}(U_x) \subset W$. Hence we obtain that $W = \bigcup\{U_x : x \in W\} = \bigcup\{\mathrm{Cl}(U_x) : x \in W\}$ and, $f(W) =$

$\bigcup\{f(U_x) : x \in W\} \subset \bigcup\{\mathrm{pInt}(f(\mathrm{Cl}(U_x))) : x \in W\} \subset \mathrm{pInt}(f(\bigcup\{\mathrm{Cl}(U_x) : x \in W\}) = \mathrm{pInt}(f(W))$. Thus f is preopen. □

Theorem 2.4. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:*

(i) *f is weakly preopen.*
(ii) *$f(\mathrm{Int}_\theta(A)) \subset \mathrm{pInt}(f(A))$ for every subset A of X.*
(iii) *$\mathrm{Int}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{pInt}(B))$ for every subset B of Y.*
(iv) *$f^{-1}(\mathrm{pCl}(B)) \subset \mathrm{Cl}_\theta(f^{-1}(B))$ for every subset B of Y.*
(v) *For each $x \in X$ and each open subset U of X containing x, there exists a preopen set V containing $f(x)$ such that $V \subset f(\mathrm{Cl}(U))$.*
(vi) *For each closed subset F of X, $f(\mathrm{Int}(F)) \subset \mathrm{pInt}(f(F))$.*
(vii) *For each open subset U of X, $f(\mathrm{Int}(\mathrm{Cl}(U))) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$.*
(viii) *For every preopen subset U of X, $f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$.*
(ix) *For every α-open subset U of X, $f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$.*

Proof. (i) $\to$ (ii): Let A be any subset of X and $x \in \mathrm{Int}_\theta(A)$. Then, there exists an open set U such that $x \in U \subset \mathrm{Cl}(U) \subset A$. Then, $f(x) \in f(U) \subset f(\mathrm{Cl}(U)) \subset f(A)$. Since f is weakly preopen, $f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U))) \subset \mathrm{pInt}(f(A))$. It implies that $f(x) \in \mathrm{pInt}(f(A))$. This shows that $x \in f^{-1}(\mathrm{pInt}(f(A)))$. Thus $\mathrm{Int}_\theta(A) \subset f^{-1}(\mathrm{pInt}(f(A)))$, and so, $f(\mathrm{Int}_\theta(A)) \subset \mathrm{pInt}(f(A))$.

(ii) $\to$ (i): Let U be an open set in X. As $U \subset \mathrm{Int}_\theta(\mathrm{Cl}(U))$ implies, $f(U) \subset f(\mathrm{Int}_\theta(\mathrm{Cl}(U))) \subset \mathrm{pInt}(f(Cl(U)))$. Hence f is weakly preopen.

(ii) $\to$ (iii): Let B be any subset of Y. Then by (ii), $f(\mathrm{Int}_\theta(f^{-1}(B)) \subset \mathrm{pInt}(B)$. Therefore $\mathrm{Int}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{pInt}(B))$.

(iii) $\to$ (ii): This is obvious.

(iii) $\to$ (iv): Let B be any subset of Y. Using (iii), we have

$$\begin{aligned} X - \mathrm{Cl}_\theta(f^{-1}(B)) &= \mathrm{Int}_\theta(X - f^{-1}(B)) = \mathrm{Int}_\theta(f^{-1}(Y - B)) \subset f^{-1}(\mathrm{pInt}(Y - B)) \\ &= f^{-1}(Y - \mathrm{pCl}(B)) = X - (f^{-1}(\mathrm{pCl}(B))\,. \end{aligned}$$

Therefore, we obtain $f^{-1}(\mathrm{pCl}(B)) \subset \mathrm{Cl}_\theta(f^{-1}(B))$.

(iv) $\to$ (iii): Similarly we obtain, $X - f^{-1}(\mathrm{pInt}(B)) \subset X - \mathrm{Int}_\theta(f^{-1}(B))$, for every subset B of Y, i.e., $\mathrm{Int}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{pInt}(B))$.

(i) $\to$ (v): Let $x \in X$ and U be an open set in X with $x \in U$. Since f is weakly preopen. $f(x) \in f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$. Let $V = \mathrm{pInt}(f(\mathrm{Cl}(U)))$. Hence $V \subset f(\mathrm{Cl}(U))$, with V containing $f(x)$.

(v) $\to$ (i): Let U be an open set in X and let $y \in f(U)$. It follows from (v) that $V \subset f(\mathrm{Cl}(U))$ for some V preopen in Y containing y. Hence we have, $y \in V \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$. This shows that $f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$, i.e., f is a weakly preopen function.

(i) $\to$ (vi) $\to$ (vii) $\to$ (viii) $\to$ (iv) $\to$ (i): This is obvious. □

Theorem 2.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a bijective function. Then the following statements are equivalent.*

(i) *f is weakly preopen,*
(ii) *$\mathrm{pCl}(f(U)) \subset f(\mathrm{Cl}(U))$ for each U open in X,*
(iii) *$\mathrm{pCl}(f(\mathrm{Int}(F)) \subset f(F)$ for each F closed in X.*

Proof. (i) $\to$ (iii): Let F be a closed set in X. Then we have $f(X - F) = Y - f(F) \subset \mathrm{pInt}(f(\mathrm{Cl}(X - F)))$ and so $Y - f(F) \subset Y - \mathrm{pCl}(f(\mathrm{Int}(F)))$. Hence $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(F)$.

(iii) $\rightarrow$ (ii): Let U be a open set in X. Since $Cl(U)$ is a closed set and $U \subset \mathrm{Int}(\mathrm{Cl}(U))$ by (iii) we have $\mathrm{pCl}(f(U)) \subset \mathrm{pCl}(f(\mathrm{Int}(\mathrm{Cl}(U))) \subset f(\mathrm{Cl}(U))$.

(ii) $\rightarrow$ (iii): Similar to (iii) $\rightarrow$ (ii).

(iii) $\rightarrow$ (i): Clear. □

Theorem 2.6. *If $f : (X,\tau) \rightarrow (Y,\sigma)$ is weakly preopen and strongly continuous, then f is preopen.*

Proof. Let U be an open subset of X. Since f is weakly preopen $f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$. However, because f is strongly continuous, $f(U) \subset \mathrm{pInt}(f(U))$ and therefore $f(U)$ is preopen. □

Example 2.7. A preopen function need not be strongly continuous.
Let $X = \{a,b,c\}$, and let τ be the indiscrete topology for X. Then the identity function of (X,τ) onto (X,τ) is a preopen function which is not strongly continuous.

Theorem 2.8. *A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is preopen if f is weakly preopen and relatively weakly open.*

Proof. Assume f is weakly preopen and relatively weakly open. Let U be an open subset of X and let $y \in f(U)$. Since f is relatively weakly open, there is an open subset V of Y for which $f(U) = f(\mathrm{Cl}(U)) \cap V$. Because f is weakly preopen, it follows that $f(U) \subset \mathrm{pInt}(f(\mathrm{Cl}(U)))$. Then $y \in \mathrm{pInt}(f(Cl(U))) \cap V \subset f(\mathrm{Cl}(U)) \cap V = f(U)$ and therefore $f(U)$ is preopen. □

Theorem 2.9. *If $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra preclosed, then f is a weakly preopen function.*

Proof. Let U be an open subset of X. Then, we have $f(U) \subset f(\mathrm{Cl}(U))) = \mathrm{pInt}(f(\mathrm{Cl}(U)))$. □

The converse of Theorem 2.9 does not hold.

Example 2.10. A weakly preopen function need not be contra preclosed is given from Example 2.2.

Next, we define a dual form, called complementary weakly preopen function.

Definition 2.11. A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is called complementary weakly preopen (written as c.w.p.o) if for each open set U of X, $f(\mathrm{Fr}(U))$ is preclosed in Y, where $\mathrm{Fr}(U)$ denotes the frontier of U.

Example 2.12. A weakly preopen function need not be c.w.p.o.
Let $X = \{a,b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is clearly weakly preopen, but is not c.w.p.o., since $F_r(\{b\}) = Cl(\{b\}) - \{b\} = \{a\}$ and $f(F_r(\{b\})) = \{x\}$ is not a preclosed set in Y.

Example 2.13. c.w.p.o. does not imply weakly preopen.
Let $X = \{a,b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$, $Y = \{x,y\}$ and $\sigma = \{\emptyset, \{y\}, Y\}$. Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is not weakly preopen, but f is c.w.p.o., since the frontier of every open set is the empty set and $f(\emptyset) = \emptyset$ is preclosed.

Examples 2.12 and 2.13 demonstrate the independence of complementary weakly preopenness and weakly preopenness.

Theorem 2.14. *Let $PO(X,\tau)$ closed under intersections. If $f:(X,\tau)\to(Y,\sigma)$ is bijective weakly preopen and c.w.p.o, then f is preopen.*

Proof. Let U be an open subset in X with $x\in U$, since f is weakly preopen, by Theorem 2.4 there exists a pre-open set V containing $f(x)=y$ such that $V\subset f(\mathrm{Cl}(U))$. Now $\mathrm{Fr}(U)=\mathrm{Cl}(U)-U$ and thus $x\notin\mathrm{Fr}(U)$. Hence $y\notin f(\mathrm{Fr}(U))$ and therefore $y\in V-f(\mathrm{Fr}(U))$. Put $V_y=V-f(\mathrm{Fr}(U))$ a pre-open set since f is c.w.p.o. Since $y\in V_y$, $y\in f(Cl(U))$. But $y\notin f(\mathrm{Fr}(U))$ and thus $y\notin f(\mathrm{Fr}(U))=f(\mathrm{Cl}(U))-f(U)$ which implies that $y\in f(U)$. Therefore $f(U)=\cup\{V_y : V_y\in PO(Y,\sigma), y\in f(U)\}$. Hence f is preopen. □

The following theorem is a variation of a result of C.Baker [4] in which contra-closedness is replaced with weakly preopen and closed by contra-M-pre-closed, where, $f:(X,\tau)\to(Y,\sigma)$ is said to be contra-M-preclosed provided that $f(F)$ is pre-open for each pre-closed subset F of X.

Theorem 2.15. *If $f:(X,\tau)\to(Y,\sigma)$ is weakly preopen, $PC(Y,\sigma)$ is closed under unions and if for each pre-closed subset F of X and each fiber $f^{-1}(y)\subset X-F$ there exists an open subset U of X for which $F\subset U$ and $f^{-1}(y)\cap\mathrm{Cl}(U)=\emptyset$, then f is contra-M-preclosed.*

Proof. Assume F is a pre-closed subset of X and let $y\in Y-f(F)$. Thus $f^{-1}(y)\subset X-F$ and hence there exists an open subset U of X for which $F\subset U$ and $f^{-1}(y)\cap\mathrm{Cl}(U)=\emptyset$. Therefore $y\in Y-f(\mathrm{Cl}(U))\subset Y-f(F)$. Since f is weakly preopen $f(U)\subset\mathrm{pInt}(f(\mathrm{Cl}(U)))$. By complement, we obtain $y\in\mathrm{pCl}(Y-f(\mathrm{Cl}(U)))\subset Y-f(F)$. Let $B_y=\mathrm{pCl}(Y-f(\mathrm{Cl}(U)))$. Then B_y is a pre-closed subset of Y containing y. Hence $Y-f(F)=\bigcup\{B_y : y\in Y-f(F)\}$ is pre-closed and therefore $f(F)$ is pre-open. □

Theorem 2.16. *If $f:(X,\tau)\to(Y,\sigma)$ is an a.o.S function, then it is a weakly preopen function.*

Proof. Let U be an open set in X. Since f is a.o.S and $\mathrm{Int}(\mathrm{Cl}(U))$ is regular open, $f(\mathrm{Int}(\mathrm{Cl}(U)))$ is open in Y and hence $f(U)\subset f(\mathrm{Int}(\mathrm{Cl}(U))\subset\mathrm{Int}(f(\mathrm{Cl}(U)))\subset\mathrm{pInt}(f(\mathrm{Cl}(U)))$. This shows that f is weakly preopen. □

The converse of Theorem 2.16 is not true in general.

Example 2.17. A weakly preopen function need not be a.o.S. Let $X=Y=\{a,b,c\}$, $\tau=\{\emptyset,\{a\},\{c\},\{a,c\},X\}$, and $\sigma=\{\emptyset,\{b\},\{a,b\},\{b,c\},X\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity. Then f is not a.o.S since $\mathrm{Int}(f(\mathrm{Int}(\mathrm{Cl}(\{a\}))))=\emptyset$. But f is weakly preopen.

Lemma 2.18. *If $f:(X,\tau)\to(Y,\sigma)$ is a continuous function, then for any subset U of X, $f(Cl(U))\subset Cl(f(U))$* [30].

Theorem 2.19. *If $f:(X,\tau)\to(Y,\sigma)$ is a weakly preopen and continuous function, then f is a β-open function.*

Proof. Let U be a open set in X. Then by weak preopenness of f, $f(U)\subset\mathrm{pInt}(f(\mathrm{Cl}(U)))$. Since f is continuous $f(\mathrm{Cl}(U))\subset\mathrm{Cl}(f(U))$. Hence we obtain that, $f(U)\subset\mathrm{pInt}(f(\mathrm{Cl}(U)))\subset\mathrm{pInt}(\mathrm{Cl}(f(U)))\subset\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f(U))))$. Therefore, $f(U)\subset\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f(U)))$ which shows that $f(U)$ is a β-open set in Y. Thus, f is a β-open function. □

Since every strongly continuous function is continuous, we have the following corollary.

Corollary 2.20. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective weakly preopen and strongly continuous function. Then f is a β-open function.*

Theorem 2.21. *If $f : (X, \tau) \to (Y, \sigma)$ is a bijective weakly preopen of a space X onto a pre-connected space Y, then X is connected.*

Proof. Let us assume that $X$is not connected. Then there exist non-empty open sets U_1 and U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = X$. Hence we have $f(U_1) \cap f(U_2) = \emptyset$ and $f(U_1) \cup f(U_2) = Y$. Since f is bijective weakly preopen, we have $f(U_i) \subset \mathrm{pInt}(f(\mathrm{Cl}(U_i)))$ for $i = 1, 2$ and since U_i is open and also closed, we have $f(\mathrm{Cl}(U_i) = f(U_i)$ for $i = 1, 2$. Hence $f(U_i)$ is preopen in Y for $i = 1, 2$. Thus, Y has been decomposed into two non-empty disjoint preopen sets. This is contrary to the hypothesis that Y is a pre-connected space. Thus X is connected. □

Definition 2.22. A space X is said to be hyperconnected [22] if every nonempty open subset of X is dense in X.

Theorem 2.23. *If X is a hyperconnected space, then a function $f : (X, \tau) \to (Y, \sigma)$ is weakly preopen if and only if $f(X)$ is preopen in Y.*

Proof. The sufficiency is clear. For the necessity observe that for any open subset U of $X, f(U) \subset f(X) = \mathrm{pInt}(f(X) = \mathrm{pInt}(f(\mathrm{Cl}(U)))$. □

3. Weakly preclosed functions

Now, we define the generalized form of weakly closed and preclosed functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly preclosed if $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(F)$ for each closed set F in X.

Clearly, (i) Every closed function is α-closed and every α-closed function is preclosed, but the reverse implications are not true in general.
(ii) every weakly closed as well as preclosed function is weakly preclosed function, but the converse is not generally true, as the next example shows.

Example 3.2.
(i) An injective function from a discrete space into an indiscrete space is preopen and preclosed, but neither α-open nor α-closed [20].
(ii) Let $X = \{x, y, z\}$ and $\tau = \{\emptyset, \{x\}, \{x, y\}, X\}$. Then a function $f : (X, \tau) \to (X, \tau)$ which is defined by $f(x) = x$, $f(y) = z$ and $f(xz) = y$ is α-open and α-closed but neither open nor closed [20].
(iii) Let $f : (X, \tau) \to (Y, \sigma)$ be the function from Example 2.2. Then it is shown that f is weakly preclosed but is not weakly closed.

From the observation above and Example 3.2 we have the following diagram,

$$\begin{array}{ccc} \text{closed function} & \longrightarrow & \alpha\text{-closed function} \\ \downarrow & & \downarrow \\ \text{weakly preclosed function} & \longleftarrow & \text{preclosed function} \end{array}$$

Theorem 3.3. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent.*

(i) *f is weakly preclosed.*
(ii) *$\mathrm{pCl}(f(U)) \subset f(\mathrm{Cl}(U))$ for every open set U of X.*
(iii) *$\mathrm{pCl}(f(U)) \subset f(\mathrm{Cl}(U))$ for each open set U in X,*

(iv) $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(F)$ *for each closed subset* F *in* X,
(v) $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(F)$ *for each preclosed subset* F *in* X,
(vi) $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(F)$ *for every* α*-closed subset* F *in* X,
(vii) $\mathrm{pCl}(f(U)) \subset f(\mathrm{Cl}(U))$ *for each regular open subset* U *of* X,
(viii) *For each subset* F *in* Y *and each open set* U *in* X *with* $f^{-1}(F) \subset U$, *there exists a preopen set* A *in* Y *with* $F \subset A$ *and* $f^{-1}(F) \subset \mathrm{Cl}(U)$,
(ix) *For each point* y *in* Y *and each open set* U *in* X *with* $f^{-1}(y) \subset U$, *there exists a preopen set* A *in* Y *containing* y *and* $f^{-1}(A) \subset \mathrm{Cl}(U)$,
(x) $\mathrm{pCl}(f(\mathrm{Int}(\mathrm{Cl}(U)))) \subset f(\mathrm{Cl}(U))$ *for each set* U *in* X,
(xi) $\mathrm{pCl}(f(\mathrm{Int}(\mathrm{Cl}_\theta(U)))) \subset f(\mathrm{Cl}_\theta(U))$ *for each set* U *in* X,
(xii) $\mathrm{pCl}(f(U)) \subset f(\mathrm{Cl}(U))$ *for each preopen set* U *in* X.

Proof. (i) → (ii) Let U be any open subset of X. Then $\mathrm{pCl}(f(U)) = \mathrm{pCl}(f(\mathrm{Int}(U))) \subset \mathrm{pCl}(f(\mathrm{Int}(\mathrm{Cl}(U))) \subset f(\mathrm{Cl}(U))$.

(ii) → (i) Let F be any closed subset of X. Then $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(\mathrm{Cl}(\mathrm{Int}(F))) \subset f(\mathrm{Cl}(F)) = f(F)$.

It is clear that: (i) → (iii) → (iv) → (v) → (vi) → (i), (i) →(xi), (viii) → (ix), and (i) → (x) → (xii) → (vii) → (i).

(vii) → (viii) Let F be a subset in Y and let U be open in X with $f^{-1}(F) \subset U$. Then $f^{-1}(F) \cap \mathrm{Cl}(X - \mathrm{Cl}(U)) = \emptyset$ and consequently, $F \cap f(\mathrm{Cl}(X - \mathrm{Cl}(U))) = \emptyset$. Since $X - \mathrm{Cl}(U)$ is regular open, $F \cap \mathrm{pCl}(f(X - \mathrm{Cl}(U))) = \emptyset$ by (vii). Let $A = Y - \mathrm{pCl}(f(X - \mathrm{Cl}(U)))$. Then A is preopen with $F \subset A$ and $f^{-1}(A) \subset X - f^{-1}(\mathrm{pCl}(f(X - \mathrm{Cl}(U)))) \subset X - f^{-1}f(X - \mathrm{Cl}(U)) \subset \mathrm{Cl}(U)$.

(xi) → (i) It is suffices see that $\mathrm{Cl}_\theta(U) = \mathrm{Cl}(U)$ for every open sets U in X.

(ix) → (i) Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F$, there exists a preopen A in Y with $y \in A$ and $f^{-1}(A) \subset \mathrm{Cl}(X - F) = X - \mathrm{Int}(F)$ by (ix). Therefore $A \cap f(\mathrm{Int}(F)) = \emptyset$, so that $y \in Y - \mathrm{pCl}(f(\mathrm{Int}(F)))$. Thus (ix) → (i). Finally, for

(xi) → (xii) Note that $\mathrm{Cl}_\theta(U) = \mathrm{Cl}(U)$ for each preopen subset U of X. □

Remark 3.4. By Theorem 2.5, if $f : (X, \tau) \to (Y, \sigma)$ is a bijective function, then f is weakly preopen if and only if f is weakly preclosed.

Next we investigate conditions under which weakly preclosed functions are preclosed.

Theorem 3.5. *If* $f : (X, \tau) \to (Y, \sigma)$ *is weakly preclosed and if for each closed subset* F *of* X *and each fiber* $f^{-1}(y) \subset X - F$ *there exists a open* U *of* X *such that* $f^{-1}(y) \subset U \subset \mathrm{Cl}(U) \subset X - F$. *Then* f *is preclosed.*

Proof. Let F is any closed subset of X and let $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \emptyset$ and hence $f^{-1}(y) \subset X - F$. By hypothesis, there exists a open U of X such that $f^{-1}(y) \subset U \subset \mathrm{Cl}(U) \subset X - F$. Since f is weakly preclosed by Theorem 3.5, there exists a preopen V in Y with $y \in V$ and $f^{-1}(V) \subset \mathrm{Cl}(U)$. Therefore, we obtain $f^{-1}(V) \cap F = \emptyset$ and hence $V \cap f(F) = \emptyset$, this shows that $y \notin \mathrm{pCl}(f(F))$. Therefore, $f(F)$ is preclosed in Y and f is preclosed. □

Theorem 3.6. (i) *If* $f : (X, \tau) \to (Y, \sigma)$ *is preclosed and contra-closed, then* f *is weakly preclosed.*

(ii) *If* $f : (X, \tau) \to (Y, \sigma)$ *is contra-open, then* f *is weakly preclosed.*

Proof. (i) Let F be a closed subset of X. Since f is preclosed $\mathrm{Cl}(\mathrm{Int}(f(F))) \subset f(F)$ and since f is contra-closed $f(F)$ is open. Therefore $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset \mathrm{pCl}(f(F)) \subset \mathrm{Cl}(\mathrm{Int}(f(F))) \subset f(F)$.

(ii) Let F be a closed subset of X. Then, $\mathrm{pCl}(f(\mathrm{Int}(F))) \subset f(\mathrm{Int}(F)) \subset f(F)$. □

Theorem 3.7. *If $f : (X, \tau) \to (Y, \sigma)$ is one-one and weakly preclosed , then for every subset F of Y and every open set U in X with $f^{-1}(F) \subset U$, there exists a preclosed set B in Y such that $F \subset B$ and $f^{-1}(B) \subset \mathrm{Cl}(U)$.*

Proof. Let F be a subset of Y and let U be a open subset of X with $f^{-1}(F) \subset U$. Put $B = \mathrm{pCl}(f(\mathrm{Int}(\mathrm{Cl}(U))))$, then B is a preclosed subset of Y such that $F \subset B$ since $F \subset f(U) \subset f(\mathrm{Int}(\mathrm{Cl}(U)) \subset \mathrm{pCl}(f(\mathrm{Int}(\mathrm{Cl}(U))) = B$. And since f is weakly preclosed, $f^{-1}(B) \subset \mathrm{Cl}(U)$. □

Taking the set F in Theorem 3.7 to be y for $y \in Y$ we obtain the following result,

Corollary 3.8. *If $f : (X, \tau) \to (Y, \sigma)$ is one-one and weakly preclosed , then for every point y in Y and every open set U in X with $f^{-1}(y) \subset U$, there exists a preclosed set B in Y containing y such that $f^{-1}(B) \subset \mathrm{Cl}(U)$.*

Recall that, a set F in a space X is θ-compact if for each cover Ω of F by open U in X, there is a finite family $U_1, \ldots, U_n$ in Ω such that $F \subset \mathrm{Int}(\cup\{\mathrm{Cl}(U_i) : i = 1, 2, \ldots, n\})$ [27].

Theorem 3.9. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly preclosed with all fibers θ-closed, then $f(F)$ is preclosed for each θ-compact F in X.*

Proof. Let F be θ-compact and let $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \emptyset$ and for each $x \in F$ there is an open U_x containing x in X and $\mathrm{Cl}(U_x) \cap f^{-1}(y) = \emptyset$. Clearly $\Omega = \{U_x : x \in F\}$ is an open cover of F and since F is θ-compact, there is a finite family $\{U_{x_1}, \ldots, U_{x_n}\}$ in Ω such that $F \subset \mathrm{Int}(A)$, where $A = \cup\{\mathrm{Cl}(U_{x_i}) : i = 1, \ldots, n\}$. Since f is weakly preclosed by Theorem 2.5 there exists a preopen B in Y with $f^{-1}(y) \subset f^{-1}(B) \subset \mathrm{Cl}(X - A) = X - \mathrm{Int}(A) \subset X - F$. Therefore $y \in B$ and $B \cap f(F) = \emptyset$. Thus $y \in Y - \mathrm{pCl}(f(F))$. This shows that $f(F)$ is preclosed. □

Two non empty subsets A and B in X are strongly separated [27], if there exist open sets U and V in X with $A \subset U$ and $B \subset V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. If A and B are singleton sets we may speak of points being strongly separated . We will use the fact that in a normal space, disjoint closed sets are strongly separated.

Recall that a space X is said to be pre-Hausdorff or in short pre-T_2 [15] if for every pair of distinct points x and y, there exist two preopen sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Theorem 3.10. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly preclosed surjection and all pairs of disjoint fibers are strongly separated, then Y is pre-T_2.*

Proof. Let y and z be two points in Y. Let U and V be open sets in X such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ respectively with $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. By weak preclosedness (Theorem 3.3 (vii)) there are preopen sets F and B in Y such that $y \in F$ and $z \in B$, $f^{-1}(F) \subset \mathrm{Cl}(U)$ and $f^{-1}(B) \subset \mathrm{Cl}(V)$. Therefore $F \cap B = \emptyset$, because $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$ and f surjective. Then Y is pre-T_2. □

Corollary 3.11. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly preclosed surjection with all fibers closed and X is normal, then Y is pre-T_2.*

Corollary 3.12. *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous weakly preclosed surjection with X compact T_2 space and Y a T_1 space, then Y is compact pre-T_2 space.*

Proof. Since f is a continuous surjection and Y is a T_1 space, Y is compact and all fibers are closed. Since X is normal Y is also pre-T_2. □

Definition 3.13. A topological space X is said to be quasi H-closed [8] (resp. P--closed), if every open (resp. pre-closed) cover of X has a finite subfamily whose closures cover X. A subset A of a topological space X is quasi H-closed relative to X (resp. P-closed relative to X) if every cover of A by open (resp. pre-closed) sets of X has a finite subfamily whose closures cover A.

Lemma 3.14. *A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if for each $B \subset Y$, $f^{-1}(\mathrm{Cl}(B)) \subset \mathrm{Cl}(f^{-1}(B))$ [17].*

Theorem 3.15. *Let X be an extremally disconnected space and $PO(X, \tau)$ closed under finite intersections. Let $f : (X, \tau) \to (Y, \sigma)$ be an open weakly preclosed function which is one-one and such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is P-closed relative to Y then $f^{-1}(G)$ is quasi H-closed.*

Proof. Let $\{V_\beta : \beta \in I\}$, I being the index set be an open cover of $f^{-1}(G)$. Then for each $y \in G \cap f(X)$, $f^{-1}(y) \subset \bigcup\{\mathrm{Cl}(V_\beta) : \beta \in I(y)\} = H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $\mathrm{Cl}(V_\beta)$ is open, hence H_y is open in X. So by Corollary 3.8, there exists a preclosed set U_y containing y such that $f^{-1}(U_y) \subset \mathrm{Cl}(H_y)$. Then, $\{U_y : y \in G \cap f(X)\} \cup \{Y - f(X)\}$ is a preclosed cover of G, $G \subset \bigcup\{\mathrm{Cl}(U_y) : y \in K\} \cup \{\mathrm{Cl}(Y - f(X))\}$ for some finite subset K of $G \cap f(X)$. Hence and by Lemma 3.15, $f^{-1}(G) \subset \bigcup\{f^{-1}(\mathrm{Cl}(U_y) : y \in K\} \cup \{f^{-1}(\mathrm{Cl}(Y - f(X))\} \subset \bigcup\{\mathrm{Cl}(f^{-1}(U_y) : y \in K\} \cup \{\mathrm{Cl}(f^{-1}(Y - f(X)))\} \subset \{\mathrm{Cl}(f^{-1}(U_y)) : y \in K\}$, so $f^{-1}(G) \subset \bigcup\{\mathrm{Cl}(V_\beta) : \beta \in I(y),\ y \in K\}$. Therefore $f^{-1}(G)$ is quasi H-closed. □

Corollary 3.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be as in Theorem 3.15. If Y is p-closed, then X is quasi-H-closed.*

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