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# ON PRODUCT OF PROJECTIONS

MOHAMMAD SAL MOSLEHIAN

Abstract. An operator with infinite dimensional kernel is positive iff it is a positive scalar times a certain product of three projections.

## 1. Introduction

Representing a certain operator on a Hilbert space as a product of "nice" operators such as selfadjoint, positive, unitary, idempotent and symmetry ones have been considered by many mathematicians. Following are some significant results in the infinite dimensional case (results in the finite dimensional case is different; (cf. [7])):

(i) A normal operator is the product of 4 selfadjoint operators [6].

(ii) Every isometry is either unitary or a shift or a product of two operators of these two kinds [2].

(iii) Each unitary operator is the product of 4 symmetries [3].

(iv) An invertible operator is a product of 7 positive operators [5].

(v) Every normal operator is the product of three diagonal operators [1].

(vi) Each symmetry (unitary) operator is the product of 6 (16) positive invertible operators [8].

More results may be seen in [9] which is an excellent survey of factorizations into "good" operators.

We aim to prove that an operator $T$ on a separable infinite dimensional Hilbert space with infinite kernel is positive iff $T = \lambda PQP$, for some positive scalar $\lambda$ and projections $P$ and $Q$.

## 2. Main result

It is well-known that if $T$ is a linear transformation in a finite dimensional complex (or real) vector space having nonzero kernel, then $T$ is the product of a finite number of projections [4].

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Let's now $T$ be an operator acting on a separable infinite dimensional Hilbert space $H$ with infinite null space $H_\circ$. If $\{\xi_1, \xi_2, \cdots\}$ is an orthonormal basis of $H_\circ$, it can be extended to an orthonormal basis for $H$. If $K$ is the closed linear span of $\{\xi_2, \xi_4, \xi_6, \cdots\}$, then obviously $\dim K = \dim K^\perp$.

Suppose now $V$ is a unitary from $K^\perp$ onto $K$ and $W$ is the unitary from $H = K \oplus K^\perp$ onto $K \oplus K$ defined by $W(y \oplus z) = y \oplus Vz$. The operator $WTW^*$ could then be represented by $\begin{bmatrix} 0 & R \\ 0 & S \end{bmatrix}$ on $K \oplus K$. Let's identify $T$ to $WTW^*$.

If $T$ is positive, then $R = 0$ and so $T = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$. Moreover $0 \leq S_1 = \frac{S}{\|T\|} \leq I$. Applying functional calculus, we can show that $\begin{bmatrix} I - S_1 & (S_1 - S_1^2)^{\frac{1}{2}} \\ (S_1 - S_1^2)^{\frac{1}{2}} & S_1 \end{bmatrix} = Q$ is a projection. It follows from

$$\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} = \| T \| \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I - S_1 & (S_1 - S_1^2)^{\frac{1}{2}} \\ (S_1 - S_1^2)^{\frac{1}{2}} & S_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

we conclude that $T = \| T \| PQP$ is a positive scalar times a product of projections $Q$ and $P = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$.

Conversely, if $T = \lambda PQP$, for some positive scalar $\lambda$ and projections $P$ and $Q$, then for $\eta \in H, \langle T\eta, \eta \rangle = \lambda \langle QP\eta, QP\eta \rangle \geq 0$. Hence $T$ is positive.

In the finite dimensional case $H = \mathbf{C}^2$, however, the fact that $T = \| T \| PQP$ implies that $\det(T) = \| T \|^2 \det(P)^2 \det(Q)$ which is not true for $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

**Remark.** Thanking Pei Yuan Wu, we note that an operator $T$ is expressible as $PQP$ for some projections $P$ and $Q$ if and only if $T$ is the direct some of the identity operator (on some space $K$) and a positive contraction $A$ on $L$ with nullity at least one half of the dimension of $L$.

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Department of Mathematics, Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, Iran
and
Institute for Studies in Theoretical Physics and Mathematics (IPM), Iran
*E-mail*: msalm@math.um.ac.ir